TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 179, May 1973

ON THE CONVERGENCE OF BEST UNIFORM DEVIATIONS

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ABSTRACT. If a function f is continuous on a closed Jordan curve Γ and meromorphic inside Γ, then the polynomials of best uniform approximation to f on Γ converge interior to Γ. Furthermore, the limit function can in each case be explicitly determined in terms of the mapping function for the interior of Γ. Applications and generalizations of this result are also given.

1. **Introduction.** For a continuous complex valued function f defined on a set E in the plane let $\|f\|_E = \sup_{z \in E}|f(z)|$. Also, if E is compact, for $n \in Z^+$ let $p_n(f, E)$ denote the polynomial of degree n of best uniform approximation to f on E.

In general, the difference $[f(z) - p_n(f, E)(z)]$ does not converge to zero on E, and so one is naturally led to the question of whether or not this difference converges for each $z \in E$ in general. One might also inquire as to the convergence or divergence of the sequence $\{[f(z) - p_n(f, E)(z)]/\|f - p_n(f, E)\|_E\}$, which is of course a broader question.

By expanding on the work of Carathéodory and Fejér [1], Schur and Goluzin [2, p. 497], we are enabled to show that if f is continuous on Γ, a closed Jordan curve, and meromorphic in the interior of Γ, then the sequence $\{f - p_n(f, \Gamma)\}_{n=0}^{\infty}$ converges in the interior of Γ in the following sense. If $\{a_i\}_{i=1}^{m}$ are the poles of f in the interior of Γ with respective multiplicities $\{l_i\}_{i=1}^{m}$, then the sequence

$$\left\{\prod_{i=1}^{m}\left(\frac{\phi(z) - a_i}{1 - \bar{a}_i\phi(z)}\right)^{l_i}[f(z) - p_n(f, \Gamma)(z)]\right\}_{n=0}^{\infty}$$

converges uniformly on compact subsets of the interior of Γ, where $w = \phi(z)$ is an analytic function that maps the interior of Γ onto the open unit disc. Furthermore, we show that the limit function for the latter sequence can be explicitly determined in terms of the function ϕ.

2. **Main theorem.** We will first state our main theorem in the case where Γ is the unit circle U, and then in §6 present the general case. Let D denote the interior of U. If f is as above, we can write

$$f(z) = \sigma(z)\Big/\prod_{i=1}^{m}(z - a_i)^{l_i},$$

where σ is analytic in D and continuous in $D + U$, and $\sigma(a_i) \neq 0$ for $i = 1, 2, \ldots, m$.

Received by the editors February 4, 1972.

AMS (MOS) subject classifications (1970). Primary 41A10, 41A50; Secondary 65D05, 65D15.

Key words and phrases. Best uniform approximation, closed Jordan curve.

Theorem 1. *Let f be as above; then for $z \in D$,*

$$\lim_{n\to\infty}[f(z) - p_n(f, U)(z)] \prod_{i=1}^{m} \left(\frac{z - a_i}{1 - \bar{a}_i z}\right)^{l_i} = \frac{1}{\lambda} \prod_{j=1}^{K} \left(\frac{z - c_j}{1 - \bar{c}_j z}\right), \tag{1}$$

where λ is a prescribed root of an explicitly determined polynomial, $K \leqq (\sum_{i=1}^{m} l_i) - 1$, and the c_j's, $j = 1, 2, \ldots, K$, can be explicitly determined in terms of λ. Furthermore, the convergence is uniform on compact subsets of D.

Remark. Whether or not (1) holds for $z \in U$ is unknown. The few examples of best approximation to meromorphic functions on U that are known [5] indicate uniform convergence on U.

It should also be remarked that if f is as in Theorem 1, we can now calculate $\lim_{n\to\infty} \|f - p_n(f, U)\|_U$, and we can find (using Taylor series for instance) a sequence of polynomials $\{q_n\}_{n=0}^{\infty}$ such that

$$\lim_{n\to\infty} \|f - q_n\|_U = \lim_{n\to\infty} \|f - p_n(f, U)\|_U.$$

3. **Preliminary results.** We shall first show that if f is as in Theorem 1, then there exists a unique function g, analytic in D, which minimizes

$$\left\|[f(z) - g(z)] \prod_{i=1}^{m} \left(\frac{z - a_i}{1 - \bar{a}_i z}\right)^{l_i}\right\|_D,$$

or more precisely, which minimizes

$$\left\|f_1(z) - g(z) \prod_{i=1}^{m} \left(\frac{z - a_i}{1 - \bar{a}_i z}\right)^{l_i}\right\|_D,$$

where $f_1(z) = \sigma(z)/ \prod_{i=1}^{m} (1 - \bar{a}_i z)^{l_i}$. We shall then show that

$$g(z) = \lim_{n\to\infty} p_n(f, U)(z) \quad \text{for } z \in D.$$

To this end we note that proving the existence and uniqueness of such a function g is equivalent to proving the existence and uniqueness of a function F analytic in D, with the prescribed values $F^{(j)}(a_i) = f_1^{(j)}(a_i)$, for $i = 1, 2, \ldots, m$ and $j = 0, 1, \ldots, l_i - 1$, and which minimizes $\|F\|_D$. For if F is such a function we can write

$$F(z) = f_1(z) - g(z) \prod_{i=1}^{m} \left(\frac{z - a_i}{1 - \bar{a}_i z}\right)^{l_i}, \tag{2}$$

where g is analytic in D. Should there exist a function g_1 analytic in D for which

$$\left\|f_1(z) - g_1(z) \prod_{i=1}^{m} \left(\frac{z - a_i}{1 - \bar{a}_i z}\right)^{l_i}\right\|_D < \left\|f_1(z) - g(z) \prod_{i=1}^{m} \left(\frac{z - a_i}{1 - \bar{a}_i z}\right)^{l_i}\right\|_D;$$

then let

$$F_1(z) = f_1(z) - g_1(z) \prod_{i=1}^{m} \left(\frac{z - a_i}{1 - \bar{a}_i z}\right)^{l_i}.$$

The function F_1 will then be analytic in D. Assume the values $F_1^{(j)}(a_i) = f_1^{(j)}(a_i)$ for $i = 1, 2, \ldots, m$ and $j = 0, 1, \ldots, l_i - 1$, and $\|F_1\|_D < \|F\|_D$, thus contradicting our assumption about F. Conversely, if we start with a "minimal" g and define F by (2), we can likewise show that F has the desired minimal property.

We shall, therefore, presently concern ourselves with the problem of determining the function F. We begin by establishing its existence.

Lemma 1. *Let $\{a_i\}_{i=1}^m$ and $\{l_i\}_{i=1}^m$ be as in Theorem* 1 *and $\{B_i^{(j)}\}_{i=1;\,j=0}^{i=m;\,j=l_i-1}$ be any set of $\sum_{i=1}^m l_i$ constants. Then there exists a function F analytic in D with $F^{(j)}(a_i) = B_i^{(j)}$ for $i = 1, 2, \ldots, m$ and $j = 0, 1, \ldots, l_i - 1$ which minimizes $\|F\|_D$ for all such functions.*

Proof. Clearly, the class of all such functions is nonempty. In fact, one can easily construct a polynomial P, which interpolates the given values and so since $\|P\|_D < \infty$, a straightforward application of Montel's compactness criterion [3, Vol. I, p. 415] establishes our lemma.

The following theorem completely describes the function F.

Theorem 2. *Let f_1 be as before; then there exists a unique function F analytic in D such that*

$$F^{(j)}(a_i) = f_1^{(j)}(a_i) \quad \text{for } i = 1, 2, \ldots, m \text{ and } j = 0, 1, \ldots, l_i - 1,$$

and such that $\|F\|_D$ is a minimum. Furthermore,

$$F(z) = \frac{1}{\lambda} \prod_{j=1}^{K} \left(\frac{z - c_j}{1 - \bar{c}_j z}\right),$$

where λ and the c_j's are as in Theorem 1.

Corollary 1. *As an immediate consequence of the above theorem, we have that there exists a unique function g analytic in D which minimizes*

$$\left\| f_1(z) - g(z) \prod_{i=1}^{m} \left(\frac{z - a_i}{1 - \bar{a}_i z}\right)^{l_i} \right\|_D.$$

Furthermore,

$$g(z) = f(z) - \frac{1}{\lambda}\left[\prod_{i=1}^{m} \left(\frac{1 - \bar{a}_i z}{z - a_i}\right)^{l_i}\right]\left[\prod_{j=1}^{K} \left(\frac{z - c_j}{1 - \bar{c}_j z}\right)\right].$$

4. **Proof of Theorem 2.** As we shall see in the next section, Theorem 1 follows directly from Theorem 2, whose proof we will concern ourselves with here.

Lemma 2. *Using the notations and definitions of Lemma* 1, *a function F analytic in D, with values $F^{(j)}(a_i) = B_i^{(j)}$ for $i = 1, 2, \ldots, m$ and $j = 0, 1, \ldots, l_i - 1$, will minimize $\|F\|_D$ for all such functions if and only if the function*

$$(3) \qquad F_1(z) = \left(\frac{1 - \bar{a}_1 z}{z - a_1}\right)\left[\frac{\lambda F(z) - \lambda B_1^{(0)}}{1 - \lambda^2 B_1^{(0)} F(z)}\right],$$

where $\lambda = 1/\|F\|_D$, has the property that if $G(z)$ is analytic in D, and $G^{(j)}(a_i) = F_1^{(j)}(a_i)$ for $i = 1, 2, \ldots, m$ and $j = 0, 1, \ldots, l_1 - 2$ if $i = 1$ and $j = 0, 1, \ldots, l_i - 1$ if $i > 1$, then $\|F_1\|_D \leqq \|G\|_D$.

Proof. Suppose $F^{(j)}(a_i) = B_i^{(j)}$ for $i = 1, 2, \ldots, m$ and $j = 0, 1, \ldots, l_i - 1$ and that $\|F\|_D$ is a minimum. Let F_1 be as above. Clearly F_1 is analytic in D and $\|F_1\|_D = 1$. If for some other function G analytic in D, with $G^{(j)}(a_i) = F_1^{(j)}(a_i)$ for $i = 1, 2, \ldots, m$ and $j = 0, 1, \ldots, l_1 - 2$ if $i = 1$ and $j = 0, 1, \ldots, l_i - 1$ for $i > 1$, we have $\|G\|_D < \|F_1\|_D$. Then let

$$F^*(z) = \frac{1}{\lambda}\left[\left(\frac{z - a_1}{1 - \bar{a}_1 z}\right)G(z) + \lambda B_1^{(0)}\right] \Big/ \left[1 + \lambda \overline{B}_1^{(0)}\left(\frac{z - a_1}{1 - \bar{a}_1 z}\right)G(z)\right].$$

F^* is analytic in D and $\|F^*\|_D < \|F\|_D$. However,

$$\begin{aligned} &F(z) - F^*(z) \\ &= \frac{1}{\lambda}\, \frac{(1 + |\lambda B_1^{(0)}|^2)((z - a_1)/(1 - \bar{a}_1 z))[F_1(z) - G(z)]}{[1 + \lambda \overline{B}_1^{(0)}((z - a_1)/(1 - \bar{a}_1 z))F_1(z)][1 + \lambda \overline{B}^{(0)}((z - a_1)/(1 - \bar{a}_1 z))G(z)]} \\ &= \phi(z) \prod_{i=1}^{m} (z - a_i)^{l_i}, \end{aligned}$$

where $\phi(z)$ is analytic in D. Thus,

$$F^{*(j)}(a_i) = B_i^{(j)} \quad \text{for } i = 1, 2, \ldots, m \text{ and } j = 0, 1, \ldots, l_i - 1,$$

thus contradicting our hypothesis about F.

In order to prove the "sufficient" portion, let F_1 be as defined in the statement of our lemma and let F_1 have the desired minimizing property. Now suppose H is a function analytic in D with values $H^{(j)}(a_i) = B_i^{(j)}$ for $i = 1, 2, \ldots, m$ and $j = 0, 1, \ldots, l_i - 1$, and such that $\|H\|_D < \|F\|_D$. Let

$$H_1(z) = \left(\frac{1 - \bar{a}_1 z}{z - a_1}\right)\left(\frac{\lambda H(z) - \lambda B_1^{(0)}}{1 - \lambda^2 \overline{B}_1^{(0)} H(z)}\right),$$

where $\lambda = 1/\|F\|_D$. The function H_1 is thus analytic in D. $\|H_1\|_D < \|F_1\|_D$ and

$$F_1(z) - H_1(z) = \left(\frac{1 - \bar{a}_1 z}{z - a_1}\right)\left[\frac{(1 - |\lambda B_1^{(0)}|^2)(F(z) - H(z))}{(1 - \lambda^2 \overline{B}_1^{(0)} F(z))(1 - \lambda^2 \overline{B}_1^{(0)} H(z))}\right].$$

Thus we see that H_1 will take on the values $H_1^{(j)}(a_i) = F_1^{(j)}(a_i)$ for $i = 1, 2, \ldots, m$ and $j = 0, 1, \ldots, l_1 - 2$ if $i = 1$ and $j = 0, 1, \ldots, l_i - 1$ if $i > 1$. This contradicts our assumption about F_1, and so our lemma follows.

We will now proceed with the proof of Theorem 2.

Proof of Theorem 2. Let F be analytic in D with values $F^{(j)}(a_i) = f_1^{(j)}(a_i)$ for $i = 1, 2, \ldots, m$ and $j = 0, 1, \ldots, l_i - 1$, and suppose $\|F\|_D$ is a minimum for all such functions.

Let us first rename the a_i's as follows: let $b_k = a_i$ for $\sum_{j=1}^{j=i-1} l_j < k \leqq \sum_{j=1}^{j=i} l_j$. Then define a finite sequence of functions analytic in D by

$$F_0(z) = \lambda F(z) \quad \text{where } \lambda = 1/\|F\|_D,$$

$$F_k(z) = \left(\frac{1 - \bar{b}_k z}{z - b_k}\right)\left[\frac{F_{k-1}(z) - F_{k-1}(b_k)}{1 - \overline{F_{k-1}(b_k)} F_{k-1}(z)}\right],$$

for $k = 1, 2, \ldots, M$ where $M = (\sum_{i=1}^{m} l_i) - 1$. We then have that $\|F_k\|_D = 1$ for $k = 0, 1, \ldots, M$; and furthermore, if G is analytic in D and of the form

$$G(z) = F_k(z) + h(z) \prod_{j=k+1}^{j=M+1} (z - b_j),$$

where h is regular in D, then $\|F_k\|_D \leqq \|G\|_D$. This follows from Lemma 2. In particular, $\|F_M\|_D$ is minimal for all functions G analytic in D which satisfy the single condition $G(b_{M+1}) = F_M(b_{M+1})$. It thus follows as a consequence of the maximum principal that F_M is identically this constant. Furthermore, by our assumptions we have that

$$|F_M(b_{M+1})| = 1. \tag{4}$$

Looking back we find that $F_M(b_{M+1})$ can be expressed as a rational function in λ whose coefficients are determined by the a_i's and the values $f_1^{(j)}(a_i)$, $i = 1, 2, \ldots, m$ and $j = 0, 1, \ldots, l_i - 1$. From (4) we thus obtain a polynomial Λ which we can explicitly determine, and of which λ is a root, i.e. $\Lambda(\lambda) = 0$. The inverse of (2) yields

$$F_{k-1}(z) = \left[\frac{((z - \bar{b}_k)/(1 - \bar{b}_k z))F_k(z) + F_{k-1}(b_k)}{1 + \overline{F_{k-1}(b_k)}((z - \bar{b}_k)/(1 - \bar{b}_k z))F_k(z)}\right], \tag{5}$$

and allows us to calculate F_0 and then F once we have determined λ. We note that the values $F_{k-1}(b_k)$, $k = 1, 2, \ldots, M$, are functions of λ, and that as defined by (5), each function F_{k-1} will be analytic in D with $\|F_{k-1}\|_D = 1$ (as desired) provided that the function F_k is and that $|F_{k-1}(b_k)| < 1$. Thus λ will be the largest positive root of the polynomial Λ which satisfies the condition $|F_{k-1}(b_k)| < 1$, for $k = 1, 2, \ldots, M$. We shall give an example of how Λ, λ and thus F can be determined later.

From (5) it is also evident that the function F_{k-1} will be rational and $\|F_{k-1}\|_D = 1$ for $k = 1, 2, \ldots, M$, since $F_M(z) \equiv \pm 1$. In particular the function F will have the form

$$F(z) = \frac{1}{\lambda} \prod_{j=1}^{K} \left(\frac{z - c_j}{1 - \bar{c}_j z} \right),$$

where $|c_j| < 1$ for $j = 1, 2, \ldots, K$ and $K \leqq M = (\sum_{i=1}^{M} l_i) - 1$. Note that due to possible cancellation of terms we might have $K < M$.

The uniqueness of F now follows easily. Suppose G is another function analytic in D which assumes the values $G^{(j)}(a_i) = F^{(j)}(a_i)$ for $i = 1, 2, \ldots, m$ and $j = 0, 1, \ldots, l_i - 1$, and such that $\|G\|_D = \|F\|_D$. Let λ be as before and define a sequence $\{G_k\}_{k=1}^{M}$ from G as we did from F. We will then conclude that $|G_M(b_{M+1})| = 1$, and furthermore, we will find that $G_M(b_{M+1})$ can be expressed as the same rational function of λ that we obtained for $F_M(b_{M+1})$, namely one whose coefficients depend only on the prescribed values of F at the points $\{a_i\}_{i=1}^{m}$. As a result we find that $G_M(b_{M+1}) = F_M(b_{M+1})$, and then working backwards we obtain $G_0 \equiv F_0$, and thus $G \equiv F$.

Proof of Theorem 1. Let g be as in Corollary 1, the unique analytic function of "best approximation" to f on D. We may then write

$$\begin{aligned} \|f - g\|_U &= \left\| f_1(z) - g(z) \prod_{i=1}^{m} \left(\frac{z - a_i}{1 - \bar{a}_i z} \right)^{l_i} \right\|_D \\ &\leqq \left\| f_1(z) - p_n(f, U)(z) \prod_{i=1}^{m} \left(\frac{z - a_i}{1 - \bar{a}_i z} \right)^{l_i} \right\|_D \\ &= \|f - p_n(f, U)\|_D, \qquad \text{for } n = 0, 1, 2, \ldots. \end{aligned} \tag{6}$$

Furthermore, the function g is not only analytic in D but continuous in $\bar{D} = D + U$, and so by Walsh's theorem [4, p.98], given any $\varepsilon > 0$, there exists a polynomial q such that $\|g - q\|_U < \varepsilon$. Thus for n sufficiently large (greater than the degree of q), we have

$$\|f - p_n(f, U)\|_U \leqq \|f - q\|_U \leqq \|f - g\|_U + \varepsilon,$$

which together with (6) implies that

$$\lim_{n \to \infty} \|f - p_n(f, U)\|_U = \|f - g\|_U. \tag{7}$$

From (7) and the uniqueness of g it now follows that every subsequence of $\{p_n(f, U)\}_{n=0}^{\infty}$ contains itself, a subsequence that converges uniformly on compact subsets of D to g. However, the sequence $\{p_n(f, U)\}_{n=0}^{\infty}$ is equicontinuous on every compact subset of D, and so our theorem follows.

5. **An example.** We now give a simple application of Theorem 1. Let $f(z) = 1/[z(z - 1/2)]$. In order to find $\lim_{n\to\infty}[f - p_n(f, U)]$ we first find that function F analytic in D, with values:

$F(0) = f_1(0) = 1$,

$F(1/2) = f_1(1/2) = 4/3$, for which $\|F\|_D$ is a minimum, where

$f_1(z) = 1/(1 - z/2)$.

Let $\lambda = 1/\|F\|_D$, $F_0 = \lambda F$, and $F_1(z) = (1/z)[(F_0(z) - \lambda)/(1 - \lambda F_0(z))]$. The function F_1 must then be a constant, and

$$\left|F_1\left(\frac{1}{2}\right)\right| = \left|2\left[\frac{\lambda((4/3) - 1)}{1 - \lambda^2(4/3)}\right]\right| = 1.$$

In order to determine λ we must then find the largest positive root of the quadratic equation $2\lambda/[3(1 - \lambda^2(4/3))] = \pm 1$ which satisfies $|F_0(0)| < 1$ or $\lambda < 1$. This yields $\lambda = ((13)^{1/2} - 1)/4 = .65\ldots.$ Thus $F_1(z) \equiv F_1(1/2) = 2\lambda/(3 - 4\lambda^2) = 1$, and $F(z) = (1/\lambda)[(z - \lambda)/(1 - \lambda z)]$. It then follows that for $z \in D$,

$$\lim_{n\to\infty}[f(z) - p_n(f, U)(z)]\left[\frac{z(z - 1/2)}{(1 - z/2)}\right] = \frac{1}{\lambda}\left(\frac{z - \lambda}{1 - \lambda z}\right),$$

and so

$$\lim_{n\to\infty}\|f - p_n(f, U)\|_U = \frac{1}{\lambda} = 1.53\ldots.$$

6. **The general case.** We now consider the case where f is a function that is continuous on a closed Jordan curve Γ and meromorphic inside Γ.

Letting Ω denote the interior of Γ we have, by Riemann's mapping theorem, that there exists a function $w = \phi(z)$ that is regular in Ω and maps Ω conformally onto the open unit disc D. Furthermore, since the boundary of Ω is Γ, a closed Jordan curve, ϕ may be extended continuously to Γ so that it will map Γ onto the unit circle U in a one-to-one manner [3, Vol. III, p. 70].

The function $f \circ \phi^{-1}$ will then be meromorphic in D and continuous on U. If we let $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ be the poles of f in Ω, then $\{a_1, a_2, \ldots, a_m\}$ where $a_i = \phi(\alpha_i)$ for $i = 1, 2, \ldots, m$ will be the poles of $f \circ \phi^{-1}$ in D. Applying Theorem 1 to $f \circ \phi^{-1}$ on U we have

$$\begin{aligned}\lim_{n\to\infty}[f \circ \phi^{-1}(w) - p_n(f \circ \phi^{-1}, U)(w)] \prod_{i=1}^{m}\left(\frac{w - a_i}{1 - \bar{a}_i w}\right)^{l_i} \\ = \frac{1}{\lambda}\prod_{j=1}^{K}\left(\frac{w - c_j}{1 - \bar{c}_j w}\right) \quad \text{for } |w| < 1,\end{aligned} \tag{8}$$

where λ, K and the c_j's, $j = 1, 2, \ldots, K$, are as described in Theorem 1, and the convergence is uniform on compact subsets of D.

Theorem 3. *Using the above notation,*

$$\lim_{n\to\infty}[f(z) - p_n(f,\Gamma)(z)] \prod_{i=1}^{m} \left(\frac{\phi(z) - a_i}{1 - \bar{a}_i\phi(z)}\right)^{l_i}$$
$$= \frac{1}{\lambda}\prod_{j=1}^{K}\left(\frac{\phi(z) - c_j}{1 - \bar{c}_j\phi(z)}\right) \quad \textit{for } z \in \Omega,$$

where the convergence is uniform on compact subsets of Ω.

Proof. Let $P_n = p_n(f \circ \phi^{-1}, U)$, $Q_n = p_n(f,\Gamma)$ for $n = 0, 1, \ldots,$ and $R(w) = \frac{1}{\lambda}\prod_{j=1}^{K}(w - c_j)/(1 - \bar{c}_j w)$.

If $z \in \Omega$ and $w = \phi(z)$ we can write

$$\left|R \circ \phi(z) - [f(z) - Q_n(z)] \prod_{j=1}^{K}\left(\frac{\phi(z) - a_i}{1 - \bar{a}_i\phi(z)}\right)^{l_i}\right|$$
$$\leqq \left|R(w) - [f \circ \phi^{-1}(w) - P_n(w)] \prod_{j=1}^{K}\left(\frac{w - a_i}{1 - \bar{a}_i w}\right)\right| \tag{9}$$
$$+ |P_n(w) - Q_n \circ \phi^{-1}(w)|.$$

Consequently, our theorem will follow if it is shown that the sequence $\{P_n - Q_n \circ \phi^{-1}\}_{n=0}^{\infty}$ converges to zero uniformly on compact subsets of D. In order to accomplish this we first prove that

$$\lim_{n\to\infty}\|f - Q_n\|_\Gamma = \lim_{n\to\infty}\|f \circ \phi^{-1} - P_n\|_U. \tag{10}$$

To this end, we observe that since ϕ^{-1} is regular in D and continuous in $D + U$, it can be arbitrarily approximated by polynomials on U. This follows from Walsh's theorem [4, p. 98]. Thus given any $\varepsilon_1 > 0$ there exists a polynomial S_m such that $\|\phi^{-1} - S_m\|_U < \varepsilon_1$, and given any $\varepsilon_2 > 0$, by choosing ε_1 sufficiently small we will have

$$\|f \circ \phi^{-1} - Q_n \circ S_m\|_U \leqq \|f \circ \phi^{-1} - Q_n \circ \phi^{-1}\|_U + \varepsilon_2 \tag{11}$$
$$= \|f - Q_n\|_\Gamma + \varepsilon_2.$$

But $Q_n \circ S_m$ is a polynomial of degree nm, so we have

$$\|f \circ \phi^{-1} - P_{nm}\|_U \leqq \|f \circ \phi^{-1} - Q_n \circ S_m\|_U,$$

which then implies that

$$\lim_{n\to\infty}\|f \circ \phi^{-1} - P_n\|_U \leqq \lim_{n\to\infty}\|f - Q_n\|_\Gamma.$$

Similarly given any $\varepsilon_3 > 0$ there exists a polynomial r_k such that $\|\phi - r_k\|_\Gamma < \varepsilon_3$, and given any $\varepsilon_4 > 0$, by choosing ε_3 sufficiently small we will have

$$|f - P_n \circ r_k|_\Gamma \leqq \|f - P_n \circ \phi\|_\Gamma + \varepsilon_4 = \|f \circ \phi^{-1} - P_n\|_U + \varepsilon_4.$$

Repeating our previous argument we obtain (10).

Now let us suppose that on some nonempty compact subset of D, $\{P_n - Q_n \circ \phi^{-1}\}_{n=0}^{\infty}$ does not uniformly converge to zero. Then since the sequences $\{P_n\}_{n=0}^{\infty}$ and $\{Q_n \circ \phi^{-1}\}_{n=0}^{\infty}$ are equicontinuous on each compact subset of D, there exists a subsequence $\{P_{k_n} - Q_{k_n} \circ \phi^{-1}\}_{n=0}^{\infty}$, such that:

(i) $\{P_{k_n}\}_{n=0}^{\infty}$ converges uniformly on compact sets of D to a function L_1 analytic in D,

(ii) $\{Q_{k_n} \circ \phi^{-1}\}_{n=0}^{\infty}$ converges uniformly on compact sets of D to a function L_2 analytic in D, and

(iii)

$$\|L_1 - L_2\|_D > 0. \tag{12}$$

By (10) we have

$$\begin{aligned}
&\left\|[f \circ \phi^{-1}(w) - L_1(w)] \prod_{i=1}^{m} \left(\frac{w - a_i}{1 - \bar{a}_i w}\right)^{l_i}\right\|_D \\
&\quad = \left\|[f(z) - L_2 \circ \phi(z)] \prod_{i=1}^{m} \left(\frac{\phi(z) - a_i}{1 - \bar{a}_i \phi(z)}\right)^{l_i}\right\|_\Omega \\
&\quad = \left\|[f \circ \phi^{-1}(w) - L_2(w)] \prod_{i=1}^{m} \left(\frac{w - a_i}{1 - \bar{a}_i w}\right)\right\|_D.
\end{aligned} \tag{13}$$

Now by (8),

$$L_1(w) = f \circ \phi^{-1}(w) - R(w) \prod_{i=1}^{m} \left(\frac{1 - \bar{a}_i w}{w - a_i}\right)^{l_i}.$$

Furthermore, by Theorem 1, $L_1(w)$ is the unique "analytic function of best approximation" to $f \circ \phi^{-1}$ on D. This uniqueness together with (13) implies $L_1 \equiv L_2$ and thus our theorem follows.

7. **Approximation to arbitrary continuous functions.** Although Theorem 3 is not in general applicable for an arbitrary continuous function f on a closed Jordan curve Γ (since f is not in general defined in the interior of Γ), a somewhat weaker version can be obtained.

Let Γ and ϕ be as in §6. For a function f continuous on Γ let

$$\rho(f, \Gamma) = \lim_{n \to \infty} \|f - p_n(f, \Gamma)\|_\Gamma.$$

Theorem 4. *Let f be continuous on Γ; then given any $\varepsilon > 0$ there exists a rational function R of the form*

$$R(w) = \frac{1}{\lambda}\left[\prod_{i=1}^{m} \left(\frac{1 - \bar{a}_i w}{w - a_i}\right)^{l_i}\right]\left[\prod_{j=1}^{K} \left(\frac{w - c_j}{1 - \bar{c}_j w}\right)\right],$$

such that: (i) $|\rho(f,\Gamma) - \|R \circ \phi\|_\Gamma| < \varepsilon$, *and*

(ii) *there exists a polynomial* q *such that* $\|(f-q) - R \circ \phi\|_\Gamma < \varepsilon$.

Proof. If $\rho(f,\Gamma) = 0$ the theorem is true if we let $R \equiv 0$. Thus we may assume $\rho(f,\Gamma) > 0$. It is well known [3, Vol. III, p.100] that if f is continuous on a closed Jordan curve Γ, then there exists a sequence of rational functions that converge to f uniformly on Γ. In particular, there exists a rational function V for which $\|f - V\|_\Gamma < \varepsilon/2$. Moreover, V is of the form $V(z) = \sigma(z)/\prod_{i=1}^{m}(z-\alpha_i)^{l_i}$, where $\sigma(z)$ is regular on Γ and in its interior and the α_i's lie in the interior of Γ. Since we have assumed that $\rho(f,\Gamma) > 0$ it will then follow that V has at least one pole inside Γ if ε is sufficiently small. Let us thus assume that this is the case.

Applying Theorem 3 to V on Γ, let $R^*(z)$ be the limit of the sequence

$$\left\{[V(z) - p_n(V,\Gamma)(z)] \prod_{i=1}^{m}\left(\frac{\phi(z)-a_i}{1-\bar{a}_i\phi(z)}\right)^{l_i}\right\}_{n=0}^{\infty}$$

for z in the interior of Γ, where ϕ is as before and $a_i = \phi(\alpha_i)$ for $i = 1, 2, \ldots, m$. The function $R^*(z)$ is then of the form

$$R^*(z) = R \circ \phi(z) \prod_{i=1}^{m}\left(\frac{\phi(z)-a_i}{1-\bar{a}_i\phi(z)}\right)^{l_i},$$

where

$$R(w) = \frac{1}{\lambda}\left[\prod_{i=1}^{m}\left(\frac{1-\bar{a}_i w}{w-a_i}\right)^{l_i}\right]\left[\prod_{j=1}^{K}\left(\frac{w-c_j}{1-\bar{c}_j w}\right)\right].$$

Now for any polynomial p and for $z \in \Gamma$,

$$\begin{aligned} |V(z) - p(z)| &\leqq |V(z) - f(z)| + |f(z) - p(z)| \\ &\leqq \varepsilon/2 + |f(z) - p(z)|. \end{aligned}$$

Similarly,

$$|f(z) - p(z)| \leqq \varepsilon/2 + |V(z) - p(z)|.$$

Thus $|\rho(V,\Gamma) - \rho(f,\Gamma)| < \varepsilon$. But $\rho(V,\Gamma) = \|R \circ \phi\|_\Gamma$ so (i) follows.

The function $(V - R \circ \phi)$ is continuous on Γ and regular interior to Γ so again by [4] there exists a polynomial q such that $\|V - (R \circ \phi) - q\|_\Gamma < \varepsilon/2$. Thus,

$$\begin{aligned} \|(f-q) - (R \circ \phi)\|_\Gamma &\leqq \|f - V\|_\Gamma + \|V - (R \circ \phi) - q\|_\Gamma \\ &\leqq \varepsilon/2 + \varepsilon/2 = \varepsilon, \end{aligned}$$

and so (ii) follows.

Remark. If f and Γ are as in Theorem 4, and if one can find a sequence $\{R_n\}_{n=0}^{\infty}$ of rational functions converging to f uniformly on Γ, and if one can calculate the mapping function ϕ for the interior of Γ, then it is now possible to determine a sequence of polynomials $\{q_k\}_{k=0}^{\infty}$ such that $\lim_{k\to\infty} \|f - q_k\|_\Gamma = \rho(f, \Gamma)$.

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