TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 178, April 1973

GENERALIZED DEDEKIND ETA-FUNCTIONS AND GENERALIZED DEDEKIND SUMS(1)

BY

BRUCE C. BERNDT

ABSTRACT. A transformation formula under modular substitutions is derived for a very large class of generalized Eisenstein series. The result also gives a transformation formula for generalized Dedekind eta-functions. Various types of Dedekind sums arise, and reciprocity laws are established.

1. **Introduction.** In [12], J. Lewittes proved transformation formulae for the analytic continuation of a very large class of Eisenstein series. From another viewpoint, these results give transformation formulae for a large class of functions which generalize the classical Dedekind eta-function $\eta(z)$. However, the formulae [12, Theorem 3, equation (51)] are so complicated that even in the simplest case of the Dedekind eta-function it is exceedingly difficult to deduce the usual transformation formulae in terms of Dedekind sums.

Our objective here is to take Lewittes' proof and give a different account of the last parts of his proof. Our new proof will yield more elegant transformation formulae in which Dedekind sums or various generalizations of Dedekind sums appear. From this new version of Lewittes' theorem, we shall show that results of several other authors are special cases. In addition, we shall deduce several new results as well.

Firstly, we easily deduce the transformation formulae of Dedekind's eta-function. Secondly, we derive transformation formulae for a large class of functions which generalize $\eta(z)$. This class includes those functions studied by C. Meyer [14], U. Dieter [6], and B. Schoeneberg [17] and which are connected with F. Klein's functions [10]. Appearing in our transformation formulae are the generalized Dedekind sums $s(h, k; x, y)$ first defined by H. Rademacher in 1964. It is interesting to observe that in the work of Meyer [14], [15], Dieter [6], and Schoeneberg [17], only rational values of the parameters x and y appear, while in our generalization

Received by the editors March 8, 1972.

AMS (MOS) subject classifications (1970). Primary 10D05; Secondary 10A40.

Key words and phrases. Eisenstein series, Dedekind eta-function, generalized Dedekind eta-function, Dedekind sum, generalized Dedekind sum, Bernoulli numbers and polynomials, reciprocity law, Lambert series, modular transformation.

(1) Research partially supported by NSF grant GP21335.

x and y may assume any real values. We next show how the reciprocity theorem for generalized Dedekind sums [16] may be derived from the transformation formulae. Another special case of our general theorem is the result of T. Apostol [2] wherein the transformation formulae of certain Lambert series are derived. S. Iseki [9] considered a more general class of functions than Apostol but derived a transformation formula only in the special case $V(z) = -1/z$. However, we shall easily derive the transformation formula for any modular substitution. Another generalization of the classical Dedekind sums appears in Apostol's formulae, and L. Carlitz ([4], [5]) derived a reciprocity theorem for these sums. Still another generalization of Dedekind sums appears in our transformation formulae from which we derive a reciprocity theorem which includes Carlitz's as a special case.

2. **An improved version of Lewittes' theorem.** We review some notation from Lewittes' paper [12]. Put $z = x + iy$ and $s = \sigma + it$ with x, y, σ and t real. For any complex number w, we choose that branch of $\log w$ with $-\pi \leq \arg w < \pi$. Let $V(z) = Vz = (az + b)/(cz + d)$ be an arbitrary modular transformation. Let r_1 and r_2 be arbitrary real numbers, and define R_1 and R_2 by

$$R_1 = ar_1 + cr_2 \quad \text{and} \quad R_2 = br_1 + dr_2.$$

Let Z denote the ring of rational integers. Let $\mathcal{H}$ denote the upper half-plane $\{z: y > 0\}$. For $z \in \mathcal{H}$ and $\sigma > 2$, define the Eisenstein series $G(z, s, r_1, r_2)$ by

$$G(z, s, r_1, r_2) = {\sum_{m,n}}' ((m + r_1)z + n + r_2)^{-s},$$

where m and n range over all pairs of integers except for the possible pair $m = -r_1$, $n = -r_2$. For $z \in \mathcal{H}$ and arbitrary s, define the following generalization of Dedekind's eta-function by

$$A(z, s, r_1, r_2) = \sum_{m > -r_1} \sum_{k=1}^{\infty} k^{s-1} e^{2\pi i k r_2 + 2\pi i k(m + r_1)z}.$$

Put $H(z, s, r_1, r_2) = A(z, s, r_1, r_2) + e^{\pi i s} A(z, s, -r_1, -r_2)$. For α real and $\sigma > 1$, Lewittes defined $\zeta(s, \alpha)$ by

$$\zeta(s, \alpha) = \sum_{n > -\alpha} (n + \alpha)^{-s}.$$

Observe that $\zeta(s, \alpha) = \zeta(s, \{\alpha\} + \chi(\alpha))$, where $\{\alpha\}$ denotes the fractional part of α, and $\chi(\alpha)$ denotes the characteristic function of the integers. These two latter notations will be repeatedly used in the sequel. Since $0 < \{\alpha\} + \chi(\alpha) \leq 1$ $\zeta(s, \{\alpha\} + \chi(\alpha))$ denotes the classical Hurwitz zeta-function.

Theorem 1. *Assume that $c > 0$. Let $Q = \{z: x > -d/c$ and $y > 0\}$. Let $\rho = \rho(R_1, R_2, c, d) = \{R_2\}c - \{R_1\}d$. Then $G(z, s, r_1, r_2)$ can be analytically continued to the entire complex s-plane, and for $z \in Q$ and all s,*

$$(1)\qquad \begin{aligned}&(cz+d)^{-s}G(Vz,s,r_1,r_2)\\&=G(z,s,R_1,R_2)-2i\sin(\pi s)\chi(R_1)\zeta(s,-R_2)+\frac{e^{-\pi i s}}{\Gamma(s)}L(z,s,R_1,R_2,c,d),\end{aligned}$$

where

$$(2)\qquad \begin{aligned}&L(z,s,R_1,R_2,c,d)\\&=\sum_{j=1}^{c}\int_C u^{s-1}\frac{e^{-(cz+d)(j-\{R_1\})u/c}}{e^{-(cz+d)u}-1}\,\frac{e^{\{(jd+\rho)/c\}u}}{e^u-1}\,du,\end{aligned}$$

where C is a loop beginning at $+\infty$, proceeding in the upper half-plane, encircling the origin in the positive direction so that $u=0$ is the only zero of $(e^{-(cz+d)u}-1)\cdot(e^u-1)$ lying "inside" the loop, and then returning to $+\infty$ in the lower half-plane. Here we choose the branch of u^s with $0<\arg u<2\pi$.

Proof. From (26) and (29) of [12], we have for $z\in Q$ and $\sigma>2$,

$$(3)\qquad \begin{aligned}&(cz+d)^{-s}G(Vz,s,r_1,r_2)\\&=G(z,s,R_1,R_2)-2i\sin(\pi s)\,(\chi(R_1)\zeta(s,-R_2)+K(z,s,R_1,R_2,c,d)),\end{aligned}$$

where

$$\begin{aligned}&\Gamma(s)K(z,s,R_1,R_2,c,d)\\&=\sum_{m>R_1}\ \sum_{n>R_2+(m-R_1)d/c}\int_0^\infty u^{s-1}e^{-(m-R_1)zu-(n-R_2)u}\,du.\end{aligned}$$

If we put $m'=m-[R_1]-1$ and $n'=n-[R_2+(m-R_1)d/c]-1$, then the above becomes

$$\sum_{m'=0}^{\infty}\sum_{n'=0}^{\infty}\int_0^\infty u^{s-1}\exp(-(m'+1-\{R_1\})zu-(n'+1-\{R_2\}+[(m'd+d+\rho)/c])u)\,du.$$

If we put $m'=qc+j$, $0\le j\le c-1$, $0\le q<\infty$, and replace n' by n, we get

$$\sum_{j=0}^{c-1}\sum_{q=0}^{\infty}\sum_{n=0}^{\infty}\int_0^\infty u^{s-1}\exp(-(qc+j+1-\{R_1\})zu-(n+1-\{R_2\}+qd+[(jd+d+\rho)/c])u)\,du$$

$$\begin{aligned}&=\sum_{j=0}^{c-1}\int_0^\infty u^{s-1}\exp(-(j+1-\{R_1\})zu-(1-\{R_2\}+[(jd+d+\rho)/c])u)\,du\\&\qquad\sum_{q=0}^{\infty}\sum_{n=0}^{\infty}e^{-(cz+d)qu-nu}\,du\\&=\sum_{j=1}^{c}\int_0^\infty u^{s-1}\frac{\exp(-(j-\{R_1\})zu-(1-\{R_2\}+[(jd+\rho)/c])u)}{(1-e^{-(cz+d)u})(1-e^{-u})}\,du\\&=\sum_{j=1}^{c}\int_0^\infty u^{s-1}\frac{e^{-(cz+d)(j-\{R_1\})u/c}}{1-e^{-(cz+d)u}}\,\frac{e^{\{(jd+\rho)/c\}u}}{e^u-1}\,du,\end{aligned}$$

where, since $x > -d/c$, we have interchanged the order of summation and integration by absolute convergence. By a classical method of Riemann [18, pp. 18–19], the integrals above may be transformed into loop integrals. Letting C be as defined in the statement of the theorem, we find that

$$\Gamma(s)K(z, s, R_1, R_2, c, d)$$

(4)
$$= \sum_{j=1}^{c} \frac{1}{e^{2\pi i s}-1} \int_C u^{s-1} \frac{e^{-(cz+d)(j-\{R_1\})u/c}}{1-e^{-(cz+d)u}} \frac{e^{\{(jd+\rho)/c\}u}}{e^u-1}\, du.$$

If we now substitute (4) into (3) and simplify slightly, we arrive at (1) for $z \in Q$ and $\sigma > 2$. However, these loop integrals converge uniformly on any compact set in the s-plane and thus represent entire functions. Hence, by analytic continuation, (1) is valid for $z \in Q$ and for all s.

We now restate Theorem 1 in terms of the function $H(z, s, r_1, r_2)$. The proof is like that of equation (51) in Lewittes' paper [12], and so we omit the proof.

Theorem 2. *For $z \in Q$ and all s,*

$$(cz+d)^{-s}H(Vz, s, r_1, r_2)$$

$$= H(z, s, R_1, R_2) - \chi(r_1)e^{\pi i s}(2\pi i)^{-s}(cz+d)^{-s}\Gamma(s)(\zeta(s, r_2) + e^{\pi i s}\zeta(s, -r_2))$$

(5)
$$+ \chi(R_1)(2\pi i)^{-s}\Gamma(s)(\zeta(s, -R_2) + e^{\pi i s}\zeta(s, R_2))$$

$$+ (2\pi i)^{-s}L(z, s, R_1, R_2, c, d),$$

where $L(z, s, R_1, R_2, c, d)$ is given by (2).

Theorems 1 and 2 may be considerably simplified if s is an integer. For then,

(6)
$$L(z, s, R_1, R_2, c, d) = 2\pi i R_0,$$

where R_0 is the sum of the residues of the integrands at $u = 0$. Upon the substitution of (6) into (1) and (5), Theorems 1 and 2 will then be valid for all $z \in \mathcal{H}$ by analytic continuation. Thus, put $s = -m$, where m is an arbitrary nonnegative integer. Now [1, p. 804],

$$\frac{ue^{xu}}{e^u-1} = \sum_{n=0}^{\infty} B_n(x)\frac{u^n}{n!}, \qquad |u| < 2\pi,$$

where $B_n(x)$ denotes the nth Bernoulli polynomial. Hence,

$$u^{-m-1}\frac{e^{-(cz+d)(j-\{R_1\})u/c}}{e^{-(cz+d)u}-1} \frac{e^{\{(jd+\rho)/c\}u}}{e^u-1}$$

$$= -\frac{u^{-m-3}}{cz+d}\sum_{\mu=0}^{\infty} B_\mu\left(\frac{j-\{R_1\}}{c}\right)\frac{(-(cz+d)u)^\mu}{\mu!} \sum_{\nu=0}^{\infty} \bar{B}_\nu\left(\frac{jd+\rho}{c}\right)\frac{u^\nu}{\nu!},$$

where $\overline{B}_n(x) = B_n(\{x\})$. Thus,

$$
\begin{aligned}
R_0 &= -\frac{1}{cz+d}\sum_{j=1}^{c}\sum_{\mu+\nu=m+2} B_\mu\left(\frac{j-\{R_1\}}{c}\right)\overline{B}_\nu\left(\frac{jd+\rho}{c}\right)\frac{(-(cz+d))^\mu}{\mu!\nu!} \\
&= \frac{1}{(m+2)!}\sum_{j=1}^{c}\sum_{k=0}^{m+2}\binom{m+2}{k}B_k\left(\frac{j-\{R_1\}}{c}\right)\overline{B}_{m+2-k}\left(\frac{jd+\rho}{c}\right)(-(cz+d))^{k-1}.
\end{aligned} \tag{7}
$$

3. **The transformation formulae for** $\log \eta(z)$. Let $s = r_1 = r_2 = 0$. Put $A(z, 0, 0, 0) = A(z)$ and observe that $H(z, 0, 0, 0) = 2A(z)$. We find then that (5) and (6) yield

$$
\begin{aligned}
A(Vz) &= A(z) + \lim_{s\to 0}(2\pi i)^{-s}\Gamma(s)\zeta(s)(1-e^{\pi i s}(cz+d)^{-s}) + \pi i R_0 \\
&= A(z) + \tfrac{1}{2}\pi i - \tfrac{1}{2}\log(cz+d) + \pi i R_0.
\end{aligned}
$$

Using the product definition of $\eta(z)$, it is easy to show that $\log \eta(z) = \pi i z/12 - A(z)$. Hence,

$$\log \eta(Vz) = \log \eta(z) + \pi i(V(z)-z)/12 - \tfrac{1}{2}\pi i + \tfrac{1}{2}\log(cz+d) - \pi i R_0. \tag{8}$$

From (7),

$$R_0 = \sum_{j=1}^{c}\{-\tfrac{1}{2}\overline{B}_2(jd/c)(cz+d)^{-1} + B_1(j/c)\overline{B}_1(jd/c) - \tfrac{1}{2}B_2(j/c)(cz+d)\}. \tag{9}$$

Since [1, p. 804, Equation 23.1.10]

$$B_n(cx) = c^{n-1}\sum_{j=0}^{c-1}B_n(x+j/c), \tag{10}$$

we find that

$$\sum_{j=1}^{c}B_2(j/c) = \sum_{j=0}^{c-1}B_2(j/c) = B_2/c = 1/6c. \tag{11}$$

Since $(c, d) = 1$,

$$\sum_{j=1}^{c}\overline{B}_2(jd/c) = \sum_{j=1}^{c}\overline{B}_2(j/c) = \sum_{j=0}^{c-1}B_2(j/c) = 1/6c. \tag{12}$$

Lastly,

$$
\begin{aligned}
\sum_{j=1}^{c}B_1(j/c)\overline{B}_1(jd/c) &= \sum_{j=1}^{c-1}\overline{B}_1(j/c)\overline{B}_1(jd/c) + B_1(1)B_1(0) \\
&= s(d, c) - 1/4,
\end{aligned} \tag{13}
$$

since $B_1(x) = x - \tfrac{1}{2}$ and

$$s(d, c) = \sum_{j \bmod c}((j/c))((jd/c)),$$

where

$$((x)) = \begin{cases} \{x\} - \tfrac{1}{2} & \text{if } x \text{ is not an integer,} \\ 0 & \text{if } x \text{ is an integer.} \end{cases}$$

Putting (11)–(13) into (9), we find that

$$R_0 = -1/12c(cz+d) - (cz+d)/12c + s(d,c) - \tfrac{1}{4}.$$

Hence, from (8) we obtain

$$\begin{aligned} \log \eta(Vz) &= \log \eta(z) - \pi i/4 + \tfrac{1}{2}\log(cz+d) - \pi i s(d,c) \\ &\quad + (\pi i/12)\{V(z) - z + (cz+d)/c + 1/c(cz+d)\} \\ &= \log \eta(z) - \pi i/4 + \tfrac{1}{2}\log(cz+d) - \pi i s(d,c) + \pi i(a+d)/12c, \end{aligned} \tag{14}$$

upon the simplification of the expression in brackets with the aid of the fact $ad - bc = 1$.

4. Transformation formulae for generalized Dedekind eta-functions. Put $s = 0$ and suppose that r_1 and r_2 are arbitrary. From (7),

$$\begin{aligned} R_0 = \sum_{j=1}^{c} \Bigg\{ &-\frac{1}{2}\bar{B}_2\left(\frac{jd+\rho}{c}\right)(cz+d)^{-1} \\ &+ B_1\left(\frac{j-\{R_1\}}{c}\right)\bar{B}_1\left(\frac{jd+\rho}{c}\right) - \frac{1}{2}B_2\left(\frac{j-\{R_1\}}{c}\right)(cz+d)\Bigg\}. \end{aligned} \tag{15}$$

Using (10) again, we have

$$\sum_{j=1}^{c} B_2\left(\frac{j-\{R_1\}}{c}\right) = \frac{1}{c}B_2(1-\{R_1\}) = \frac{1}{c}B_2(\{R_1\}), \tag{16}$$

since [1, p. 804, Equation 23.1.8]

$$B_n(1-x) = (-1)^n B_n(x). \tag{17}$$

Since $(c, d) = 1$, $jd + [\rho]$ runs through a complete residue system (mod c) as j does, and so by (10),

$$\sum_{j=1}^{c} \bar{B}_2\left(\frac{jd+\rho}{c}\right) = \sum_{j=0}^{c-1} \bar{B}_2\left(\frac{j+\{\rho\}}{c}\right) = \frac{1}{c}B_2(\{\rho\}). \tag{18}$$

If $(c, d) = 1$, and x and y are arbitrary real numbers, the generalized Dedekind sum $s(d, c; x, y)$ is defined by [16]

$$s(d, c; x, y) = \sum_{j \bmod c} \left(\left(d\frac{j+y}{c} + x\right)\right)\left(\left(\frac{j+y}{c}\right)\right).$$

Let $g(d, c; \{R_2\}, -\{R_1\})$ denote the second sum on the right side of (15). We see that $g(d, c; \{R_2\}, -\{R_1\})$ is very closely related to $s(d, c; \{R_2\}, -\{R_1\})$.

Various possibilities occur. If neither R_1 or ρ are integers, then

$$g(d, c; \{R_2\}, -\{R_1\}) = s(d, c; \{R_2\}, -\{R_1\}).$$

If $R_1 \in Z$, but $\rho \notin Z$, then $g(d, c; \{R_2\}, -\{R_1\})$ contains an "extra" term corresponding to $j = c$. Suppose that ρ is an integer. Let j' be the unique integer such that $1 \leq j' \leq c$ and $j'd + \rho \equiv 0 \pmod{c}$. Then there is an "extra" term corresponding to $j = j'$. Unless both R_1 and ρ are integers and $j' = c$, we have in general that

$$\begin{aligned} g(d, c; \{R_2\}, -\{R_1\}) &= s(d, c; \{R_2\}, -\{R_1\}) \\ &\quad + \tfrac{1}{2}\chi(R_1)\bar{B}_1(R_2) - \tfrac{1}{2}\chi(\rho)B_1((j' - \{R_1\})/c). \end{aligned} \tag{19}$$

Now if both R_1 and ρ are integers and $j' = c$, the "extra" terms coincide. In such a case R_2 must be integral. If $R_1, R_2 \in Z$, then $r_1, r_2 \in Z$, since $ad - bc = 1$. Conversely, if $r_1, r_2 \in Z$, then $R_1, \rho \in Z$. We see that we are then in the case of $\log \eta(z)$ discussed earlier. Thus, in the remainder of this section, assume that at least one of the pair r_1, r_2 is not an integer. Observe that if $\rho \in Z$, but $R_1 \notin Z$, then

$$B_1((j' - \{R_1\})/c) = \bar{B}_1((j' - \{R_1\})/c).$$

Also, from the definition of ρ, it is easily seen that $\rho \in Z$ if and only if $r_1 \in Z$. A short calculation shows that $j' \equiv ar_1 - [R_1] \pmod{c}$. Hence, from the definition of R_1, $(j' - \{R_1\})/c \equiv -r_2 \pmod{1}$. From the above remarks and (17), we conclude that

$$\chi(\rho)B_1((j' - \{R_1\})/c) = \chi(r_1)\bar{B}_1(-r_2) = -\chi(r_1)\bar{B}_1(r_2).$$

Finally, putting the above into (19) and then using (16), (18) and (19) in (15), we find that

$$\begin{aligned} R_0 &= -\bar{B}_2(\rho)/2c(cz+d) - (cz+d)\bar{B}_2(R_1)/2c \\ &\quad + s(d, c; R_2, -R_1) + \tfrac{1}{2}\chi(R_1)\bar{B}_1(R_2) + \tfrac{1}{2}\chi(r_1)\bar{B}_1(r_2). \end{aligned} \tag{20}$$

Let

$$\begin{aligned} f_m(r_1, r_2, c, d) = \lim_{s \to -m} (2\pi i)^{-s}(&-\chi(r_1)e^{\pi i s}(cz+d)^{-s}\Gamma(s)(\zeta(s, r_2) + e^{\pi i s}\zeta(s, -r_2)) \\ &+ \chi(R_1)\Gamma(s)(\zeta(s, -R_2) + e^{\pi i s}\zeta(s, R_2))). \end{aligned}$$

From (5), we see that we must determine $f_0(r_1, r_2, c, d)$ for the various possibilities.

Case 1. Let $r_1 \notin Z$, $r_2 \in Z$. Suppose that $R_2 \in Z$. By the definition of R_2, $br_1 \in Z$. If also $R_1 \in Z$, then $ar_1 \in Z$. Since $r_1 \notin Z$ and $(a, b) = 1$, we have a contradiction. Hence, $R_1 \notin Z$, and so $f_0(r_1, r_2, c, d) = 0$. Assume then that $R_2 \notin Z$. Now for $0 < \alpha \leq 1$ [19, p. 271], $\zeta(0, \alpha) = \tfrac{1}{2} - \alpha$ and $\zeta'(0, \alpha) = \log \Gamma(\alpha) -$

$\frac{1}{2}\log(2\pi)$. Also, if $0 < \alpha < 1$, $\{-\alpha\} = 1 - \{\alpha\}$. Thus, if $R_1 \in Z$,

$$\begin{aligned} f_0(r_1, r_2, c, d) &= \lim_{s\to 0} \Gamma(s)(\zeta(s, -R_2) + e^{\pi is}\zeta(s, R_2)) \\ &= \lim_{s\to 0} \Gamma(s)(\tfrac{1}{2} - (1 - \{R_2\}) + (\log\Gamma(1 - \{R_2\}) - \tfrac{1}{2}\log(2\pi))s + \cdots \\ &\qquad + \tfrac{1}{2} - \{R_2\} + (\log\Gamma(\{R_2\}) - \tfrac{1}{2}\log(2\pi) + \pi i(\tfrac{1}{2} - \{R_2\}))s + \cdots) \\ &= \log\Gamma(1 - \{R_2\}) + \log\Gamma(\{R_2\}) - \log(2\pi) - \pi i\bar{B}_1(R_2) \\ &= \log(\pi/\sin\pi\{R_2\}) - \log(2\pi) - \pi i\bar{B}_1(R_2) \\ &= -\log(2\sin\pi\{R_2\}) - \pi i\bar{B}_1(R_2) \\ &= -\log(1 - e^{-2\pi i\{R_2\}}) - 2\pi i\bar{B}_1(R_2). \end{aligned}$$

Case 2. Let $r_1 \in Z$, $r_2 \notin Z$. Suppose that $R_2 \in Z$. By an argument completely analogous to that in Case 1, $R_1 \notin Z$. Proceeding as in Case 1, we find that

$$\begin{aligned} f_0(r_1, r_2, c, d) &= \log(1 - e^{-2\pi i\{-r_2\}}) + 2\pi i\bar{B}_1(-r_2) \\ &\qquad - \chi(R_1)(\log(1 - e^{-2\pi i\{R_2\}}) + 2\pi i\bar{B}_1(R_2)) \\ &= \log(1 - e^{-2\pi i\{r_2\}}) - \chi(R_1)(\log(1 - e^{-2\pi i\{R_2\}}) + 2\pi i\bar{B}_1(R_2)). \end{aligned}$$

Case 3. Let $r_1, r_2 \notin Z$. Suppose that $R_2 \in Z$. If also $R_1 \in Z$, it follows that $r_1, r_2 \in Z$ since $ad - bc = 1$. We have a contradiction, and so $R_1 \notin Z$. We see that $f_0(r_1, r_2, c, d)$ is the same as in Case 1.

In summary, in all cases we have

$$\begin{aligned} f_0(r_1, r_2, c, d) &= \chi(r_1)\log(1 - e^{-2\pi ir_2}) \\ &\qquad - \chi(R_1)(\log(1 - e^{-2\pi iR_2}) + 2\pi i\bar{B}_1(R_2)). \end{aligned} \tag{21}$$

Put

$$\alpha(r_1, r_2) = \begin{cases} e^{\pi i\bar{B}_1(r_2)}(1 - e^{-2\pi ir_2}), & r_1 \in Z,\ r_2 \notin Z, \\ 1, & \text{otherwise}. \end{cases}$$

Combining (20) and (21) with (5), we conclude that

$$\begin{aligned} H(Vz, 0, r_1, r_2) &= H(z, 0, R_1, R_2) \\ &\quad + 2\pi i\left\{-\frac{1}{2c(cz+d)}\bar{B}_2(\rho) - \frac{(cz+d)}{2c}\bar{B}_2(R_1) + s(d, c; R_2, -R_1)\right\} \\ &\quad + \log\alpha(r_1, r_2) - \log\alpha(R_1, R_2). \end{aligned} \tag{22}$$

The transformation formulae for $H(z, 0, r_1, r_2)$ given by (22) appear to be new in general. However, Meyer [14], Dieter [6], and Schoeneberg [17] have derived the result for a subset of these functions, and their formulae were given implicitly by Hecke [8]. Previously, J. Lehner [11] and J. Livingood [13] had achieved results for special cases of the aforementioned results.

We now justify our claim that the results of Meyer, Dieter, and Schoeneberg are special cases of (22). We shall employ some of the notation in [17].

For $z \in \mathcal{H}$, $g, h \in Z$, and a positive integer N, define

$$\eta_{g,h}(z;\, N) = \alpha(g/N,\, h/N)\, e^{\pi i z \bar{B}_2(g/N)} \prod_{m=1;\; m\equiv g(N)}^{\infty} (1 - e^{2\pi i(h+zm)/N}) \prod_{m=1;\; m\equiv -g(N)}^{\infty} (1 - e^{2\pi i(-h+zm)/N}). \tag{23}$$

Let $g' = ag + ch$, $h' = bg + dh$, and

$$b_{g,h}(N) = \begin{cases} 1, & g \equiv h \equiv 0 \ (N), \\ 0, & \text{otherwise.} \end{cases}$$

Then,

$$\begin{aligned} &\log \eta_{g,h}(Vz;\, N) - \log \eta_{g',h'}(z;\, N) \\ &= (\pi i a/c)\bar{B}_2(g/N) + (\pi i d/c)\bar{B}_2(g'/N) - 2\pi i s(d,\, c;\, h'/N,\, -g'/N) \\ &\quad + b_{g,h}(N)(\log(cz+d) - \tfrac{1}{2}\pi i). \end{aligned} \tag{24}$$

To prove (24) we may without loss of generality assume that $0 \le g, h < N$. Furthermore, note that $\eta_{0,0}(z;\, N) = \eta^2(z)$, and (24) reduces to (14). Hence, we shall assume that at least one of the pair g, h is not zero. Taking logarithms of both sides of (23) and then putting, respectively, $m = jN + g$, $j > -g/N$, and $m = jN - g$, $j > g/N$, in the resulting two sums, we have

$$\begin{aligned} \log \eta_{g,h}(z;\, N) &= \log \alpha(g/N,\, h/N) + \pi i z B_2(g/N) \\ &\quad - \sum_{j>-g/N} \sum_{k=1}^{\infty} \frac{1}{k} e^{2\pi i k h/N + 2\pi i k(j+g/N)z} \\ &\quad - \sum_{j>g/N} \sum_{k=1}^{\infty} \frac{1}{k} e^{-2\pi i k h/N + 2\pi i k(j-g/N)z} \\ &= \log \alpha(g/N,\, h/N) + \pi i z B_2(g/N) - H(z,\, 0,\, g/N,\, h/N). \end{aligned} \tag{25}$$

Observe that $R_1 = g'/N$, $R_2 = h'/N$, and $\{\rho\} = \{-g/N\}$. Using these facts, substituting (25) into (22), and then simplifying slightly with the use of (17), we arrive at (24).

5. **The reciprocity formula for generalized Dedekind sums.** We shall assume that at least one of the pair r_1, r_2 is not an integer for it is well known that the

reciprocity formula for the ordinary Dedekind sums may be derived from (14). Our assumption on the pair r_1, r_2 implies that at least one of the pair R_1, R_2 is not an integer.

Let $Vz = (az + b)/(cz + d)$, $V^*z = (bz - a)/(dz - c)$, and $Tz = -1/z$, where $c, d > 0$. Replacing z by $-1/z$ in (22), we obtain

$$\begin{aligned} &H(V^*z, 0, r_1, r_2) - H(Tz, 0, R_1, R_2) \\ &= \log \alpha(r_1, r_2) - \log \alpha(R_1, R_2) \\ &\quad + 2\pi i \left\{ -\frac{z}{2c(dz-c)} \bar{B}_2(\rho) - \frac{dz-c}{2cz} \bar{B}_2(R_1) + s(d, c; R_2, -R_1) \right\}. \end{aligned} \tag{26}$$

Next apply (22) with V replaced by V^*. Observe that R_1 and R_2 are replaced by R_2 and $-R_1$, respectively. Thus, we get

$$\begin{aligned} &H(V^*z, 0, r_1, r_2) - H(z, 0, R_2, -R_1) \\ &= \log \alpha(r_1, r_2) - \log \alpha(R_2, -R_1) \\ &\quad + 2\pi i \left\{ -\frac{1}{2d(dz-c)} \bar{B}_2(\rho) - \frac{dz-c}{2d} \bar{B}_2(R_2) + s(-c, d; -R_1, -R_2) \right\}. \end{aligned} \tag{27}$$

Lastly, apply (22) to the transformation T, and set $r_1 = R_1$ and $r_2 = R_2$. Accordingly, we obtain

$$\begin{aligned} &H(Tz, 0, R_1, R_2) - H(z, 0, R_2, -R_1) \\ &= \log \alpha(R_1, R_2) - \log \alpha(R_2, -R_1) \\ &\quad + 2\pi i \left\{ -\frac{1}{2z} \bar{B}_2(-R_1) - \frac{z}{2} \cdot \bar{B}_2(R_2) + s(0, 1; -R_1, -R_2) \right\}. \end{aligned} \tag{28}$$

Since $\bar{B}_2(x) = \bar{B}_2(-x)$ from (17), we have upon combining (26)–(28),

$$\left(-\frac{z}{2c(dz-c)} + \frac{1}{2d(dz-c)} \right) \bar{B}_2(\rho) - \left(\frac{dz-c}{2cz} + \frac{1}{2z} \right) \bar{B}_2(R_1) + \left(\frac{dz-c}{2d} - \frac{z}{2} \right) \bar{B}_2(R_2)$$
$$+ ((R_1))((R_2)) + s(d, c; R_2, -R_1) - s(-c, d; -R_1, -R_2) = 0.$$

Since $s(-c, d; -R_1, -R_2) = -s(c, d; -R_1, R_2)$, the above reduces to

$$\begin{aligned} &s(d, c; R_2, -R_1) + s(c, d; -R_1, R_2) \\ &= ((-R_1))((R_2)) + \bar{B}_2(\rho)/2cd + d\bar{B}_2(-R_1)/2c + c\bar{B}_2(R_2)/2d, \end{aligned} \tag{29}$$

which is the reciprocity formula for generalized Dedekind sums first proved by Rademacher [16]. When $r_1 = g/N$ and $r_2 = h/N$, proofs of (29) were previously given by Meyer [15] and Dieter [6]. Another proof of (29) has recently been given by E. Grosswald [7].

6. **Transformation formulae of some Lambert series.** In Theorem 2, put $r_1 = r_2 = 0$ and $s = -m$, where $m > 0$ is even. Here,

$$H(z, -m, 0, 0) = 2A(z, -m, 0, 0) = 2A(z, -m)$$

$$= 2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} k^{-m-1}e^{2\pi iknz} = 2\sum_{k=1}^{\infty} k^{-m-1}\frac{e^{2\pi ikz}}{1-e^{2\pi ikz}},$$

which is a Lambert series in the variable $e^{2\pi iz}$. From the functional equation of $\zeta(s)$ [18, p. 24],

$$f_m(0, 0, c, d) = 2(2\pi i)^m(1-(cz+d)^m)\lim_{s\to -m}\Gamma(s)\zeta(s)$$

$$= 2(2\pi i)^m(1-(cz+d)^m)\lim_{s\to -m} 2^{s-1}\pi^s \sec(\tfrac{1}{2}\pi s)\zeta(1-s)$$

$$= (1-(cz+d)^m)\zeta(m+1).$$

Hence, from (5)–(7),

$$(cz+d)^m A(Vz, -m)$$

$$= A(z, -m) + \tfrac{1}{2}(1-(cz+d)^m)\zeta(m+1)$$

(30)
$$+\frac{(2\pi i)^{m+1}}{2(m+2)!}\sum_{j=1}^{c}\sum_{k=0}^{m+2}\binom{m+2}{k}B_k(j/c)\bar{B}_{m+2-k}(jd/c)(-(cz+d))^{k-1}.$$

The transformation formulae (30) were first proved by Apostol [2]. However, due to a miscalculation of residues, the term $\tfrac{1}{2}(1-(cz+d)^m)\zeta(m+1)$ was omitted. See also [9, p. 661]. Consequently, the result is also misstated by Carlitz ([4], [5]), but the other results in [4], [5] are unaffected by this and remain correct. In the notation of Carlitz ([4], [5]), the double sum on the right side of (30) is $-f(d, c, z)$. To show this, all one needs is (17) and the observation that $B_k(j/c)$ may be replaced by $\bar{B}_k(j/c)$. For $k \neq 1$, this is clear. For $k = 1$, this is also clear for $1 \le j \le c-1$, but for $j = c$, $B_1(1) \neq \bar{B}_1(1)$. However, in the latter case $\bar{B}_{m+1}(d) = B_{m+1} = 0$, since $m > 0$ is even. Hence, in all cases $B_k(j/c)$ may be replaced by $\bar{B}_k(j/c)$ in (30).

7. **Additional new transformation formulae.** Let $s = -m$, where $m > 0$ is even, and let r_1 and r_2 be arbitrary. Now, for $0 < \alpha \le 1$ and $\sigma < 0$ [18, p. 37],

$$\Gamma(s)\zeta(s, \alpha) = \frac{(2\pi)^s}{\sin(\pi s)}\sum_{n=1}^{\infty}\frac{\sin(2\pi n\alpha + \pi s/2)}{n^{1-s}}.$$

It follows that for α real and $\sigma < 0$,

$$\Gamma(s)(\zeta(s, \{\alpha\} + \chi(\alpha)) + e^{\pi is}\zeta(s, \{-\alpha\} + \chi(\alpha)))$$

(31)
$$= \frac{(2\pi)^s}{2i\sin(\pi s)}\sum_{n=1}^{\infty}\frac{e^{-2\pi in\alpha+3\pi is/2} - e^{-2\pi in\alpha-\pi is/2}}{n^{1-s}}$$

$$= (2\pi)^s e^{\pi is/2}\phi(1-s, -\alpha),$$

where for α real and $\sigma > 1$,

$$\phi(s, \alpha) = \sum_{n=1}^{\infty} e^{2\pi i n \alpha} n^{-s}.$$

In fact, $\phi(s, \alpha)$ is a special case of Lerch's zeta-function. It follows from (31) that

$$f_m(r_1, r_2, c, d) = -\chi(r_1)(cz + d)^m \phi(m + 1, -r_2) + \chi(R_1)\phi(m + 1, R_2).$$

Hence, from (5)–(7),

$$(cz + d)^m H(Vz, -m, r_1, r_2) = H(z, -m, R_1, R_2) - \chi(r_1)(cz + d)^m \phi(m + 1, -r_2)$$

$$+ \chi(R_1)\phi(m + 1, R_2) + (2\pi i)^{m+1} g(d, c; z; R_1, R_2)/(m + 2)!, \tag{32}$$

where

$$g(d, c; z; R_1, R_2) = \sum_{j=1}^{c} \sum_{k=0}^{m+2} \binom{m+2}{k} B_k\left(\frac{j - \{R_1\}}{c}\right) \bar{B}_{m+2-k}\left(\frac{jd + \rho}{c}\right)(-(cz + d))^{k-1}.$$

Iseki [9] has proved (32) in the special case $V(z) = -1/z$. Another proof of Iseki's result has been given by Apostol [3].

8. The reciprocity formula for $g(d, c; z; R_1, R_2)$. Let V, V^*, and T be as in §5. Put $c_m = (2\pi i)^{m+1}/(m + 2)!$. Replacing z by $-1/z$ in (32), we obtain

$$(-c/z + d)^m H(V^*z, -m, r_1, r_2) - H(Tz, -m, R_1, R_2)$$

$$= -\chi(r_1)(-c/z + d)^m \phi(m + 1, -r_2) \tag{33}$$

$$+ \chi(R_1)\phi(m + 1, R_2) + c_m g(d, c; Tz; R_1, R_2).$$

Apply (32) to V^* to get

$$(dz - c)^m H(V^*z, -m, r_1, r_2) - H(z, -m, R_2, -R_1)$$

$$= -\chi(r_1)(dz - c)^m \phi(m + 1, -r_2) \tag{34}$$

$$+ \chi(R_2)\phi(m + 1, -R_1) + c_m g(-c, d; z; R_2, -R_1).$$

Lastly, let $r_1 = R_1$, $r_2 = R_2$, and apply (32) to the transformation T. Then,

$$z^m H(Tz, -m, R_1, R_2) - H(z, -m, R_2, -R_1)$$

$$= -\chi(R_1) z^m \phi(m + 1, -R_2) + \chi(R_2)\phi(m + 1, -R_1) + c_m g(0, 1; z; R_2, -R_1). \tag{35}$$

Now,

$$g(0, 1; z; R_2, -R_1) = \sum_{k=0}^{m+2} \binom{m+2}{k} B_k(1 - \{R_2\}) \bar{B}_{m+2-k}(-R_1)(-z)^{k-1}$$

$$= (-1/z)(\bar{B}(-R_1) - zB(1 - \{R_2\}))^{m+2}, \tag{36}$$

upon the use of a standard symbolic notation for Bernoulli polynomials. Combining (33)–(36), we find that

$$
\begin{aligned}
&c_m z^m g(d, c; Tz; R_1, R_2) - c_m g(-c, d; z; R_2, -R_1) \\
(37) \qquad &- (c_m/z)(\bar{B}(-R_1) - zB(1 - \{R_2\}))^{m+2} \\
&+ \chi(R_1) z^m (\phi(m+1, R_2) - \phi(m+1, -R_2)) = 0.
\end{aligned}
$$

From the Fourier series [1, p. 805],

$$B_{2n+1}(x) = \frac{2(-1)^{n+1}(2n+1)!}{(2\pi)^{2n+1}} \sum_{k=1}^{\infty} \frac{\sin(2\pi kx)}{k^{2n+1}},$$

where $n \geq 1$ and $0 \leq x \leq 1$, we find that

$$\phi(m+1, R_2) - \phi(m+1, -R_2) = -(2\pi i)^{m+1}\bar{B}_{m+1}(R_2)/(m+1)!.$$

Hence, (37) reduces to

$$
\begin{aligned}
(38) \qquad z^m g(d, c; Tz; R_1, R_2) &= g(-c, d; z; R_2, -R_1) \\
&+ (1/z)(\bar{B}(-R_1) - zB(1 - \{R_2\}))^{m+2} + \chi(R_1) z^m (m+2) \bar{B}_{m+1}(R_2).
\end{aligned}
$$

In the case when $r_1 = r_2 = 0$, (38) reduces to a reciprocity theorem proved by Carlitz ([4], [5]) for the function $f(d, c, z)$ mentioned in §6.

REFERENCES

1. Milton Abramowitz and Irene A. Stegun (editors), *Handbook of mathematical functions, with formulas, graphs, and mathematical tables*, 3rd printing with corrections, Nat. Bur. Standards Appl. Math. Series, 55, Superintendent of Documents, U. S. Government Printing Office, Washington, D. C., 1965. MR 31 #1400.

2. Tom M. Apostol, *Generalized Dedekind sums and transformation formulae of certain Lambert series*, Duke Math. J. 17 (1950), 147–157. MR 11, 641.

3. ———, *A short proof of Shô Iseki's functional equation*, Proc. Amer. Math. Soc. 15 (1964), 618–622. MR 29 #2233.

4. Leonard Carlitz, *Some theorems on generalized Dedekind sums*, Pacific J. Math. 3 (1953), 513–522. MR 15, 12.

5. ———, *Dedekind sums and Lambert series*, Proc. Amer. Math. Soc. 5 (1954), 580–584. MR 16, 14.

6. U. Dieter, *Das Verhalten der Kleinschen Funktionen* $\log \sigma_{g,h}(\omega_1, \omega_2)$ *gegenüber Modultransformationen und verallgemeinerte Dedekindsche Summen*, J. Reine Angew. Math. 201 (1959), 37–70. MR 21 #3397.

7. E. Grosswald, *Dedekind-Rademacher sums and their reciprocity formula*, J. Reine Angew. Math. 251 (1971), 161–173. MR 45 #1849.

8. E. Hecke, *Bestimmung der Klassenzahl einer neuen Reihe von algebraischen Zahlkörpern*, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II 1921, 1–23.

9. Shô Iseki, *The transformation formula for the Dedekind modular function and related functional equations*, Duke Math. J. 24 (1957), 653–662. MR 19, 943.

10. F. Klein, *Über die elliptischen Normalkurven der n-ten Ordnung und zugehörige Modulfunctionen der n-ten Stufe*, Abh. Sächs. Kgl. Ges. Wiss. Leipzig 13 (1885), 337–400.

11. Joseph Lehner, *A partition function connected with the modulus five*, Duke Math. J. 8 (1941), 631–655. MR 3, 166.

12. Joseph Lewittes, *Analytic continuation of Eisenstein series*, Trans. Amer. Math. Soc. 177 (1972), 469–490.

13. John Livingood, *A partition function with the prime modulus $P > 3$*, Amer. J. Math. 67 (1945), 194–208. MR 6, 259.

14. C. Meyer, *Die Berechnung der Klassenzahl Abelscher Körper über quadratischen Zahlkörpern*, Akademie-Verlag, Berlin, 1957. MR 19, 531.

15. ———, *Über einige Anwendungen Dedekindscher Summen*, J. Reine Angew. Math. 198 (1957), 143–203. MR 21 #3396.

16. Hans Rademacher, *Some remarks on certain generalized Dedekind sums*, Acta Arith. 9 (1964), 97–105. MR 29 #1172.

17. B. Schoeneberg, *Zur Theorie der verallgemeinerten Dedekindschen Modulfunktionen*, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II 1969, 119–128. MR 42 #7595.

18. E. C. Titchmarsh, *The theory of the Riemann zeta-function*, Clarendon Press, Oxford, 1951. MR 13, 741.

19. E. T. Whittaker and G. N. Watson, *A course of modern analysis*, 4th. ed., Cambridge Univ. Press, Cambridge, 1962.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, URBANA, ILLINOIS 61801