INTERSECTIONS OF QUASI-LOCAL DOMAINS

BY

BRUCE PREKOWITZ⁽¹⁾

ABSTRACT. Let $R = \bigcap V_i$ be an intersection of quasi-local domains with a common quotient field K . Our goal is to find conditions on the V_i 's in order to get some or all of V_i 's to be localizations of R . We show for example that if V_1 is a 1-dimensional valuation domain and if $V_1 \not\subseteq V_2$, then both V_1 and V_2 are localizations of $R = V_1 \cap V_2$.

It is well known that for any domain R we have $R = \bigcap R_{M_\alpha}$ where M_α ranges over the maximal ideals of R . However, if a domain R is represented as an intersection of quasi-local domains we cannot conclude that the domains occurring in the intersection are actually localizations of R . Our purpose will be to find some conditions sufficient to ensure that some or all of the quasi-local domains occurring in the intersection are indeed localizations of R .

1. Preliminary remarks. Suppose $R \subseteq V$ where V is a quasi-local domain with maximal ideal M . Then $M \cap R$ is called the *center of V on R* . We observe that if P is the center of V on R , then $R_P \subseteq V$. In fact, V is a localization of R if and only if $V = R_P$.

We also note that if $R \subseteq V$ are domains with the same quotient field, then the nonzero ideals of V contract to nonzero ideals of R .

All rings under consideration are commutative domains with identity. Our terminology will be pretty much that of [1, Chapter 2, §4].

2. Finite intersections. A well-known result (see [1, p. 78] or [2, p. 38]) states that if $R = V_1 \cap \dots \cap V_n$ where the V_i 's are all valuation domains with the same quotient field, and if P_i is the center of V_i on R , then $V_i = R_{P_i}$ for all i . Our first result is a slight generalization of this theorem. The proof, although basically the same as that in [1], will be repeated in slightly more generality. First we will need a preliminary lemma, the proof of which is found in [1].

Lemma A. *Let x be a unit in a quasi-local domain V . Then there exists an*

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integer k (depending on x) such that for any integer m prime to k , $1 + x + \dots + x^{m-1}$ is a unit in V .

Proof. See [1, p. 77].

Theorem 1. Suppose $R = V_1 \cap \dots \cap V_n \cap W$ where the V_i 's are valuation domains and W is a domain with a finite number of maximal ideals. Suppose that the V_i 's and W all have the same quotient field. Let $N_1, \dots, N_r$ be the maximal ideals of W and let $Q_j = N_j \cap R$. Then $W = \bigcap_{j=1}^r R_{Q_j}$.

Proof. Let $x \in W$. If x is a unit of W_{N_j} choose an integer b_j as in Lemma A. Similarly if x is a unit of V_i , choose an integer k_i . Pick $m > 1$ prime to all the k 's and b 's. Consider $s = (1 + x + \dots + x^{m-1})^{-1}$. It follows that if $x \in V_i$ (or W_{N_j}) then s is a unit of V_i (or W_{N_j}). If $x \notin V_i$, then $y = x^{-1}$ is a nonunit of V_i and $s = y^{m-1}(1 + y + \dots + y^{m-1})^{-1}$ is a nonunit of V_i . Hence $s \in R - Q_j$ for all j . If $x \notin V_i$, then $sx = y^{m-2}(1 + y + \dots + y^{m-1})^{-1} \in V_i$. If $x \in V_i$ (or W_{N_j}), then $sx \in V_i$ (or W_{N_j}). Thus $sx \in R$, and hence $x = sx/s \in \bigcap_{j=1}^r R_{Q_j}$. So $W \subseteq \bigcap_{j=1}^r R_{Q_j}$. The reverse inclusion is immediate since $R_{Q_j} \subseteq W_{N_j}$ for all j .

As a corollary we get the aforementioned result.

Corollary 2. Suppose $R = V_1 \cap \dots \cap V_n$ where the V_i 's are all valuation domains with the same quotient field. Let P_i be the center of V_i on R . Then $V_i = R_{P_i}$ for all i .

We would naturally like to know under what additional hypotheses to Theorem 1 can we conclude that $V_i = R_{P_i}$ for all i . The following result tells us that if the V_i 's are all 1-dimensional and if the intersection is irredundant, then we have the desired conclusion.

Theorem 3. Let $R = V \cap W$ where V is a 1-dimensional valuation domain and W is a domain with a finite number of maximal ideals. Suppose that V and W have the same quotient field K , and that $W \not\subseteq V$. Let $N_1, \dots, N_r$ be the maximal ideals of W and let $Q_j = N_j \cap R$. Let P be the center of V on R . Then $V = R_P$ and $W = \bigcap_{j=1}^r R_{Q_j}$.

Proof. The conclusion is trivial in the case $W = K$. So now assume that $W \neq K$. That $W = \bigcap_{j=1}^r R_{Q_j}$ is just Theorem 1. Hence the quotient field of R is K . Of course $R_P \subseteq V$ is immediate, and so we need only show that $V \subseteq R_P$. Let $x \in V$. So $x = a/b$ where $a, b \in R$. Now since K is the quotient field of R it follows that $P, Q_1, \dots, Q_r$ are all nonzero primes of R . So clearly we can choose a and b so that $b \in P \cap Q_1 \cap \dots \cap Q_r$. Now choose $y \in W - V$. By the same argument as that used in the proof of Theorem 1, for a suitable integer $m > 1$, $u = (1 + y + \dots + y^{m-1})^{-1}$ is a unit of $W_{N_1}, \dots, W_{N_r}$ and u is a nonunit of V . Hence $u \in P - (Q_1 \cup \dots \cup Q_r)$. If γ is a valuation associated with V , then the value group of γ has an Archimedean order since V is 1-dimensional [3, Chapter

2]. So there exists an integer $k > 0$ such that $k\gamma(u) > \gamma(b)$. Now $b + u^k \notin Q_1 \cup \dots \cup Q_r$. Thus $b/(b + u^k) \in R_{Q_1} \cap \dots \cap R_{Q_r} = W$. Since $\gamma(b + u^k) = \gamma(b)$, $b/(b + u^k)$ is a unit of V . Thus $b/(b + u^k) \in R - P$. Now $xb/(b + u^k) = a/(b + u^k) \in V$ since both factors are in V . Since $b + u^k \notin Q_1 \cup \dots \cup Q_r$, $a/(b + u^k) \in W$. Thus $a/(b + u^k) \in V \cap W = R$ and hence $x \in R_P$.

Ohm's example [4, p. 330] shows that the hypothesis that W has only finitely many maximal ideals cannot be dropped from Theorem 3. Heinzer [5] has recently generalized Theorem 3 by replacing the requirement that W have only finitely many maximal ideals with the hypothesis that W has a nonzero Jacobson radical. Ohm and Heinzer show in [6] that if V is a rational valuation domain, then the conclusion $V = R_P$ holds with no conditions on the Jacobson radical of W .

Remark. Suppose $R = V_1 \cap \dots \cap V_n$ where the V_i 's are quasi-local domains. Let P_i be the center of V_i on R . Then $P_1 \cup \dots \cup P_n$ is the set of non-units of R . Thus any maximal ideal of R must be equal to one of the P_i 's, and hence R has at most n maximal ideals.

Let $R = \bigcap_I V_i$. We say the intersection is *irredundant* if for all $j \in I$ we have $\bigcap_{I - \{j\}} V_i \not\subseteq V_j$.

Corollary 4. Suppose $R = V_1 \cap \dots \cap V_n \cap W$ is an irredundant intersection where the V_i 's are 1-dimensional valuation domains and W is a domain with a finite number of maximal ideals. Suppose that the V_i 's and W all have the same quotient field. Let P_i be the center of V_i on R . Let $N_1, \dots, N_r$ be the maximal ideals of W and let $Q_j = N_j \cap R$. Then $V_i = R_{P_i}$ for $i = 1, 2, \dots, n$ and $W = \bigcap_{j=1}^r R_{Q_j}$.

Proof. $W = \bigcap_{j=1}^r R_{Q_j}$ follows from Theorem 1. $R = V_i \cap (\bigcap_{j \neq i} V_j \cap W)$ is an irredundant intersection and $\bigcap_{j \neq i} V_j \cap W = \bigcap_{j \neq i} V_j \cap W_{N_1} \cap \dots \cap W_{N_r}$ has only finitely many maximal ideals. Thus $V_i = R_{P_i}$ follows from Theorem 3.

3. General intersections. A number of results in [1] come under the assumption that certain quasi-local domains occurring in a particular intersection are either valuation or 1-dimensional. In an attempt to unify and generalize some of these results, we make the following definition. A domain V is called an *LS domain* (linear spectrum) if the prime ideals of V are linearly ordered by inclusion. We note the following characterization of LS-domains.

Proposition 5. The following are equivalent for a domain R :

- (i) R is an LS-domain.
- (ii) Every radical ideal of R is prime.
- (iii) If $a, b \in R$, then there exists an integer $m > 0$ such that a divides b^m or b divides a^m .

Proof. (i) $\Rightarrow$ (ii). Let I be a radical ideal of R . So $I = \bigcap P_\lambda$ where P_λ ranges over the primes of R which contain I . But since $\{P_\lambda\}$ is a chain of primes, $\bigcap P_\lambda$ is prime.

(ii) $\Rightarrow$ (iii). $\sqrt{(a)} \cap \sqrt{(b)} = P$ is a radical ideal of R and is therefore prime. But then $\sqrt{(a)}$ and $\sqrt{(b)}$ must be comparable. Suppose $\sqrt{(a)} \subseteq \sqrt{(b)}$. Then $a \in \sqrt{(b)}$ and so there exists an integer $m > 0$ such that $a^m \in (b)$.

(iii) $\Rightarrow$ (i). Suppose that the primes of R are not totally ordered. So there exists P and Q primes of R such that $P \not\subseteq Q$ and $Q \not\subseteq P$. Pick $x \in P - Q$ and $y \in Q - P$. By (iii) there exists $m > 0$ such that $x^m \in (y) \subseteq Q$ or $y^m \in (x) \subseteq P$. So $x \in Q$ or $y \in P$, a contradiction.

Suppose $R = \bigcap V_i$ is an intersection of quasi-local domains all with the same quotient field K . We say the intersection is *locally finite* if, for any $0 \neq x \in R$, x is a unit in all but finitely many of the V 's. We say the intersection is *strongly locally finite* if, for any $0 \neq x \in K$, x is a unit in all but finitely many of the V 's.

Observation. Suppose $R = \bigcap V_i$ is a locally finite intersection of quasi-local domains lying between R and its quotient field K . Then if $0 \neq x \in K$, we can write $x = a/b$ where $a, b \in R - \{0\}$. Then the only V_i 's in which x is not a unit are those in which either a or b is a nonunit. There are only finitely many such V_i 's by our local finiteness assumption. Hence the intersection is strongly locally finite. Thus we see that if R has the same quotient field as the V_i 's then local finiteness is equivalent to strong local finiteness. We also note that trivially any finite intersection is strongly locally finite.

The following theorem generalizes a theorem in [1, p. 80] and a lemma of Griffin [7, p. 721].

Theorem 6. Suppose $R = \bigcap V_i$ is a strongly locally finite intersection of LS-domains all with the same quotient field K . Let S be a multiplicatively closed subset of $R - \{0\}$. In V_i let N_i be the prime maximal with respect to exclusion of S . Then $R_S = \bigcap V_{i_{N_i}}$, and this is a strongly locally finite intersection of LS-domains all with quotient field K .

Proof. That $V_{i_{N_i}}$ is an LS-domain with quotient field K is obvious. Since $N_i \cap S = \emptyset$, S consists of units in $V_{i_{N_i}}$. Hence $R_S \subseteq \bigcap V_{i_{N_i}}$. To show the reverse inclusion, let $x \in \bigcap V_{i_{N_i}}$. Since $x \in K$, x is in all but finitely many of the V_i 's, say $x \notin V_1, \dots, V_n$. Since $x \in V_{1_{N_1}}$, $x = a/b$ where $a, b \in V_1$, $b \notin N_1$. Of course b is a nonunit of V_1 since $x \notin V_1$. Let $P = \sqrt{(b)}$ = the intersection of those primes of V_1 which contain b . P is prime by Proposition 5. Claim $P \cap S \neq \emptyset$. If not, then $P \subseteq N_1$ which is maximal with respect to exclusion of S . But then $b \in P \subseteq N_1$ contrary to our condition $b \notin N_1$. So let

$t_1 \in P \cap S \subseteq \sqrt{(b)}$. Now there exists an integer $n > 0$ such that $t_1^n/b \in V_1$. Let $s_1 = t_1^n \in S$. So $s_1x = t_1^n/b \cdot a \in V_1$. In like manner choose $s_2, \dots, s_n \in S$ such that $s_jx \in V_j$. Then $s = s_1s_2 \cdots s_n \in S$ and we have $sx \in V_1 \cap \cdots \cap V_n$. But since $x \in \bigcap_{i > n} V_i$ we get $sx \in \bigcap_i V_i = R$. So $x = sx/s \in R_S$. That the new intersection is strongly locally finite is obvious since $V_i \subseteq V_{iN_i}$ for all i and since $R = \bigcap V_i$ is strongly locally finite.

Theorem 7. Suppose that $R = \bigcap V_i$ is a locally finite intersection of LS-domains between R and its quotient field K . Let P_i be the center of V_i on R and let $\{M_{i\gamma}\}$ be the chain of nonzero primes of V_i . Let $P_{i\gamma} = M_{i\gamma} \cap R$. Suppose that for some fixed j we have $P_{i\gamma} \not\subseteq P_j$ for all $i\gamma$ such that $i \neq j$. Then $V_j = R_{P_j}$.

Proof. By Theorem 6, $R_{P_j} = \bigcap V_{iN_i}$ where N_i is maximal in V_i with respect to exclusion of $R - P_j$. Eliminating obviously redundant copies of K from this intersection we have $R_{P_j} = \bigcap_{\Gamma} V_{kN_k}$ where k ranges over $\Gamma = \{i \mid N_i \neq (0)\}$. Now if $k \in \Gamma$, we have $N_k V_{kN_k} \cap R_{P_j} = QR_{P_j}$ where Q is some prime of R contained in P_j . Since $N_k \neq (0)$ we have $N_k \cap R = \text{some } P_{k\gamma}$. So $[(N_k V_{kN_k} \cap V_k) \cap R] = N_k \cap R = \text{some } P_{k\gamma} = Q R_{P_j} \cap R = Q \subseteq P_j$. Consequently some $P_{k\gamma} \subseteq P_j$ and hence $k = j$ by hypothesis. So $\Gamma = \{j\}$ and $R_{P_j} = V_{jN_j} = V_j$.

Corollary 8. Suppose $R = \bigcap V_i$ is a locally finite intersection of 1-dimensional quasi-local domains between R and its quotient field K . Let P_i be the center of V_i on R . Suppose $P_i \not\subseteq P_j$ for all pairs i, j with $i \neq j$. Then $V_i = R_{P_i}$ for all i .

An example by Graham Evans [1, p. 78] shows that the incomparability assumption in Corollary 8 cannot be dropped.

Proposition 9. Suppose $R = \bigcap V_i$ is an intersection of quasi-local domains all with quotient field K . Let P_i be the center of V_i on R . If for some j , R_{P_j} is a valuation domain with quotient field K , then $V_j = R_{P_j}$.

Proof. Of course $R_{P_j} \subseteq V_j \subseteq K$ and since R_{P_j} is a valuation domain it follows that V_j is a localization of R_{P_j} . Hence V_j is a localization of R and so $V_j = R_{P_j}$.

Theorem 10. Suppose $R = \bigcap V_i$ is a locally finite intersection of valuation domains lying between R and its quotient field K . Let P_i be the center of V_i on R . If, for some j , $\text{rk}(P_j) = 1$, then $V_j = R_{P_j}$.

Proof. By Theorem 6, $R_{P_j} = \bigcap V_{iN_i}$ where N_i is maximal in V_i with respect to exclusion of $R - P_j$. Eliminating the obviously redundant copies of K

we have $R_{P_j} = \bigcap_{\Gamma} V_{iN_i}$ where $\Gamma = \{i \mid N_i \neq (0)\}$. If $k \in \Gamma$, then $N_k V_{kN_k} \cap R_{P_j} = QR_{P_j}$ for some prime of R , $Q \subseteq P_j$. But $Q = N_k \cap R \neq (0)$ since $N_k \neq (0)$. Hence $Q = P_j$. Now if $0 \neq x \in P_j$, then x is a nonunit of V_{kN_k} for all $k \in \Gamma$ and thus Γ must be finite. So $R_{P_j} = V_{1N_1} \cap \dots \cap V_{nN_n}$. By Corollary 2, $R_{P_j} = V_{kN_k}$ holds for each $k \in \Gamma$. Hence R_{P_j} is a valuation domain and thus $V_j = R_{P_j}$ by Proposition 9.

If under the hypotheses of Theorem 10, V_j is a rational valuation domain and is irredundant in the intersection, then by the aforementioned result of Heinzer and Ohm [6], $V_j = R_{P_j}$. This is essentially a result due to Krull [1, p. 81], at least when all of the V_i 's are rational valuation domains. In Ohm's example [4, p. 330] we see that under the hypotheses of Theorem 10, if $\text{rk}(P_j) = 2$, then $V_j = R_{P_j}$ need not hold even if V_j is irredundant in the intersection.

If we weaken the hypotheses of Theorem 10 so as to only require that the V_i 's be LS-domains, we get a similar result for those V_i 's which are not redundant in the intersection. First we note a proposition which allows us to refine a redundant intersection to an irredundant one.

Proposition 11. *Suppose $R = \bigcap_{i \in \Gamma} V_i$ is a locally finite intersection of quasi-local domains between R and its quotient field K . Then $R = \bigcap_{i \in \Gamma^*} V_i$ is an irredundant intersection for some $\Gamma^* \subseteq \Gamma$.*

Proof. Let $\mathcal{S} = \{\Gamma_\nu \subseteq \Gamma \mid R = \bigcap_{i \in \Gamma_\nu} V_i\}$. $\mathcal{S}$ is partially ordered by inclusion. Suppose $\{\Gamma_{\nu_j}\}$ is a chain in $\mathcal{S}$. Claim: $\bar{\Gamma} = \bigcap \Gamma_{\nu_j} \in \mathcal{S}$. Then Zorn's lemma is applicable and a minimal element Γ^* in $\mathcal{S}$ will clearly satisfy our requirements. To verify our claim, we must show that $\bigcap_{i \in \bar{\Gamma}} V_i = R$. Clearly $R \subseteq \bigcap_{i \in \bar{\Gamma}} V_i$. Suppose $x \in K$, $x \notin R$. Since the intersection is actually strongly locally finite it follows that x is in all but finitely many of the V_i 's. Say $V_1, \dots, V_n$ are exactly those V_i 's which exclude x . Clearly for some $1 \leq k \leq n$ we have $k \in \Gamma_{\nu_j}$ for all j . If not, there exists $\Gamma_{\nu_{j(k)}}$ not containing k for each $1 \leq k \leq n$. Then $\Gamma_{\nu_{j_0}} = \min\{\Gamma_{\nu_{j(k)}}\}$ does not contain any k such that $1 \leq k \leq n$. So we have $x \in \bigcap_{i \in \Gamma_{\nu_{j_0}}} V_i = R$, a contradiction. Hence for some k such that $1 \leq k \leq n$ we have $k \in \bar{\Gamma} = \bigcap_j \Gamma_{\nu_j}$, and so $x \notin \bigcap_{i \in \bar{\Gamma}} V_i$. Thus $\bigcap_{i \in \bar{\Gamma}} V_i \subseteq R$.

Theorem 12. *Suppose $R = \bigcap_{i \in \Gamma} V_i$ is a locally finite intersection of LS-domains between R and its quotient field. Suppose V_j is a 1-dimensional valuation domain such that $\bigcap_{i \in \Gamma - \{j\}} V_i \not\subseteq V_j$. Let P_j be the center of V_j on R . If P_j has rank = 1, then $V_j = R_{P_j}$.*

Proof. By Theorem 6, $R_{P_j} = \bigcap_{\Gamma} V_{iN_i}$ where N_i is maximal in V_i with respect to exclusion of $R - P_j$. Of course $V_{iN_i} = V_j$. By hypothesis $V_j \not\subseteq$

$\bigcap_{\Gamma - \{j\}} V_i$ and so $V_{jN_j} = V_j \not\subseteq \bigcap_{\Gamma - \{j\}} V_{iN_i}$. By Proposition 11 we can find $\Gamma^* \subseteq \Gamma$ such that $R_{P_j} = \bigcap_{\Gamma^*} V_{iN_i}$ is an irredundant, locally finite intersection; of course $j \in \Gamma^*$. Since $\text{rk}(P_j) = 1$ we have $N_i V_{iN_i} \cap R_{P_j} = P_j R_{P_j}$ for all $i \in \Gamma^*$. (This contraction is $\neq (0)$ since R_{P_j} has quotient field K and $N_i \neq (0)$.) By local finiteness, there can be only finitely many elements in Γ^* . So we can write $R_{P_j} = V_j \cap (V_{1N_1} \cap \dots \cap V_{tN_t})$ where $V_{1N_1} \cap \dots \cap V_{tN_t} \not\subseteq V_j$. But $V_{1N_1} \cap \dots \cap V_{tN_t}$ has only finitely many maximal ideals, and so by Theorem 3, $V_j = R_{P_j}$.

Suppose that $R = V_1 \cap \dots \cap V_n$ where the V_i 's are quasi-local domains. Let P_i be the center of V_i on R and suppose that $V_i = R_{P_i}$ for all i . If N is prime in R , then $N \subseteq P_1 \cup \dots \cup P_n$; and hence $N \subseteq P_k$ for some k . So NR_{P_k} is prime in V_k and $NR_{P_k} \cap R = N$. Consequently any prime of R is the contraction of a prime of one of the V_i 's. The following theorem gives us a condition sufficient to ensure that this will be the case even if $V_i = R_{P_i}$ does not hold for all i .

Theorem 13. Suppose $R = V_1 \cap \dots \cap V_n$ where the V_i 's are LS-domains all with quotient field K . Let P be prime in R . Then $P = N \cap R$ for N prime in some V_i .

Proof. By Theorem 6, $R_P = V_{1N_1} \cap \dots \cap V_{nN_n}$ where N_i is maximal in V_i with respect to disjointness from $R - P$. Let $Q_i = N_i \cap R \subseteq P$. Now $PR_P = (N_1 V_{1N_1} \cap R_P) \cup \dots \cup (N_n V_{nN_n} \cap R_P) =$ the set of nonunits of R_P . Intersecting both sides with R yields $P = Q_1 \cup \dots \cup Q_n$. Hence $P = Q_i$ for some i .

Theorem 14. Suppose $R = \bigcap V_i$ is a strongly locally finite intersection of LS-domains all with the same quotient field K . Let P be a rank 1 prime of R . Then $P = N \cap R$ for N prime in some V_i .

Proof. Once again Theorem 6 is applicable and we have $R_P = \bigcap V_{iN_i}$ where N_i is maximal in V_i with respect to excluding $R - P$. So for each i we have $(0) \subseteq N_i \subseteq P$. Suppose $N_i \cap R = (0)$ for all i . Then Theorem 6 tells us that $R_{(0)} = \bigcap V_{iN_i} = R_P$ a contradiction, since $\text{rk}(P) = 1$. Thus for some i we have $(0) \neq N_i \cap R \subseteq P$ and so $N_i \cap R = P$.

4. Some counterexamples. Suppose we are given $R = V \cap W$ where V and W are quasi-local domains with the same quotient field. Let P and Q be the centers of V and W on R respectively. If V is a valuation domain, then $W = R_Q$ by Theorem 1. If the intersection is irredundant and if V is a 1-dimensional valuation domain, we also have $V = R_P$ by Theorem 3. In the following example V is a 2-dimensional valuation domain and W is a 3-dimensional regular local

ring. Here we have $V \neq R_P$; and thus we see that the hypothesis of 1-dimensionality in Theorem 3 is essential.

Our first example is a generalization due to Stephen McAdam of one of my examples. As an application, I will give the details of my original example.

Example 1. Let T be a domain with prime ideals $\bar{Q}, \bar{M}, \bar{N}$ such that $\bar{Q} \subseteq \bar{M} \cap \bar{N}$, $\bar{M} \not\subseteq \bar{N}$, $\bar{N} \not\subseteq \bar{M}$, and such that $T_{\bar{Q}}$ is not a valuation domain. Let V be a valuation overring of T with primes $\bar{M}$ and $\bar{Q}$ lying over $\bar{M}$ and $\bar{Q}$ respectively and with $\bar{M}$ maximal in V [2, p. 37]. Let $R = V \cap T_{\bar{N}}$ and let $P = \bar{M} \cap R$. We will show that $V \neq R_P$.

Claim $\bar{Q} T_{\bar{N}} \subseteq P$: Let $x \in \bar{Q} T_{\bar{N}}$. So $x = q/s$ where $q \in \bar{Q}$, $s \in T - \bar{N}$. Suppose $s/q \in V$. Then $s \in qV \cap T \subseteq \bar{Q} \cap T = \bar{Q} \subseteq \bar{N}$, a contradiction. Thus $s/q \notin V$ and so $q/s \in \bar{M}$ since V is a valuation domain. Hence $\bar{Q} T_{\bar{N}} \subseteq \bar{M} \cap T_{\bar{N}} = \bar{M} \cap R = P$. In particular we may view $\bar{Q} T_{\bar{N}}$ as a prime ideal in R as well as in $T_{\bar{N}}$.

Now by Theorem 1, $T_{\bar{N}} = T_{(\bar{N} T_{\bar{N}} \cap R)}$. Thus

$$T_{\bar{Q}} = (T_{\bar{N}})_{(\bar{Q} T_{\bar{N}})} = [R_{(\bar{N} T_{\bar{N}} \cap R)}]_{(\bar{Q} T_{\bar{N}})} = R_{(\bar{Q} T_{\bar{N}})} \supseteq R_P.$$

But $T_{\bar{Q}}$ is not a valuation domain and hence neither is R_P . Consequently $R_P \neq V$.

Let k be a field and let x, y, z be algebraically independent over k . Let $\mathbf{R}$ denote the additive group of reals. Consider the valuation γ on $k(x, y, z)$ given by

- (i) $\gamma(\alpha) = (0, 0)$ for all $0 \neq \alpha \in k$,
- (ii) $\gamma(x) = (\pi, 0)$,
- (iii) $\gamma(y) = (1, 0)$,
- (iv) $\gamma(z) = (0, 1)$,

where the value group is a subgroup of $\mathbf{R} \times \mathbf{R}$ lexicographically ordered, i.e., $(a, b) > (c, d)$ if and only if either $a > c$ or $a = c$ and $b > d$. Since the values of x, y , and z are linearly independent over $\mathbf{Z}$, our valuation is well defined on $k(x, y, z)$. Now in Example 1, let $V = \{r \in k(x, y, z) \mid \gamma(r) \geq (0, 0)\}$. Let $T = k[x, y, z]$, $\bar{Q} = (x, y)T$, $\bar{M} = (x, y, z)T$, $\bar{N} = (x, y, z + 1)T$. So V is a 2-dimensional valuation domain and $W = T_{\bar{N}}$ is a 3-dimensional regular local ring. Since $x/y \in V - W$ and $1/z \in W - V$, $R = V \cap W$ is an irredundant intersection.

Suppose that $R = V \cap W$ is an irredundant intersection of quasi-local domains, and that we have $V = R_P$, $W = R_N$ where P and N are the centers of V and W on R respectively. Then certainly P and N are not comparable. In Example 1, P is the center of V on R and $N = \bar{N} T_{\bar{N}} \cap R$ is the center of $W = T_{\bar{N}}$ on R . But $z \in P - N$ and $z + 1 \in N - P$ and hence P and N are not comparable. Thus we also see by Example 1 that incomparability of P and N is not sufficient to ensure that $V = R_P$ and $W = R_N$.

Theorems 1 and 7 both give one sided results and can be combined to give the following:

Corollary 15. *Suppose $R = V \cap W$ is an irredundant intersection of quasi-local domains with the same quotient field. Let P and Q be the centers of V and W on R respectively. Suppose that V is a valuation domain, W is one-dimensional, and $Q \not\subseteq P$. Then $V = R_P$ and $W = R_Q$.*

In Example 2 we will show that the hypothesis $Q \not\subseteq P$ is essential in this corollary. In this example V is a 2-dimensional valuation domain and W is a one-dimensional quasi-local domain. $V \cap W = R$ is irredundant but $Q \subseteq P$ and $V \neq R_P$.

Before giving the details of this example we will define and verify some known facts about a certain class of domains.

Suppose R is a domain contained in a field K . We define the *composite power series ring* $[[R, K))$ as: $[[R, K)) = \{f \in K[[x]] \mid \text{the constant term of } f \text{ lies in } R\}$. $[[R, K))$ is a domain with quotient field $K((x))$. If J is an ideal in R , by a slight abuse of notation we let $[[J, K)) = \{f \in K[[x]] \mid \text{the constant term of } f \text{ lies in } J\}$. Clearly $[[J, K))$ is an ideal of $[[R, K))$. Furthermore, if P is prime in R , then $[[P, K))$ is prime in $[[R, K))$. We will show that all of the nonzero primes of $[[R, K))$ are gotten in this way.

Proposition 16. *Let $\mathcal{R} = [[R, K))$ be a composite power series ring in the variable x and let $(0) \neq \mathcal{P}$ be prime in $\mathcal{R}$. Then*

- (a) $x \in \mathcal{P}$,
- (b) $ax \in \mathcal{P}$ for all $a \in K$,
- (c) $[[(0), K)) \subseteq \mathcal{P}$,
- (d) $[[P, K)) = \mathcal{P}$ where $P = \mathcal{P} \cap R$.

Proof. (a) Let $0 \neq f \in \mathcal{P}$. So $f = x^n(a_0 + a_1x + a_2x^2 + \cdots)$ where $a_i \in K$, $a_0 \neq 0$, and n is a nonnegative integer. Now $a_0^{-1}x \in \mathcal{R}$ and so $(a_0^{-1}x)f = x^{n+1}(1 + a_0^{-1}a_1x + \cdots) \in \mathcal{P}$. Now $(1 + a_0^{-1}a_1x + \cdots)$ is a unit of $\mathcal{R}$. So $x^{n+1} \in \mathcal{P}$ and hence $x \in \mathcal{P}$.

(b) Since $x \in \mathcal{P}$ and $a^2x \in \mathcal{R}$ we have $x(a^2x) = (ax)^2 \in \mathcal{P}$. Thus $ax \in \mathcal{P}$.

(c) Let $f = a_1x + a_2x^2 + \cdots + a_ix^i + \cdots \in [[(0), K))$. So $f = a_1x + x(a_2x + a_3x^2 + \cdots)$. The first term is in $\mathcal{P}$ by (b) and the second term is in $\mathcal{P}$ by (a).

(d) Let $f = a_0 + a_1x + \cdots \in \mathcal{P}$. So $f - a_0 \in [[(0), K)) \subseteq \mathcal{P}$ and hence $a_0 \in \mathcal{P} \cap R = P$. Therefore $\mathcal{P} \subseteq [[P, K))$. Conversely if $f = a_0 + a_1x + \cdots \in [[P, K))$, then $f - a_0 \in [[(0), K)) \subseteq \mathcal{P}$ and $a_0 \in P \subseteq \mathcal{P}$. Thus $f = (f - a_0) + a_0 \in \mathcal{P}$ and so $[[P, K)) \subseteq \mathcal{P}$.

Consequently the primes of $\mathfrak{R}$ are (0) and the $[[P, K)]$ where P ranges over the primes of R .

Proposition 17. *If R is an n -dimensional LS-domain and K is any field containing R , then $[[R, K)]$ is an $(n + 1)$ -dimensional LS-domain.*

Proof. If $(0) \subset P_1 \subset P_2 \subset \cdots \subset P_n$ are the primes of R , then $(0) \subset [[(0), K)] \subset [[P_1, K)] \subset \cdots \subset [[P_n, K)]$ are the primes of $[[R, K)]$.

Proposition 18. *$[[R, K)]$ is a valuation domain if and only if R is a valuation domain with quotient field K .*

Proof. Suppose $[[R, K)]$ is a valuation domain. Let the quotient field of R be $k \subseteq K$. Suppose $0 \neq y \in K \subseteq K((x)) = \text{quotient field of } [[R, K)]$. Then either y or y^{-1} lies in $[[R, K)]$. So y or y^{-1} lies in $[[R, K)] \cap K = R$. Thus R is a valuation domain with quotient field K .

Conversely suppose that R is a valuation domain with quotient field K . Consider $0 \neq f \in K((x))$. So $f = x^n(a_0 + a_1x + \cdots + a_ix^i + \cdots)$ where $a_i \in K$, $a_0 \neq 0$, $n \in \mathbb{Z}$. Now $f^{-1} = x^{-n}(b_0 + b_1x + \cdots + b_ix^i + \cdots)$ where $b_i \in K$.

(i) If $n > 0$, then $f \in [[R, K)]$.

(ii) If $n < 0$, then $f^{-1} \in [[R, K)]$.

(iii) If $n = 0$, then either $a_0 \in R$ in which case $f \in [[R, K)]$ or $b_0 = a_0^{-1} \in R$ in which case $f^{-1} \in [[R, K)]$. Hence in any case we have f or f^{-1} lies in $[[R, K)]$ and thus $[[R, K)]$ is a valuation domain.

Example 2. Let k be a field and let y be an indeterminant. So $T = k[[y]]$ is a discrete valuation ring with quotient field $L = k((y))$. Let $K = k(y)$. Consider the following composite power series rings in the variable x : $V = [[T, L)]$ and $W = [[K, L)]$. By our last two propositions, we see that V is a 2-dimensional valuation domain with quotient field $L((x))$, and that W is a one-dimensional quasi-local domain with quotient field $L((x))$. Since T and K are incomparable rings, it follows that V and W are also incomparable. Hence $R = V \cap W$ is an irredundant intersection. Let $\bar{M} = yT = \text{the maximal ideal of } T$. Then $M = [[\bar{M}, L)]$ is the maximal ideal of V . $N = [[(0), L)]$ is the maximal ideal of W . So $P = M \cap R = [[\bar{M} \cap K, L)]$ and $Q = N \cap R = [[(0), L)]$. Since $Q \subseteq P$ it follows that $R_P \subseteq R_Q = W$ by Theorem 1. Since $V \not\subseteq W$ we have $V \neq R_P$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, CHICAGO, ILLINOIS
60637

DEPARTMENT OF MATHEMATICS, LOUISIANA STATE UNIVERSITY, BATON ROUGE,
LOUISIANA 70803