TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 176, February 1973

A NECESSARY AND SUFFICIENT CONDITION FOR A "SPHERE" TO SEPARATE POINTS IN EUCLIDEAN, HYPERBOLIC, OR SPHERICAL SPACE

BY

J. E. VALENTINE AND S. G. WAYMENT

ABSTRACT. The purpose of this paper is to give conditions wholly and explicitly in terms of the mutual distances of $n+3$ points in n-space which are necessary and sufficient for two of the points to lie in the same or different components of the space determined by the sphere which is determined by $n+1$ of the points. Thus in euclidean space we prove that if the cofactor $[p_ip_j^2]$ of the element $p_ip_j^2$ $(i \neq j)$ in the determinant $|p_ip_j^2|$ $(i, j = 0, 1, \cdots, n+2)$ is nonzero then p_i, p_j lie in the same or different components of $E_n - \Omega$ (where Ω denotes the sphere or hyperplane containing the remaining $n+1$ points) if and only if $\operatorname{sgn} [p_ip_j^2] = (-1)^n$ or $(-1)^{n+1}$, respectively. In hyperbolic space the result is: if the cofactor $[\sinh^2 p_ip_j/2]$ of the element $\sinh^2 p_ip_j/2$ $(i \neq j)$ in the determinant $|\sinh^2 p_ip_j/2|$ $(i, j = 0, 1, \cdots, n+1)$ is nonzero then p_i, p_j lie in the same or different components of $H_n - \Omega$ (where Ω denotes the hyperplane, sphere, horosphere, or one branch of an equidistant surface containing the remaining $n+1$ points) if and only if $\operatorname{sgn} [\sinh^2 p_ip_j/2] = (-1)^n$ or $(-1)^{n+1}$, respectively. For spherical space we obtain: if the cofactor $[\sin^2 p_ip_j/2]$ of the element $\sin^2 p_ip_j/2$ $(i \neq j)$ in the determinant $|\sin^2 p_ip_j/2|$ $(i, j = 0, 1, \cdots, n+2)$ is nonzero then p_i, p_j lie in the same or different components of $S_n - \Omega$ (where Ω denotes the sphere containing the remaining $n+1$ points which may be an $(n-1)$ dimensional subspace) if and only if $\operatorname{sgn} [\sin^2 p_ip_j/2] = (-1)^n$ or $(-1)^{n+1}$ respectively.

1. **Introduction and notation.** The symmetric determinants

$$D(p_0, p_1, p_2, \cdots, p_k) = \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & p_0p_1^2 & \cdots & p_0p_k^2 \\ 1 & p_0p_1^2 & 0 & \cdots & p_1p_k^2 \\ 1 & p_0p_2^2 & p_1p_2^2 & \cdots & p_2p_k^2 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & p_1p_k^2 & \cdots & 0 \end{vmatrix}$$

Received by the editors June 4, 1971 and, in revised form, February 3, 1972.

AMS (MOS) subject classifications (1969). Primary 5250, 5010, 5040.

Key words and phrases. Components, determinant, euclidean, hyperbolic, sphere, spherical.

have long played a fundamental role in the study of euclidean geometry. Cayley [4], for example, established anew the fact known to Lagrange that $D(p_0, p_1, p_2, p_3, p_4)$ vanishes for any five points p_0, p_1, p_2, p_3, p_4 of 3-dimensional euclidean space E_3. In an elegant article Darboux [5] gave the complete geometric equivalents of the minors of $D(p_0, p_1, p_2, p_3)$ for p_0, p_1, p_2, p_3 points of E_3 and he very nearly obtained our result in E_3. In this paper we proceed along the lines of [3], where Blumenthal and Gillam gave necessary and sufficient conditions for two points p_n, p_{n+1} of n-dimensional euclidean space E_n to lie on the same or opposite sides of the hyperplane of E_n determined by the points $p_0, p_1, \cdots, p_{n-1}$. Their principal tool was $D(p_0, p_1, \cdots, p_{n+1})$, which vanishes for $n+2$ points of E_n. We, like Blumenthal and Gillam, obtain our results without the use of coordinates. We will be concerned in the euclidean case with the unbordered principal minor $C(p_0, p_1, \cdots, p_{n+2})$ of $D(p_0, p_1, \cdots, p_{n+2})$, which vanishes for $n+3$ points of E_n, while $C(p_0, p_1, \cdots, p_{n+1})$ vanishes for $n+2$ points of E_n if and only if the $n+2$ points lie on a sphere or hyperplane and $C(p_0, p_1, \cdots, p_{n+1})$ has sign $(-1)^{n+1}$ otherwise.

Blumenthal [2] and [1] used the determinants

$$\Delta(p_0, p_1, \cdots, p_k) = \begin{vmatrix} 1 & \cos p_0p_1 & \cos p_0p_2 & \cdots & \cos p_0p_k \\ \cos p_0p_1 & 1 & \cos p_1p_2 & \cdots & \cos p_1p_k \\ \cos p_0p_2 & \cos p_1p_2 & 1 & \cdots & \cos p_2p_k \\ \vdots & \vdots & \vdots & & \vdots \\ \cos p_0p_k & \cos p_{1k} & \cos p_2p_k & \cdots & 1 \end{vmatrix}$$

and

$$\Lambda(p_0, p_1, \cdots, p_k) = \begin{vmatrix} 1 & \cosh p_0p_1 & \cosh p_0p_2 & \cdots & \cosh p_0p_k \\ \cosh p_0p_1 & 1 & \cosh p_1p_2 & \cdots & \cosh p_1p_k \\ \cosh p_0p_2 & \cosh p_1p_2 & 1 & \cdots & \cosh p_2p_k \\ \vdots & \vdots & \vdots & & \vdots \\ \cosh p_0p_k & \cosh p_1p_k & \cosh p_2p_k & \cdots & 1 \end{vmatrix}$$

to characterize spherical and hyperbolic space with space constants 1 and -1, respectively.

Haantjes [6], while studying when the local property of vanishing curvature implies an arc in certain metric spaces is a geodesic, intoduced the determinants

$$\gamma(p_0, p_1, p_2, p_3) = |\sin^2 p_ip_j/2| \qquad (i, j = 0, 1, 2, 3)$$

and

$$K(p_0, p_1, p_2, p_3) = |\sinh^2 p_ip_j/2| \qquad (i, j = 0, 1, 2, 3)$$

where p_0, p_1, p_2, p_3 are quadruples of points in the spherical and hyperbolic planes, respectively.

These determinants arise from the respective determinants $\Delta(p_0, p_1, p_2, p_3)$ and $\Lambda(p_0, p_1, p_2, p_3)$ by bordering these respective determinants with a first row and column with "intersecting" element -1, and the remaining elements of the first column one and the remaining elements in the first row zero. Upon subtracting the first column from the remaining columns one obtains the determinants

$$I_\Delta(p_0, p_1, p_2, p_3) = \begin{vmatrix} -1 & \vdots & 1 \\ \cdots & \cdot & \cdots \\ 1 & \vdots & -2\sin^2 p_ip_j/2 \end{vmatrix} \qquad (i, j = 0, 1, 2, 3)$$

and

$$I_\Lambda(p_0, p_1, p_2, p_3) = \begin{vmatrix} -1 & \vdots & 1 \\ \cdots & \cdot & \cdots \\ 1 & \vdots & 2\sinh^2 p_ip_j/2 \end{vmatrix} \qquad (i, j = 0, 1, 2, 3).$$

The determinants $\gamma(p_0, p_1, p_2, p_3)$ and $K(p_0, p_1, p_2, p_3)$ are within a constant of being principal minors of $I_\Delta(p_0, p_1, p_2, p_3)$ and $I_\Lambda(p_0, p_1, p_2, p_3)$. Valentine [10] showed that four points, p_0, p_1, p_2, p_3, of the hyperbolic plane lie on a line, circle, horocycle, or one branch of an equidistant curve if and only if $K(p_0, p_1, p_2, p_3) = 0$. Andalafte and Valentine [11] extended this result to show that $n+2$ points $p_0, p_1, \cdots, p_{n+1}$ of n-dimensional hyperbolic space lie on a hyperplane, $(n-1)$-dimensional sphere, $(n-1)$-dimensional horosphere, or one sheet of an $(n-1)$-dimensional equidistant surface if and only if $K(p_0, p_1, \cdots, p_{n+1}) = |\sinh^2 p_ip_j/2|$ vanishes $(i, j = 0, 1, \cdots, n+1)$. They showed, moreover, that $\operatorname{sgn} K(p_0, p_1, \cdots, p_k) = (-1)^k$ $(i, j = 0, 1, \cdots, k)$ in the event that $K(p_0, p_1, \cdots, p_k) \neq 0$ $(k = 1, 2, \cdots, n+1)$ and $K(p_0, p_1, \cdots, p_k) = 0$ if $k \geq n+2$. Valentine [9] showed that four points p_0, p_1, p_2, p_3 of the spherical plane lie on a circle if and only if $\gamma(p_0, p_1, p_2, p_3) = 0$. From [11] it is clear that the analogous properties of the determinants $\gamma(p_0, p_1, \cdots, p_k)$ are valid in n-dimensional spherical space.

An $(n-1)$-dimensional hyperplane and an $(n-1)$-dimensional sphere separate n-dimensional euclidean space into two components. In n-dimensional hyperbolic space an $(n-1)$-dimensional hyperplane, an $(n-1)$-dimensional sphere, an $(n-1)$-dimensional horosphere, and one sheet of an $(n-1)$- dimensional equidistant surface separate the space into two components. While in n-dimensional spherical spaces the $(n-1)$-dimensional spheres separate the space into two components. With the exception of the hyperplanes in the respective spaces, the "spheres" are determined by $n+1$ independent points in the respective n-dimensional spaces, while the $(n-1)$-dimensional hyperplanes are determined by n independent points. It is often desirable to know when two points are in the same or different components.

In this paper we give necessary and sufficient conditions for points p_{n+1}, p_{n+2} to lie in the same or different components of a "sphere" determined by independent points $p_0, p_1, \cdots, p_n$ of n-dimensional euclidean, hyperbolic, or spherical space. We give the same characterizations for points p_{n+1}, p_{n+2} relative to $(n-1)$-dimensional hyperplanes containing points $p_0, p_1, \cdots, p_n$ which contain an independent n-tuple. In the process we give the geometrical significance of the signs of the non-principal minors of the determinants $C(p_0, p_1, \cdots, p_{n+2})$, $K(p_0, p_1, \cdots, p_{n+2})$, and $\gamma(p_0, p_1, \cdots, p_{n+2})$. Throughout this paper we will adhere to the notation we have introduced here.

2. **Cofactors of** $C(p_0, p_1, \cdots, p_{n+2})$, **where** $p_0, p_1, \cdots, p_{n+2}$ **are points in** n**-dimensional euclidean space** E_n. We are concerned here with cofactors $[p_i p_j^2]$ of elements $p_i p_j^2$ $(i \neq j)$, of $C(p_0, p_1, \cdots, p_{n+2})$. We will select the cofactor of $p_{n+1}\, p_{n+2}^2$ as typical. Since hyperplanes and spheres are equivalent under the group generated by inversions and this group preserves components, we first show that sgn $[p_{n+1}\, p_{n+2}^2]$ is an inversive invariant, where $[p_{n+1}\, p_{n+2}^2]$ denotes the cofactor of the element $p_{n+1}\, p_{n+2}^2$ in the determinant $C(p_0, p_1, \cdots, p_{n+2})$. In order to obtain our result it then suffices to give the geometrical significance of sgn $[p_{n+1}\, p_{n+2}^2]$ when $p_0, p_1, \cdots, p_n$ is an independent $(n+1)$-tuple which determines a sphere S.

Theorem 2.1. *Let $p_0, p_1, \cdots, p_{n+2}$ be an $(n+3)$-tuple in n-dimensional euclidean space E_n. Then* sgn $[p_{n+1}\, p_{n+2}^2]$ *is an inversive invariant.*

Proof. From the formula for inversion, points x, x' are inverse points with respect to a circle with center o and radius r if and only if $ox \cdot ox' = r^2$.

If x', y' are inverse points of x, y, respectively, then the two triangles oxy and $ox'y'$ have the same angle at o. It follows from the euclidean law of cosines that

$$(1) \qquad (ox^2 + oy^2 - xy^2)/(2ox \cdot oy) = (ox'^2 + oy'^2 - x'y'^2)/(2ox' \cdot oy').$$

Replacing ox' and oy' in (1) by r^2/ox and r^2/oy, respectively, and solving the resulting equation for $x'y'$ we obtain

$$(2) \qquad x'y'^4 = (r^2/ox^2 \cdot oy^2) \cdot xy^2.$$

Application of (2) to all pairs of $p_0, p_1, \cdots, p_{n+2}$ yields

$$(3) \qquad p_i' p_j'^2 = (r^4/op_i^2 \cdot op_j^2) \cdot p_i p_j^2 \qquad (i, j = 0, 1, \cdots, n+2).$$

Consequently

$$(4) \qquad [p_{n+1}' p_{n+2}'^2] = |(r^4/op_i^2 \cdot op_j^2) \cdot p_i p_j^2|$$

$$(i, j = 0, 1, \cdots, n+2;\ i \neq n+1;\ j \neq n+2).$$

Factoring r^2/op_i^2 from the ith row $(i = 0, 1, \cdots, n)$ and r^2/op_{n+2}^2 from the $(n+1)$st row and factoring r^2/op_j^2 from the jth column $(j = 0, 1, \cdots, n+1)$ of the determinant on the right side of the equality sign in (4) yields

$$[p_i'p_j'^2] = \pi(r^4/op_i^2 \cdot op_j^2)|p_i p_j^2| \qquad (i = 0, 1, \cdots, n, n+2;\ j = 0, 1, \cdots, n+1)$$

and the proof is complete.

Theorem 2.2. *Let* $[p_{n+1}\,p_{n+2}^2]$ *be different from zero. Then* (1) *the points* p_0, $p_1, \cdots, p_n$ *lie on an* $(n-1)$*-dimensional hyperplane and not on an* $(n-2)$*-dimensional hyperplane or* $p_0, p_1, \cdots, p_n$ *determine an* $(n-1)$*-dimensional sphere and* (2) p_{n+1} *and* p_{n+2} *are in the same or different components of* $E_n - \Omega$ *(where* Ω *denotes the hyperplane or sphere containing* $p_0, p_1, \cdots, p_n$*) if and only if* $\operatorname{sgn}[p_{n+1}\,p_{n+2}^2] = (-1)^n$ *or* $(-1)^{n+1}$*, respectively.*

Proof. Since $C(p_0, p_1, \cdots, p_{n+2}) = 0$, from an expansion theorem for determinants (see [6, p. 372]) we have

$$\begin{aligned} C(p_0, p_1, \cdots, p_n, p_{n+1}) \cdot C(p_0, p_1, \cdots, p_n, p_{n+2}) - [p_{n+1}p_{n+2}^2]^2 \\ = C(p_0, p_1, \cdots, p_{n+2}) \cdot C(p_0, p_1, \cdots, p_n) = 0. \end{aligned} \tag{5}$$

The nonvanishing of $[p_{n+1}\,p_{n+2}^2]$ implies that neither $C(p_0, p_1, \cdots, p_n, p_{n+1})$ nor $C(p_0, p_1, \cdots, p_n, p_{n+2})$ vanishes, and hence the points $p_0, p_1, \cdots, p_n$ are in an E_{n-1}, not in an E_{n-2}, or $p_0, p_1, \cdots, p_n$ are not in an E_{n-1} and they determine an $(n-1)$-dimensional sphere $S(p_0, p_1, \cdots, p_n)$.

Case 1. The points $p_0, p_1, \cdots, p_n$ determine an $(n-1)$-dimensional sphere $S(p_0, p_1, \cdots, p_n)$.

Replacing p_{n+2} in (5) by any point x of E_n, it is seen that $[p_{n+1}x^2]$ vanishes if and only if x is a point of S and hence $\operatorname{sgn}[p_{n+1}x^2] = \operatorname{sgn}[p_{n+1}y^2]$ for any two points x, y in the same component of $E_n - S$.

If p_{n+1} and p_{n+2} are in the same component of $E_n - S$, then $\operatorname{sgn}[p_{n+1}p_{n+2}^2] = \operatorname{sgn}[p_{n+1}p_{n+1}^2]$. Since $[p_{n+1}\,p_{n+1}^2] = -C(p_0, p_1, \cdots, p_{n+1})$, for points p_{n+1}, p_{n+2} in the same component $\operatorname{sgn}[p_{n+1}\,p_{n+2}^2] = (-1)^n$.

On the other hand, suppose p_{n+1} and p_{n+2} are in different components of $E_n - S$ and denote the inverse point of p_{n+1} in S by p_{n+1}^*. Then $\operatorname{sgn}[p_{n+1}\,p_{n+2}^2] = \operatorname{sgn}[p_{n+1}\,p_{n+1}^{*2}]$. Inspection of the vanishing determinant $C(p_0, p_1, \cdots, p_{n+1}, p_{n+1}^*)$ gives

$$\begin{aligned} [p_{n+1}p_{n+1}^{*2}] = -\{(r^4/op_{n+1}^2) \cdot C(p_0, p_1, \cdots, p_{n+1}) \\ + [p_{n+1}p_{n+1}^*(r^2/op_{n+1}^2)] \cdot C(p_0, p_1, \cdots, p_n)\} \end{aligned} \tag{6}$$

where r denotes the radius of S and o denotes the center of S. From (5),

$[p_{n+1}p_{n+1}^{*2}]^2 = (r^4/op_{n+1}^2)\, C^2(p_0, p_1, \cdots, p_{n+1})$. It follows that $[p_{n+1}p_{n+1}^{*2}] = (r^2/op_{n+1}) \cdot C(p_0, p_1, \cdots, p_{n+1})$ and hence $\operatorname{sgn}[p_{n+1}p_{n+1}^{*2}] = (-1)^{n+1}$. Thus for p_{n+1} and p_{n+2} in different components of $E_n - S$, $\operatorname{sgn}[p_{n+1}p_{n+2}^2] = (-1)^{n+1}$.

To establish the converse it suffices to show that if $\operatorname{sgn}[p_{n+1}p_{n+2}^2] = (-1)^n$, then p_{n+1} and p_{n+2} are in the same component of $E_n - S$. This is trivial, for then $\operatorname{sgn}[p_{n+1}p_{n+2}^2] \neq \operatorname{sgn}[p_{n+1}p_{n+1}^{*2}]$ and hence p_{n+1}^*, p_{n+2} are not in the same components of $E_n - S$. Since none of the points $p_{n+1}, p_{n+2}, p_{n+1}^*$ lies on S, it follows that p_{n+1} and p_{n+2} are in the same component of $E_n - S$ and Case 1 of the theorem is proved.

Case 2. The points $p_0, p_1, \cdots, p_n$ lie in an $(n-1)$-dimensional hyperplane.

The hyperplane H containing $p_0, p_1, \cdots, p_n$ may be mapped onto an $(n-1)$-dimensional sphere S by an inversion. If $p_0', p_1', \cdots, p_{n+2}'$ are the inverse points of $p_0, p_1, \cdots, p_{n+2}$, then since $C(p_0, p_1, \cdots, p_{n+1}) \neq 0$, we have $C(p_0', p_1', \cdots, p_{n+1}') \neq 0$ and it follows that $p_0', p_1', \cdots, p_n'$ determines S. Applying Case 1 to the points $p_0', p_1', \cdots, p_{n+1}'$, we see p_{n+1}', p_{n+2}' lie in the same or different components of $E_n - S$ if and only if $\operatorname{sgn}[p_{n+1}'p_{n+2}'^2] = (-1)^n$ or $(-1)^{n+1}$, respectively. Since $\operatorname{sgn}[p_{n+1}'p_{n+2}'^2]$ is an inversive invariant and components of $E_n - S$ are preserved under inversions, if S is mapped back on the hyperplane H containing $p_0, p_1, \cdots, p_n$, we see that p_{n+1}, p_{n+2} are in the same or different components of $E_n - H$ if and only if $\operatorname{sgn}[p_{n+1}p_{n+2}^2] = (-1)^n$ or $(-1)^{n+1}$, respectively.

3. **Cofactors of $K(p_0, p_1, \cdots, p_{n+2})$ where $p_0, p_1, \cdots, p_{n+2}$ are points in n-dimensional hyperbolic space H_n with space constant 1.** In this section we are concerned with cofactors $[\sinh^2 p_ip_j/2]$ of elements $\sinh^2 p_ip_j/2$ $(i \neq j)$ of $K(p_0, p_1, \cdots, p_{n+2})$. Once again we select the cofactor of $\sinh^2 p_{n+1}p_{n+2}/2$ as typical. Since hyperplanes, spheres, horospheres, and sheets of equidistant surfaces are equivalent under the group generated by hyperbolic inversions and this group preserves components, we first show that $\operatorname{sgn}[\sinh^2 p_{n+1}p_{n+2}/2]$ is an inversive invariant. In order to obtain our result, it then suffices to give the geometric interpretation of $\operatorname{sgn}[\sinh^2 p_{n+1}p_{n+2}/2]$ when $p_0, p_1, \cdots, p_n$ is an independent $(n+1)$-tuple which determines a sphere S.

Theorem 3.1. *Let $p_0, p_1, \cdots, p_{n+2}$ be an $(n+3)$-tuple of points in n-dimensional hyperbolic space H_n. Then $\operatorname{sgn}[\sinh^2 p_{n+1}p_{n+2}/2]$ is an inversive invariant, where $[\sinh^2 p_{n+1}p_{n+2}/2]$ denotes the cofactor of the element $\sinh^2 p_{n+1}p_{n+2}/2$ in the determinant $K(p_0, p_1, \cdots, p_{n+2})$.*

Proof. From the hyperbolic formula for inversion [8, p. 242] points x, x' are inverse points with respect to a sphere with center o and radius r if and only if $\tanh ox/2 \cdot \tanh ox'/2 = \tanh^2 r/2$.

If x', y' are the inverse points of x, y, respectively, then the two triangles oxy and $ox'y'$ have the same angle at o. It follows from the hyperbolic law of cosines that

$$\begin{aligned}(7)\qquad &[\cosh ox \cosh oy - \cosh xy]/[\sinh ox \sinh oy]\\ &\qquad = [\cosh ox' \cosh oy' - \cosh x'y']/[\sinh ox' \sinh oy'].\end{aligned}$$

Let $X = \tanh ox/2$, $Y = \tanh oy/2$, and $R = \tanh^2 r/2$. Then $\tanh ox'/2 = R/X$, and consequently,

$$\begin{aligned}&\cosh ox = [1 + X^2]/[1 - X^2], \qquad \sinh ox = 2X/[1 - X^2],\\ (8)\qquad &\cosh ox' = [X^2 + R^2]/[X^2 - R^2],\\ &\sinh ox' = 2XR/[X^2 - R^2], \qquad 1 + 2\sinh^2 xy/2 = \cosh xy.\end{aligned}$$

Substituting the values of (8) together with the same identities when x, X are replaced by y, Y, respectively, in (7) and solving the new equation for $\sinh x'y'/2$, we obtain

$$(9)\qquad \sinh x'y'/2 = R[(1 - X^2)/(X^2 - R^2)]^{1/2}[(1 - Y^2)/(Y^2 - R^2)]^{1/2} \cdot \sinh xy/2.$$

Application of (9) to all pairs of $p_0, p_1, \cdots, p_{n+2}$ yields

$$(10)\ \sinh^2 p_i' p_j'/2 = R^2[(1 - P_i^2)/(P_i^2 - R^2)][(1 - P_j^2)/(P_j^2 - R^2)] \cdot \sinh^2 p_i p_j/2.$$

Thus

$$\begin{aligned}&[\sinh^2 p'_{n+1} p'_{n+2}]\\ (11)\qquad &= |R^2[(1 - P_i^2)/(P_i^2 - R^2)][(1 - P_j^2)/(P_j^2 - R^2)]\sinh^2 p_i p_j/2|\\ &\qquad\qquad (i, j = 0, 1, \cdots, n+2;\ i \neq n+1;\ j \neq n+2).\end{aligned}$$

Factoring $R[(1 - P_i^2)/(P_i^2 - R^2]$ from the ith row $(i = 0, 1, \cdots, n)$ and $R[(1 - P_{n+2}^2)/(P_{n+2}^2 - R^2)]$ from the $(n+1)$st row and factoring $R[(1 - P_j^2)/(P_j^2 - R^2)]$ from the jth column $(j = 0, 1, \cdots, n+1)$ of the determinant on the right side of the equality sign in (11) yields

$$\begin{aligned}&[\sinh^2 p_i' p_j'/2] = \pi\{R^2[(1 - P_i^2)/(P_i^2 - R^2)][(1 - P_j^2)/(P_j^2 - R^2)]\}|\sinh^2 p_i p_j/2|\\ &\qquad\qquad (i = 0, 1, \cdots, n, n+2;\ j = 0, 1, \cdots, n+1)\end{aligned}$$

and the proof is complete.

Theorem 3.2. *Let* $[\sinh^2 p_i p_j/2]$ *be different from zero. Then* (1) *the points* $p_0, p_1, \cdots, p_n$ *lie on an* $(n-1)$*-dimensional hyperplane and not in an* $(n-2)$*-dimensional hyperplane or* $p_0, p_1, \cdots, p_n$ *determine an* $(n-1)$*-dimensional sphere, horosphere, or one sheet of an equidistant surface and* (2) p_{n+1} *and* p_{n+2} *are in the same or different components of* $H_n - \Omega$ (*where* Ω *denotes the hyperplane,* *sphere, horosphere, or one sheet of an equidistant surface containing* $p_0, p_1, \cdots, p_n$ *if and only if* $\operatorname{sgn}[\sinh^2 p_{n+1}p_{n+2}/2] = (-1)^n$ *or* $(-1)^{n+1}$, *respectively.*

Proof. Since $K(p_0, p_1, \cdots, p_{n+2}) = 0$, from an expansion theorem for determinants (see [6, p. 372]) we have

$$K(p_0, p_1, \cdots, p_n, p_{n+1}) \cdot K(p_0, p_1, \cdots, p_n, p_{n+2}) - [\sinh^2 p_{n+1}p_{n+2}/2]^2 = K(p_0, p_1, \cdots, p_{n+2}) \cdot K(p_0, p_1, \cdots, p_n) = 0. \tag{12}$$

The nonvanishing of $[\sinh^2 p_{n+1}p_{n+2}/2]$ implies that neither $K(p_0, p_1, \cdots, p_n, p_{n+1})$ nor $K(p_0, p_1, \cdots, p_n, p_{n+2})$ vanishes, and thus the points $p_0, p_1, \cdots, p_n$ are in an H_{n-1}, not in an H_{n-2}, or $p_0, p_1, \cdots, p_n$ are not in an H_{n-1} and they lie on an $(n-1)$-dimensional sphere or horosphere or they lie on one sheet of an $(n-1)$-dimensional equidistant surface.

Case 1. The points $p_0, p_1, \cdots, p_n$ determine an $(n-1)$-dimensional sphere $S(p_0, p_1, \cdots, p_n)$.

Replacing p_{n+2} in (12) by any point x of H_n, it is seen that $[\sinh^2 p_{n+1}x/2]$ vanishes if and only if x is a point of S (see [8]) and hence $\operatorname{sgn}[\sinh^2 p_{n+1}x/2] = \operatorname{sgn}[\sinh^2 p_{n+1}y/2]$ for any two points x, y in the same component of $H_n - S$.

If p_{n+1} and p_{n+2} are in the same component of $H_n - S$, then $\operatorname{sgn}[\sinh^2 p_{n+1}p_{n+2}] = \operatorname{sgn}[\sinh^2 p_{n+1}p_{n+1}/2]$. Since $[\sinh^2 p_{n+1}p_{n+1}/2] = -K(p_0, p_1, \cdots, p_{n+1})$, for points in the same component $\operatorname{sgn}[\sinh^2 p_{n+1}p_{n+2}/2] = (-1)^n$.

Now suppose p_{n+1} and p_{n+2} are in different components of $H_n - S$, and denote the inverse point of p_{n+1} by p^*_{n+1}. Then $\operatorname{sgn}[\sinh^2 p_{n+1}p_{n+2}/2] = \operatorname{sgn}[\sinh^2 p_{n+1}p^*_{n+1}, 2]$. Inspection of the vanishing determinant $K(p_0, p_1, \cdots, p_{n+1}, p^*_{n+1})$ gives

$$\begin{aligned}[\sinh^2 p_{n+1}p^*_{n+1}/2] = -\{&R[(1 - P^{*2}_{n+1})/(P^{*2}_{n+1} - R^2)]K(p_0, p_1, \cdots, p_{n+1}) \\ &+ (\sinh^2 p_{n+1}p^*_{n+1}/2)[R(1 - p^*_{n+1})/(p^{*2}_{n+1} - R)] \\ &K(p_0, p_1, \cdots, p_n)\}\end{aligned} \tag{13}$$

where o is the center of S, $2\tanh^{-1}(R)^{1/2}$ is the radius of S, and as in the proof of Theorem 3.1, $p^*_{n+1} = \tanh op^*_{n+1}/2$. From (12),

$$[\sinh^2 p_{n+1}p^*_{n+1}/2]^2 = R^2[(1 - P^{*2}_{n+1})/(P^{*2}_{n+1} - R)]^2 \cdot K^2(p_0, p_1, \cdots, p_{n+1}).$$

It follows that

$$[\sinh^2 p_{n+1}p^*_{n+1}/2] = R[(1 - P^{*2}_{n+1})/(P^{*2}_{n+1} - R^2)]^{1/2} \cdot K(p_0, p_1, \cdots, p_{n+1})$$

and hence $\operatorname{sgn}[\sinh^2 p_{n+1}p^*_{n+1}/2] = (-1)^{n+1}$. Thus for p_{n+1} and p_{n+2} in different components of $H_n - S$, $\operatorname{sgn}[\sinh^2 p_{n+1}p_{n+2}/2] = (-1)^{n+1}$.

To establish the converse it suffices to show that if $\operatorname{sgn}[\sinh^2 p_{n+1}p_{n+2}/2] = (-1)^n$, then p_{n+1} and p_{n+2} are in the same component of $H_n - S$. This is clear, for then $\operatorname{sgn}[\sinh^2 p_{n+1}p_{n+2}/2] \neq \operatorname{sgn}[\sinh^2 p_{n+1}p^*_{n+1}/2]$ and hence p^*_{n+1}, p_{n+2} are not in the same component of $H_n - S$. Since none of the points $p_{n+1}, p_{n+2}, p^*_{n+1}$ lies on S, it follows that p_{n+1} and p_{n+2} are in the same component of S and Case 1 of the theorem is proved.

Case 2. The points $p_0, p_1, \cdots, p_n$ lie in an $(n-1)$-dimensional hyperplane, $(n-1)$-dimensional horosphere, or on one branch of an $(n-1)$-dimensional equidistant surface.

Let Ω denote the surface containing $p_0, p_1, \cdots, p_n$. Now Ω may be mapped onto an $(n-1)$-dimensional sphere S by a hyperbolic inversion. If $p'_0, p'_1, \cdots, p'_{n+2}$ are the inverse points of $p_0, p_1, \cdots, p_{n+2}$, then since $K(p_0, p_1, \cdots, p_{n+1}) \neq 0$, we have $K(p'_0, p'_1, \cdots, p'_{n+1}) \neq 0$, and it follows that $p'_0, p'_1, \cdots, p'_n$ determine S. Applying Case 1 to the points $p'_0, p'_1, \cdots, p'_{n+2}$, we see p'_{n+1}, p'_{n+2} lie in the same or different components of $H_n - S$ if and only if $\operatorname{sgn}[\sinh^2 p'_{n+1}p'_{n+2}/2] = (-1)^n$ or $(-1)^{n+1}$, respectively. Since $\operatorname{sgn}[\sinh^2 p'_{n+1}p'_{n+2}/2]$ is an inversive invariant and components of $H_n - S$ are preserved under hyperbolic inversions, if S is mapped back onto the surface Ω containing $p_0, p_1, \cdots, p_n$, we see that p_{n+1} and p_{n+2} are in the same or different components of $H_n - \Omega$ if and only if $\operatorname{sgn}[\sinh^2 p_{n+1}p_{n+2}/2] = (-1)^n$ or $(-1)^{n+1}$, respectively.

4. **Cofactors of $\gamma(p_0, p_1, \cdots, p_{n+2})$ where $p_0, p_1, \cdots, p_{n+2}$ are points in n-dimensional spherical space S_n of radius 1.** Here we are concerned with cofactors $[\sin^2 p_ip_j/2]$ of elements $\sin^2 p_ip_j/2$ $(i \neq j)$ of $\gamma(p_0, p_1, \cdots, p_{n+2})$. We again consider the cofactor of $\sin^2 p_{n+1}p_{n+2}/2$ as typical.

Theorem 4.1. *Let $p_0, p_1, \cdots, p_{n+2}$ be an $(n+3)$-tuple in n-dimensional spherical space S_n. If $[\sin^2 p_{n+1}p_{n+2}/2] \neq 0$, then* (1) *the points $p_0, p_1, \cdots, p_n$ lie on an $(n-1)$-dimensional subspace and not on an $(n-2)$-dimensional subspace or $p_0, p_1, \cdots, p_n$ determine an $(n-1)$-dimensional sphere and* (2) *p_{n+1} and p_{n+2} are in the same or different components of $S_n - \Omega$ (where Ω denotes the $(n-1)$-dimensional subspace or the sphere containing $p_0, p_1, \cdots, p_n$) if and only if $\operatorname{sgn}[\sin^2 p_{n+1}p_{n+2}/2] = (-1)^n$ or $(-1)^{n+1}$, respectively.*

Proof. Since $\gamma(p_0, p_1, \cdots, p_{n+2}) = 0$, we again have

$$\gamma(p_0, p_1, \cdots, p_n, p_{n+1})\gamma(p_0, p_1, \cdots, p_n, p_{n+2}) - [\sin^2 p_{n+1}p_{n+2}/2]^2 = \gamma(p_0, p_1, \cdots, p_{n+2})\gamma(p_0, p_1, \cdots, p_n) = 0 \tag{14}$$

(see [6, p. 372]). Part (1) of the theorem follows as in the proofs of Theorem 2.2 and 3.2.

Case 1. The points $p_0, p_1, \cdots, p_n$ determine an $(n-1)$-dimensional sphere S.

That $\operatorname{sgn}[\sin^2 p_{n+1}p_{n+2}/2] = (-1)^n$ for points p_{n+1}, p_{n+2} in the same component of $S_n - S$ follows as in Case 1 of Theorems 2.2 and 3.2.

Suppose then that p_{n+1} and p_{n+2} are in different components of $S_n - S$. Let r denote the radius of S and let o denote the center of S. Choose the point p^*_{n+1} such that $\tan op_{n+1}/2 \cdot \tan op^*_{n+1}/2 = \tan^2 r/2$ and so that p^*_{n+1} is on a great circle joining o and p_{n+1}. The triangles op_ip_{n+1} and $op_ip^*_{n+1}$ $(i = 0, 1, 2, \cdots, n)$ have the same angle at o. It follows from the spherical law of cosines that

$$(15)\quad \begin{aligned}&(\cos op_i \cos op_{n+1} - \cos p_ip_{n+1})/(\sin op_i \sin op_{n+1})\\ &\qquad = (\cos op_i \cos op^*_{n+1} - \cos p_ip^*_{n+1})/(\sin op_i \sin op^*_{n+1}).\end{aligned}$$

Letting $P_i = \tan op_i/2$ $(i = 0, 1, \cdots, n)$, $P^*_{n+1} = \tan op^*_{n+1}/2$ and $R = \tan^2 r/2$, we have

$$(16)\quad \begin{aligned}\cos op_i &= (1 - P_i^2)/(1 + P_i^2),\\ \sin op_i &= 2P_i/(1 + P_i^2) \qquad (i = 0, 1, \cdots, n)\\ \cos op^*_{n+1} &= (P^{*2}_{n+1} - R^2)/(P^{*2}_{n+1} + R^2),\\ \sin op^*_{n+1} &= 2P^*_{n+1}R/(P^{*2}_{n+1} + R^2),\\ 1 - 2\sin^2 p_ip_{n+1}/2 &= \cos p_ip_{n+1},\\ 1 - 2\sin^2 p_ip^*_{n+1}/2 &= \cos p_ip^*_{n+1} \qquad (i = 0, 1, \cdots, n).\end{aligned}$$

Substituting the values of (16) in (15) and solving the new equation for $\sin^2 p_ip^*_{n+1}/2$, we obtain

$$(17)\quad \sin^2 p_ip^*_{n+1}/2 = R[(1 + P^{2*}_{n+1})/(P^{*2}_{n+1} + R^2)]\sin^2 p_ip_{n+1}/2 \qquad (i = 0, 1, \cdots, n).$$

From the way p^*_{n+1} was chosen, p_{n+1}, p^*_{n+1} are in different components of $S_n - S$, and so $\operatorname{sgn}[\sin^2 p_{n+1}p_{n+2}/2] = \operatorname{sgn}[\sin^2 p_{n+1}p^*_{n+1}/2]$. Inspection of the vanishing determinant $\gamma(p_0, p_1, \cdots, p_{n+1}, p^*_{n+1})$, with the aid of (17), gives

$$(18)\quad \begin{aligned}[\sin^2 p_{n+1}p^*_{n+1}/2] = -\{&R[(1 + P^{*2}_{n+1})/(P^{*2}_{n+1} + R^2)]\gamma(p_0, p_1, \cdots, p_{n+1})\\ &+ \sin^2 p_{n+1}p^*_{n+1}/2[R(1 + P^{*2}_{n+1}\ /(P^{*2}_{n+1} + R^2)]\\ &\cdot \gamma(p_0, p_1, \cdots, p_n)\}.\end{aligned}$$

From (14),

$$[\sin^2 p_{n+1}p^*_{n+1}/2]^2 = R^2[(1 + P^{*2}_{n+1})/(P^{*2}_{n+1} + R^2)]^2\gamma^2(p_0, p_1, \cdots, p_{n+1}).$$

It follows that

$$[\sin^2 p_{n+1}p^*_{n+1}/2] = \{R[(1+P^{*2}_{n+1})/(P^{*2}_{n+1}+R^2)]\} \cdot \gamma(p_0, p_1, \cdots, p_{n+1})$$

and hence $\operatorname{sgn}[\sin^2 p_{n+1}p^*_{n+1}/2] = (-1)^{n+1}$. Thus for p_{n+1} and p_{n+2} in different components of $S_n - S$, $\operatorname{sgn}[\sin^2 p_{n+1}p_{n+2}/2] = (-1)^{n+1}$.

The converse follows as in Theorems 2.2 and 3.2.

Case 2. The points $p_0, p_1, \cdots, p_n$ lie on a $(n-1)$-dimensional subspace.

The argument here is similar to Case 1. The main difference being that when p_{n+1} and p_{n+2} are in different components of $S_n - \Omega$, the point p^*_{n+1} is just the reflection of p_{n+1} in Ω. Then $p_ip_{n+1} = p_ip^*_{n+1}$ $(i = 0, 1, \cdots, n)$ and the argument is much simpler.

REFERENCES

1. L. M. Blumenthal, *The geometry of a class of semimetric spaces*, Tôhoku Math. J. 43 (1937), 205–224.

2. ———, *The metric characterization of ϕ-spherical spaces*, Univ. Nac. Tucumán. Revista A. 5 (1946), 69–93. MR 8, 596; 709.

3. L. M. Blumenthal and B. E. Gillam, *Distribution of points in n-space*, Amer. Math. Monthly 50 (1943), 181–185. MR 4, 250.

4. A. Cayley, *On a theorem in the geometry of position*, Cambridge Math. J. 2 (1841), 267–271.

5. G. Darboux, *Sur les rélations entre les groupes de points, de cercles et de sphères dans le plan et dans l'espace*, Ann. Sci. de l'E.M.S. (2) 1 (1872), 323–392.

6. Thomas Muir (Revised and enlarged by William H. Metzler), *A treatise on the theory of determinants*, Dover, New York, 1928.

7. K. Menger, *Bericht über metrische geometric*, Jber. Deutsch. Math. Verein. 40 (1931), 201–219.

8. D. M. Y. Sommerville, *The elements of non-Euclidean geometry*, G. Bell, London, 1914.

9. J. E. Valentine, *An analogue of Ptolemy's theorem in spherical geometry*, Amer. Math. Monthly 77 (1970), 47–51. MR 40 #7932.

10. ———, *An analogue of Ptolemy's theorem and its converse in hyperbolic geometry*, Pacific J. Math. 34 (1970), 817–825. MR 42 #5151.

11. J. E. Valentine and E. Z. Andalafte, *A metric characterization of "spherical" surfaces in n-dimensional hyperbolic space*, J. Reine Angew. Math. 251 (1971), 142–152.

DEPARTMENT OF MATHEMATICS, UTAH STATE UNIVERSITY, LOGAN, UTAH 84321

Current address (J. E. Valentine): University of Jordan, Amman, Jordan

Current address (S. G. Wayment): Department of Mathematics, University of Texas at San Antonio, San Antonio, Texas 78228