SUCCESSIVE REMAINDERS OF THE NEWTON SERIES

BY

G. W. CROFTS AND J. K. SHAW

ABSTRACT. If f is analytic in the open unit disc D and λ is a sequence of points in D converging to 0, then f admits the Newton series expansion $f(z) = f(\lambda_1) + \sum_{n=1}^{\infty} \Delta_{\lambda}^n f(\lambda_{n+1})(z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_n)$, where $\Delta_{\lambda}^n f(z)$ is the n th divided difference of f with respect to the sequence λ . The Newton series reduces to the Maclaurin series in case $\lambda_n \equiv 0$. The present paper investigates relationships between the behavior of zeros of the normalized remainders $\Delta_{\lambda}^k f(z) = \Delta_{\lambda}^k f(\lambda_{k+1}) + \sum_{n=k+1}^{\infty} \Delta_{\lambda}^n f(\lambda_{n+1})(z - \lambda_{k+1}) \cdots (z - \lambda_n)$ of the Newton series and zeros of the normalized remainders $\sum_{n=k}^{\infty} a_n z^{n-k}$ of the Maclaurin series for f . Let C_{λ} be the supremum of numbers $c > 0$ such that if f is analytic in D and each of $\Delta_{\lambda}^k f(z)$, $0 \leq k < \infty$, has a zero in $|z| \leq c$, then $f \equiv 0$. The corresponding constant for the Maclaurin series (C_{λ} , where $\lambda_n \equiv 0$) is called the Whittaker constant for remainders and is denoted by W . We prove that $C_{\lambda} \geq W$, for all λ , and, moreover, $C_{\lambda} = W$ if $\lambda \in l_1$. In obtaining this result, we prove that functions f analytic in D have expansions of the form $f(z) = \sum_{n=0}^{\infty} \Delta_{\lambda}^n f(z_n) C_n(z)$, where $|z_n| \leq W$, for all n , and $C_n(z)$ is a polynomial of degree n determined by the conditions $\Delta_{\lambda}^j C_k(z_j) = \delta_{jk}$.

1. **Introduction.** Let f be analytic in the open unit disc D and let λ denote a sequence of points in D . The Newton series for f is given by

$$(1.1) \quad f(z) = f(\lambda_1) + \sum_{n=1}^{\infty} \Delta_{\lambda}^n f(\lambda_{n+1})(z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_n),$$

where $\Delta_{\lambda}^n f$ denotes the n th divided difference of f with respect to λ :

$$(1.2) \quad \Delta_{\lambda}^0 f(z) = f(z), \quad \Delta_{\lambda}^n f(z) = \frac{\Delta_{\lambda}^{n-1} f(z) - \Delta_{\lambda}^{n-1} f(\lambda_n)}{z - \lambda_n}, \quad n = 1, 2, 3, \dots$$

The series in (1.1) converges uniformly to f on compact subsets of D for each $\lambda \in c_0$, the space of complex sequences converging to 0 ([10], [4]). Note that (1.1) reduces to the Maclaurin series for f in case $\lambda_n \equiv 0$. If λ is the constant sequence $\lambda_n \equiv z_0$, then (1.1) is simply the Taylor series for f , expanded about z_0 . In this case, convergence is guaranteed only in $|z - z_0| < 1 - |z_0|$.

Presented to the Society, August 30, 1972; received by the editors September 5, 1972.
AMS (MOS) subject classifications (1970). Primary 30A98, 30A40; Secondary 30A16.
Key words and phrases. Newton series, zeros of remainders, extremal functions, matrix transformations.

In the present paper, we consider the behavior of zeros of the successive normalized remainders $\Delta_{\lambda}^k f(\lambda_{k+1}) + \sum_{n=k+1}^{\infty} \Delta_{\lambda}^n f(\lambda_{n+1})(z - \lambda_{k+1}) \cdots (z - \lambda_n)$, $0 \leq k < \infty$, of the Newton series. In view of the identity

$$(1.3) \quad \Delta_{\lambda}^k f(z) = \Delta_{\lambda}^k f(\lambda_{k+1}) + \sum_{n=k+1}^{\infty} \Delta_{\lambda}^n f(\lambda_{n+1})(z - \lambda_{k+1}) \cdots (z - \lambda_n),$$

which follows from a simple induction argument, our problem is equivalent to studying the zeros of the successive divided differences $\Delta_{\lambda}^k f$ of f . For $\lambda \in c_0$, let C_{λ} denote the supremum of positive numbers c such that if f is analytic in D and each of $\Delta_{\lambda}^k f(z)$, $0 \leq k < \infty$, has a zero in $|z| \leq c$, then $f \equiv 0$. The bound $C_{\lambda} \leq 1$ is part of the definition. In 1965, M. Pommiez [10] proved that $C_{\lambda} \geq .536$ for each $\lambda \in c_0$. Pommiez noted that C_{λ} might be independent of λ for suitably restricted sequences in c_0 .

In the case when λ is the null sequence ν ($\nu_n \equiv 0$), C_{ν} has been determined ([1], [2], [3], [6]). Here, one considers the successive normalized remainders $\mathcal{S}^k f(z) = \sum_{n=k}^{\infty} f^{(n)}(0) z^{n-k}/n!$ ($k = 0, 1, 2, \dots$) of the Maclaurin series, or equivalently, the zeros of the shift operator $\mathcal{S}^k f$. The constant $W(\mathcal{S})$ is the supremum of positive numbers c such that if f is analytic in D and each of $\mathcal{S}^k f(z)$, $0 \leq k < \infty$, has a zero in $|z| \leq c$, then $f \equiv 0$. Clearly, we have

$$(1.4) \quad C_{\nu} = W(\mathcal{S});$$

$W(\mathcal{S})$ is known as the Whittaker constant belonging to $\mathcal{S}$, and satisfies $.549 < W(\mathcal{S}) < .562$. The following theorem, due to J. D. Buckholtz and J. L. Frank ([3], [2]) completely characterizes $W(\mathcal{S})$.

Theorem. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ have radius of convergence 1 and let $\epsilon > 0$. Then

(i) infinitely many of the partial sums $\sum_{n=0}^k a_n z^n$ have all their zeros in the disc $|z| \leq W(\mathcal{S})^{-1} + \epsilon$,

(ii) infinitely many of the remainders $\sum_{n=k}^{\infty} a_n z^{n-k}$ have no zero in $|z| \leq (W(\mathcal{S})^{-1} + \epsilon)^{-1}$,

(iii) $W(\mathcal{S})$ cannot be replaced by a larger number in either (i) or (ii).

Because of this result, the number $P = W(\mathcal{S})^{-1}$ is called the power series constant.

Following our remarks concerning the null sequence, it is natural to ask whether some extension of (1.4) holds for the nontrivial sequences in c_0 . In this direction, our principal result is

Theorem 1. For each $\lambda \in c_0$, $C_{\lambda} \geq W(\mathcal{S})$; moreover, $C_{\lambda} = W(\mathcal{S})$ for each $\lambda \in l_1 = \{\mu: \sum |\mu_n| < \infty\}$.

This is proved in § 3. To simplify notation in Theorem 1 and its proof, we will abbreviate $W(\mathfrak{D})$ to W and drop the subscript λ from Δ_λ^n when no confusion is likely as to the particular sequence λ under consideration.

A further characterization of W , and one which we will need, is obtained from the remainder polynomials $B_n(z; z_0, z_1, z_2, \dots, z_{n-1})$. These are defined inductively by

$$B_0(z) = 1,$$

$$(1.5) \quad B_n(z; z_0, z_1, \dots, z_{n-1}) = z^n - \sum_{k=0}^{n-1} z_k^{n-k} B_k(z; z_0, z_1, \dots, z_{k-1}),$$

for $n = 1, 2, 3, \dots$, where $\{z_k\}_{k=0}^\infty$ is a sequence of complex numbers. Let $H_n = \max |B_n(0; w_0, w_1, \dots, w_{n-1})|$, where the maximum is taken over all sequences $\{w_k\}_{k=0}^{n-1}$ in $\bar{D}$. Buckholtz [1] proved that

$$(1.6) \quad W^{-1} = \lim_{n \rightarrow \infty} H_n^{1/n} = \sup_{1 \leq n < \infty} H_n^{1/n},$$

and that there exists a constant β , $0 < \beta < 1$, such that

$$(1.7) \quad W^n H_n \geq \beta$$

for $0 \leq n < \infty$ [3]. From (1.6) the numerical value of W can be (theoretically) calculated to any desired accuracy.

2. Preliminaries. The bound $C_\lambda \geq .536$ is a consequence of the following expansion theorem of Pommiez.

Theorem. Suppose that f is analytic in D and c is a number such that $0 \leq c \leq .536$. Let $\{z_n\}_{n=0}^\infty$ be a sequence of complex numbers in $|z| \leq c$ and let $\lambda \in c_0$. Then there exists a sequence $\{C_n\}$ of polynomials, C_n of degree n , such that for all $z \in D$,

$$(2.1) \quad f(z) = f(z_0) + \sum_{n=1}^{\infty} \Delta_\lambda^n f(z_n) C_n(z).$$

From (2.1) the conditions $\Delta_\lambda^n f(z_n) = 0$, $0 \leq n < \infty$, imply $f \equiv 0$. Thus $C_\lambda \geq .536$, $\lambda \in c_0$.

We seek an expansion of the form (2.1) with milder restrictions on the sequence $\{z_k\}$. Applying (2.1) to $F(z) = (z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_n)$ and noting $\Delta^k F(z) = (z - \lambda_{k+1})(z - \lambda_{k+2}) \cdots (z - \lambda_n)$, $0 \leq k \leq n-1$, and $\Delta^n F(z) = 1$, it follows that the polynomials $C_n(z)$ must satisfy

$$(2.2) \quad \begin{aligned} (z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_n) &= (z_0 - \lambda_1)(z_0 - \lambda_2) \cdots (z_0 - \lambda_n) \\ &+ \sum_{k=1}^{n-1} (z_k - \lambda_{k+1}) \cdots (z_k - \lambda_n) C_k(z) + C_n(z). \end{aligned}$$

Then clearly $C_n(z)$ depends on $z_0, z_1, \dots, z_{n-1}$ and $\lambda_1, \lambda_2, \dots, \lambda_n$. Taking $C_0(z) = 1$, (2.2) becomes

$$(2.3) \quad \begin{aligned} C_n(z; z_0, z_1, \dots, z_{n-1}; \lambda_1, \lambda_2, \dots, \lambda_n) \\ = (z - \lambda_1) \dots (z - \lambda_n) - \sum_{k=0}^{n-1} (z_k - \lambda_{k+1}) \dots (z_k - \lambda_n) \\ \cdot C_k(z; z_0, z_1, \dots, z_{k-1}; \lambda_1, \lambda_2, \dots, \lambda_k). \end{aligned}$$

We therefore take (2.3) as our recursive defining relation, without regard to the restrictions on $\{z_k\}_{k=0}^{n-1}$ imposed by Pommiez's theorem.

Lemma 1. Let $\{z_j\}_{j=0}^{\infty}$ and $\{\lambda_j\}_{j=1}^{\infty}$ be sequences of complex numbers. The following identities hold:

$$(2.4) \quad \begin{aligned} \Delta^k C_n(z; z_0, \dots, z_{n-1}; \lambda_1, \dots, \lambda_n) \\ = C_{n-k}(z; z_k, \dots, z_{n-1}; \lambda_{k+1}, \dots, \lambda_n), \quad \text{for } 0 \leq k \leq n, \end{aligned}$$

$$(2.5) \quad C_n(z_0; z_0, \dots, z_{n-1}; \lambda_1, \dots, \lambda_n) = 0, \quad n \geq 1,$$

$$(2.6) \quad \Delta^k C_n(z_k; z_0, \dots, z_{n-1}; \lambda_1, \dots, \lambda_n) = \delta_{kn}, \quad 0 \leq k, n < \infty,$$

where δ_{kn} denotes the Kronecker delta,

$$(2.7) \quad \begin{aligned} C_n(\alpha z; \alpha z_0, \dots, \alpha z_{n-1}; \alpha \lambda_1, \dots, \alpha \lambda_n) \\ = \alpha^n C_n(z; z_0, \dots, z_{n-1}; \lambda_1, \dots, \lambda_n), \quad n \geq 0, \end{aligned}$$

where α is a complex number.

$$(2.8) \quad \begin{aligned} C_n(z; z_0, \dots, z_{n-1}; \lambda_1, \dots, \lambda_n) \\ = \sum_{k=0}^n C_{n-k}(\lambda_{k+1}; z_k, \dots, z_{n-1}; \lambda_{k+1}, \dots, \lambda_n) (z - \lambda_1)(z - \lambda_2) \dots (z - \lambda_k), \\ n \geq 0, \end{aligned}$$

$$(2.9) \quad C_n(z; z_0, \dots, z_{n-1}; 0, \dots, 0) = B_n(z; z_0, \dots, z_{n-1}), \quad n \geq 0.$$

Proof. In (2.8), take $(z - \lambda_1) \dots (z - \lambda_k)$ to be 1 if $k = 0$. Note that (2.6) implies that the polynomials $C_n(z)$, $0 \leq n < \infty$, together with the sequence of linear functionals $f \rightarrow \Delta^k f(z_k)$, $0 \leq k < \infty$, form a biorthonormal system [4].

We establish (2.4) by induction on n . If $n = 0$, then $k = 0$ and the result is trivial. Let m be a positive integer and suppose that for each j such that $0 \leq j \leq m-1$ we have $\Delta^k C_j(z; z_0, \dots, z_{j-1}; \lambda_1, \dots, \lambda_j) = C_{j-k}(z; z_k, \dots, z_{j-1}; \lambda_{k+1}, \dots, \lambda_j)$,

$0 \leq k \leq j$. Note that $k > j$ implies $\Delta^k C_j(z) = 0$, since $C_j(z)$ is a polynomial of degree j in z . By (2.3) and the induction hypothesis, $k \leq m-1$ implies

$$\begin{aligned} & \Delta^k C_m(z; z_0, \dots, z_{m-1}; \lambda_1, \dots, \lambda_m) \\ &= \Delta^k (z - \lambda_1) \dots (z - \lambda_m) \\ &\quad - \sum_{j=0}^{m-1} (z_j - \lambda_{j+1}) \dots (z_j - \lambda_m) \Delta^k C_j(z; z_0, \dots, z_{j-1}; \lambda_1, \dots, \lambda_j) \\ &= (z - \lambda_{k+1}) \dots (z - \lambda_m) \\ &\quad - \sum_{j=k}^{m-1} (z_j - \lambda_{j+1}) \dots (z_j - \lambda_m) C_{j-k}(z; z_k, \dots, z_{j-1}; \lambda_{k+1}, \dots, \lambda_j) \\ &= (z - \lambda_{k+1}) \dots (z - \lambda_m) \\ &\quad - \sum_{p=0}^{m-k-1} (z_{p+k} - \lambda_{p+k+1}) \dots (z_{p+k} - \lambda_m) C_p(z; z_k, \dots, z_{p+k-1}; \lambda_{k+1}, \dots, \lambda_{p+k}) \\ &= C_{m-k}(z; z_k, \dots, z_{m-1}; \lambda_{k+1}, \dots, \lambda_m). \end{aligned}$$

Since (2.3) implies $\Delta^m C_m(z; z_0, \dots, z_{m-1}; \lambda_1, \dots, \lambda_m) = \Delta^m (z - \lambda_1) \dots (z - \lambda_m) = 1$, the proof of (2.4) is complete.

For the proof of (2.5), note first that $C_1(z_0; z_0; \lambda_1) = (z_0 - \lambda_1) - (z_0 - \lambda_1)C_0(z_0) = 0$. Let $m \geq 2$ be an integer and suppose

$$C_j(z_0; z_0, \dots, z_{j-1}; \lambda_1, \dots, \lambda_j) = 0 \quad \text{for } 1 \leq j \leq m-1.$$

Then

$$\begin{aligned} & C_m(z_0; z_0, \dots, z_{m-1}; \lambda_1, \dots, \lambda_m) \\ &= (z_0 - \lambda_1) \dots (z_0 - \lambda_m) \\ &\quad - \sum_{j=0}^{m-1} (z_j - \lambda_{j+1}) \dots (z_j - \lambda_m) C_j(z_0; z_0, \dots, z_{j-1}; \lambda_1, \dots, \lambda_j) \\ &= (z_0 - \lambda_1) \dots (z_0 - \lambda_m) - (z_0 - \lambda_1) \dots (z_0 - \lambda_m) = 0. \end{aligned}$$

The proofs of (2.7) and (2.9) are similar. Equation (2.6) follows from (2.4) and (2.5), together with the fact that $C_n(z)$ is a polynomial of degree n in z ; (2.8) is the Newton series expansion of $C_n(z; z_0, \dots, z_{n-1}; \lambda_1, \dots, \lambda_n)$.

A convenient representation for the coefficient $C_n(\lambda_1; z_0, \dots, z_{n-1}; \lambda_1, \dots, \lambda_n)$ in (2.8) is obtained by considering the infinite upper triangular matrices A and B defined as follows: for $0 \leq j, k < \infty$ let

$$A_{jk} = \begin{cases} C_{k-j}(\lambda_{j+1}; z_j, \dots, z_{k-1}; \lambda_{j+1}, \dots, \lambda_k), & j < k, \\ 1, & j = k, \\ 0, & j > k, \end{cases} \quad (2.10)$$

$$B_{jk} = \begin{cases} (z_j - \lambda_{j+1})(z_j - \lambda_{j+2}) \dots (z_j - \lambda_k), & j < k, \\ 1, & j = k, \\ 0, & j > k. \end{cases}$$

Thus A_{kn} is the k th coefficient of $C_n(z; z_0, \dots, z_{n-1}; \lambda_1, \dots, \lambda_n)$ in the expansion (2.8). It is easy to show that the matrices A and B are mutually reciprocal. For $j < n$, (2.3) implies

$$\begin{aligned} \sum_{k=j}^n A_{jk} B_{kn} &= \sum_{k=j}^n C_{k-j}(\lambda_{j+1}; z_j, \dots, z_{k-1}; \lambda_{j+1}, \dots, \lambda_k)(z_k - \lambda_{k+1}) \dots (z_k - \lambda_n) \\ &= \sum_{m=0}^{n-j} C_m(\lambda_{j+1}; z_j, \dots, z_{m+j-1}; \lambda_{j+1}, \dots, \lambda_{m+j})(z_{m+j} - \lambda_{m+j+1}) \dots (z_{m+j} - \lambda_n) \\ &= (\lambda_{j+1} - \lambda_{j+1})(\lambda_{j+1} - \lambda_{j+2}) \dots (\lambda_{j+1} - \lambda_n) = 0. \end{aligned}$$

By applying (2.8) and then (2.6) and (2.4), we obtain

$$\begin{aligned} \sum_{k=j}^n B_{jk} A_{kn} &= \sum_{k=j}^n (z_j - \lambda_{j+1}) \dots (z_j - \lambda_k) C_{n-k}(\lambda_{k+1}; z_k, \dots, z_{n-1}; \lambda_{k+1}, \dots, \lambda_n) \\ &= \sum_{m=0}^{n-j} (z_j - \lambda_{j+1}) \dots (z_j - \lambda_{j+m}) \\ &\quad \cdot C_{n-j-m}(\lambda_{m+j+1}; z_{m+j}, \dots, z_{n-1}; \lambda_{m+j+1}, \dots, \lambda_n) \\ &= C_{n-j}(z_j; z_j, \dots, z_{n-1}; \lambda_{j+1}, \dots, \lambda_n) = 0. \end{aligned}$$

Since $A_{nn} = B_{nn} = 1$, $0 \leq n < \infty$, It follows that $AB = BA = I$, where I is the identity matrix.

Let m be a nonnegative integer and define the $(m+1)$ by $(m+1)$ matrices A^m and B^m by

$$A_{jk}^m = A_{jk}, \quad B_{jk}^m = B_{jk}, \quad 0 \leq j, k \leq m.$$

Arguing as above, it follows that A^m and B^m are inverses. Hence the entries in A^m can be determined by considering the cofactors in B^m . In particular,

$$(2.11) \quad A_{0,m}^m = C_m(\lambda_1; z_0, \dots, z_{m-1}; \lambda_1, \dots, \lambda_m) \\ = (-1)^m \text{Det} \begin{bmatrix} (z_0 - \lambda_1) & (z_0 - \lambda_1)(z_0 - \lambda_2) & \dots & [(z_0 - \lambda_1)(z_0 - \lambda_2) \dots (z_0 - \lambda_m)] \\ 1 & (z_1 - \lambda_2) & \dots & [(z_1 - \lambda_2)(z_1 - \lambda_3) \dots (z_1 - \lambda_m)] \\ 0 & 1 & \dots & [(z_2 - \lambda_3)(z_2 - \lambda_4) \dots (z_2 - \lambda_m)] \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (z_{m-1} - \lambda_m) \end{bmatrix}.$$

We will use (2.11) in establishing a relationship, in analogy to (2.9), between $C_n(\lambda_1; z_0, \dots, z_{n-1}; \lambda_1, \dots, \lambda_n)$ and $B_n(0; z_0, \dots, z_{n-1})$. Note first that if $z_0 \neq 0$, then (2.3) and (1.5) give $C_1(\lambda_1; z_0; \lambda_1) = -(z_0 - \lambda_1) = ((z_0 - \lambda_1)/z_0)(-z_0) = ((z_0 - \lambda_1)/z_0)B_1(0; z_0)$. Substituting this expression into (2.3), with $n = 2$, and using (1.5), we obtain

$$C_2(\lambda_1; z_0, z_1; \lambda_1, \lambda_2) = ((z_0 - \lambda_1)/z_0)B_2(0; z_0, z_1),$$

and similarly,

$$C_3(\lambda_1; z_0, z_1, z_2; \lambda_1, \lambda_2, \lambda_3) = ((z_0 - \lambda_1)/z_0)[B_3(0; z_0, z_1, z_2) - \lambda_2 B_2(0; z_0, z_1)].$$

In general, we can prove the following result.

Theorem 2. *If $n \geq 2$ and $z_0, z_1, \dots, z_{n-1}$ are nonzero, then*

$$(2.12) \quad C_n(\lambda_1; z_0, z_1, \dots, z_{n-1}; \lambda_1, \lambda_2, \dots, \lambda_n) \\ = \left(\frac{z_0 - \lambda_1}{z_0} \right) \sum_{\substack{2 \leq j_1 < \dots < j_k \leq n-1 \\ 0 \leq k \leq n-2}} (-1)^k \lambda_{j_1} \lambda_{j_2} \dots \lambda_{j_k} \\ \cdot B_{n-k}(0; z_0, z_1, \dots, \widehat{z_{j_1}}, \dots, \widehat{z_{j_2}}, \dots, \widehat{z_{j_k}}, \dots, z_{n-1}).$$

The summation in (2.12) is taken over all possible configurations $\lambda_{p_1} \lambda_{p_2} \dots \lambda_{p_k}$ such that $2 \leq p_1 < p_2 < \dots < p_k \leq n-1$ and $0 \leq k \leq n-2$; if $k = 0$ we take $\lambda_{p_1} \dots \lambda_{p_k} = 1$. The symbol $\widehat{z_{j_i}}$ means that the variable z_{j_i} has been removed; thus

$$\begin{aligned}
& B_{n-k}(0; z_0, z_1, \dots, \widehat{z_{j_1}}, \dots, \widehat{z_{j_2}}, \dots, \widehat{z_{j_k}}, \dots, z_{n-1}) \\
& = B_{n-k}(0; z_0, z_1, \dots, z_{j_1-1}, z_{j_1+1}, \dots, z_{j_2-1}, \\
& \qquad \qquad \qquad z_{j_2+1}, \dots, z_{j_k-1}, z_{j_k+1}, \dots, z_{n-1}),
\end{aligned}$$

To prove Theorem 2, we need the following technical lemma concerning the remainder polynomials.

Lemma 2. *If $n \geq 1$, then*

$$\begin{aligned}
& B_{n+1}(0; z_0, z_1, z_2, \dots, z_n) \\
(2.13) \quad & = z_0 B_n(0; z_0, z_2, z_3, \dots, z_n) - z_0 B_n(0; z_1, z_2, \dots, z_n).
\end{aligned}$$

If $n \geq 2$, $z_0 \neq 0$ and $z_1 \neq 0$, then

$$\begin{aligned}
& B_n(0; z_0, z_1, z_3, z_4, \dots, z_n) \\
(2.14) \quad & = B_n(0; z_0, z_2, z_3, \dots, z_n) - (z_0/z_1) B_n(0; z_1, z_2, \dots, z_n).
\end{aligned}$$

Proof. Consider (2.13). If $n = 1$, then

$$z_0 B_1(0; z_0) - z_0 B_1(0; z_1) = z_0(-z_0) - z_0(-z_1) = -z_0^2 + z_0 z_1 = B_2(0; z_0, z_1).$$

Let n be a positive integer and suppose that (2.13) holds for the integers k such that $1 \leq k \leq n$. Then by (1.5),

$$\begin{aligned}
B_{n+2}(0; z_0, z_1, z_2, \dots, z_{n+1}) &= - \sum_{k=0}^{n+1} z_k^{n+2-k} B_k(0; z_0, z_1, \dots, z_{k-1}) \\
&= -z_0^{n+2} - z_1^{n+1} B_1(0; z_0) - \sum_{k=2}^{n+1} z_k^{n+2-k} z_0 [B_{k-1}(0; z_0, z_2, \dots, z_{k-1}) \\
& \qquad \qquad \qquad - B_{k-1}(0; z_1, z_2, \dots, z_{k-1})] \\
&= -z_0^{n+2} - z_0 \sum_{k=2}^{n+1} z_k^{n+2-k} B_{k-1}(0; z_0, z_2, \dots, z_{k-1}) \\
& \qquad \qquad \qquad + z_0 z_1^{n+1} + z_0 \sum_{k=2}^{n+1} z_k^{n+2-k} B_{k-1}(0; z_1, z_2, \dots, z_{k-1}) \\
&= z_0 B_{n+1}(0; z_0, z_2, z_3, \dots, z_{n+1}) - z_0 B_{n+1}(0; z_1, z_2, \dots, z_{n+1}).
\end{aligned}$$

For the proof of (2.14) note first that $B_2(0; z_0, z_2) - (z_0/z_1)B_2(0; z_1, z_2) = -z_0^2 + z_0 z_2 - (z_0/z_1)(-z_1^2 + z_1 z_2) = -z_0^2 + z_0 z_2 + z_0 z_1 - z_0 z_2 = -z_0^2 + z_0 z_1 = B_2(0; z_0, z_1)$. If (2.14) holds for the integers k such that $2 \leq k \leq n$, then

$$\begin{aligned} & B_{n+1}(0; z_0, z_1, z_3, \dots, z_{n+1}) \\ &= -z_0^{n+1} - z_1^n(-z_0) - \sum_{k=3}^{n+1} z_k^{n+2-k} B_{k-1}(0; z_0, z_1, z_3, \dots, z_{k-1}) \\ &= -z_0^{n+1} + z_0 z_1^n - \sum_{k=3}^{n+1} z_k^{n+2-k} \left[B_{k-1}(0; z_0, z_2, z_3, \dots, z_{k-1}) \right. \\ &\quad \left. - \frac{z_0}{z_1} B_{k-1}(0; z_1, z_2, \dots, z_{k-1}) \right] \\ &= -z_0^{n+1} - z_2^n B_1(0; z_0) - \sum_{k=3}^{n+1} z_k^{n+2-k} B_{k-1}(0; z_0, z_2, \dots, z_{k-1}) \\ &\quad + z_0 z_1^n + z_2^n B_1(0; z_0) + \frac{z_0}{z_1} \sum_{k=3}^{n+1} z_k^{n+2-k} B_{k-1}(0; z_1, z_2, \dots, z_{k-1}) \\ &= -z_0^{n+1} - \sum_{k=2}^{n+1} z_k^{n+2-k} B_{k-1}(0; z_0, z_2, z_3, \dots, z_{k-1}) \\ &\quad + \frac{z_0}{z_1} \left[z_1^{n+1} + z_2^n B_1(0; z_0) \frac{z_1}{z_0} + \sum_{k=3}^{n+1} z_k^{n+2-k} B_{k-1}(0; z_1, z_2, \dots, z_{k-1}) \right] \\ &= B_{n+1}(0; z_0, z_2, z_3, \dots, z_{n+1}) \\ &\quad + \frac{z_0}{z_1} \left[z_1^{n+1} + z_2^n B_1(0; z_1) + \sum_{k=3}^{n+1} z_k^{n+2-k} B_{k-1}(0; z_1, z_2, \dots, z_{k-1}) \right] \\ &= B_{n+1}(0; z_0, z_2, \dots, z_{n+1}) - (z_0/z_1) B_{n+1}(0; z_1, z_2, \dots, z_{n+1}), \end{aligned}$$

and this completes the proof.

Proof of Theorem 2. The proof is by induction; the case $n = 2$ has been established. Suppose now that $n > 2$ and (2.12) holds for the integers k such that $2 \leq k \leq n-1$. Taking $m = n$ in (2.11) and expanding the determinant by the first column, we have

$$\begin{aligned} & C_n(\lambda_1; z_0, \dots, z_{n-1}; \lambda_1, \dots, \lambda_n) \\ &= (-1)^n [(-1)^{n-1} (z_0 - \lambda_1) C_{n-1}(\lambda_2; z_1, z_2, \dots, z_{n-1}; \lambda_2, \lambda_3, \dots, \lambda_n)] \\ &\quad - (-1)^n [(-1)^{n-1} (z_0 - \lambda_2) C_{n-1}(\lambda_1; z_0, z_2, \dots, z_{n-1}; \lambda_1, \lambda_3, \dots, \lambda_n)]. \end{aligned}$$

By the induction hypothesis,

$$\begin{aligned}
& C_n(\lambda_1; z_0, \dots, z_{n-1}; \lambda_1, \dots, \lambda_n) \\
&= -(z_0 - \lambda_1) \left(\frac{z_1 - \lambda_2}{z_1} \right) \\
&\quad \sum_{\substack{3 \leq j_1 < \dots < j_k \leq n-1 \\ 0 \leq k \leq n-3}} (-1)^k \lambda_{j_1} \dots \lambda_{j_k} B_{n-k-1}(0; z_1, z_2, \dots, \widehat{z_{j_1}}, \dots, \widehat{z_{j_k}}, \dots, z_{n-1}) \\
&\quad + (z_0 - \lambda_2) \left(\frac{z_0 - \lambda_1}{z_0} \right) \\
&\quad \sum_{\substack{3 \leq j_1 < \dots < j_k \leq n-1 \\ 0 \leq k \leq n-3}} (-1)^k \lambda_{j_1} \dots \lambda_{j_k} B_{n-k-1}(0; z_0, z_2, \dots, \widehat{z_{j_1}}, \dots, \widehat{z_{j_k}}, \dots, z_{n-1}).
\end{aligned}$$

By (2.13) and (2.14), the coefficient of $((z_0 - \lambda_1)/z_0)(-1)^k \lambda_{j_1} \dots \lambda_{j_k}$ is given by

$$\begin{aligned}
& -((z_1 - \lambda_2)/z_1) z_0 B_{n-k-1}(0; z_1, z_2, \dots, \widehat{z_{j_1}}, \dots, \widehat{z_{j_k}}, \dots, z_{n-1}) \\
& \quad + (z_0 - \lambda_2) B_{n-k-1}(0; z_0, z_2, \dots, \widehat{z_{j_1}}, \dots, \widehat{z_{j_k}}, \dots, z_{n-1}) \\
&= z_0 B_{n-k-1}(0; z_0, z_2, \dots, \widehat{z_{j_1}}, \dots, \widehat{z_{j_k}}, \dots, z_{n-1}) \\
& \quad - z_0 B_{n-k-1}(0; z_1, z_2, \dots, \widehat{z_{j_1}}, \dots, \widehat{z_{j_k}}, \dots, z_{n-1}) \\
& \quad - \lambda_2 [B_{n-k-1}(0; z_0, z_2, \dots, \widehat{z_{j_1}}, \dots, \widehat{z_{j_k}}, \dots, z_{n-1}) \\
& \quad \quad - (z_0/z_1) B_{n-k-1}(0; z_1, z_2, \dots, \widehat{z_{j_1}}, \dots, \widehat{z_{j_k}}, \dots, z_{n-1})] \\
&= B_{n-k}(0; z_0, z_1, z_2, \dots, \widehat{z_{j_1}}, \dots, \widehat{z_{j_k}}, \dots, z_{n-1}) \\
& \quad - \lambda_2 B_{n-k-1}(0; z_0, z_1, \widehat{z_2}, \dots, \widehat{z_{j_1}}, \dots, \widehat{z_{j_k}}, \dots, z_{n-1}).
\end{aligned}$$

Hence,

$$\begin{aligned}
& C_n(\lambda_1; z_0, \dots, z_{n-1}; \lambda_1, \dots, \lambda_n) \\
&= \left(\frac{z_0 - \lambda_1}{z_0} \right) \sum_{\substack{3 \leq j_1 < \dots < j_k \leq n-1 \\ 0 \leq k \leq n-3}} (-1)^k \lambda_{j_1} \dots \lambda_{j_k} \\
&\quad \cdot B_{n-k}(0; z_0, z_1, \dots, \widehat{z_{j_1}}, \dots, \widehat{z_{j_k}}, \dots, z_{n-1}) \\
&\quad + \left(\frac{z_0 - \lambda_1}{z_0} \right) \sum_{\substack{3 \leq j_1 < \dots < j_k \leq n-1 \\ 0 \leq k \leq n-3}} (-1)^{k+1} \lambda_2 \lambda_{j_1} \dots \lambda_{j_k} \\
&\quad \cdot B_{n-k-1}(0; z_0, z_1, \widehat{z_2}, \dots, \widehat{z_{j_1}}, \dots, \widehat{z_{j_k}}, \dots, z_{n-1})
\end{aligned}$$

$$= \left(\frac{z_0 - \lambda_1}{z_0} \right) \sum_{\substack{2 \leq j_1 < \dots < j_k \leq n-1 \\ 0 \leq k \leq n-2}} (-1)^k \lambda_{j_1} \dots \lambda_{j_k} \\ \cdot B_{n-k}(0; z_0, z_1, \dots, \widehat{z_{j_1}}, \dots, \widehat{z_{j_k}}, \dots, z_{n-1}),$$

and this completes the proof.

We conclude this section now with a lemma to be used in § 3 below.

Lemma 3. *Let $n \geq 1$ and suppose that the complex numbers $z_0, z_1, \dots, z_{n-1}$ lie in the disc $|z| \leq W$. Then*

$$(2.15) \quad |B_n(0; z_0, \dots, z_{n-1})| \leq 1.$$

Moreover, there exist complex numbers $\zeta_0, \zeta_1, \dots, \zeta_{n-1}$ on the circle $|z| = W$ such that

$$(2.16) \quad |B_n(0; \zeta_0, \zeta_1, \dots, \zeta_{n-1})| = W^n H_n.$$

Proof. Let $w_k = z_k/W$, $0 \leq k \leq n-1$. Then $|w_k| \leq 1$, $0 \leq k \leq n-1$, and (2.7), (2.9) and (1.6) imply $|B_n(0; z_0, \dots, z_{n-1})| = W^n |B_n(0; w_0, \dots, w_{n-1})| \leq W^n H_n \leq 1$. For (2.16), the maximum principle permits us to choose points $w'_0, w'_1, \dots, w'_{n-1}$ on the circle $|z| = 1$ such that $|B_n(0; w'_0, \dots, w'_{n-1})| = H_n$. If $\zeta_k = W w'_k$, $0 \leq k \leq n-1$, then $|B_n(0; \zeta_0, \dots, \zeta_{n-1})| = |W^n B_n(0; w'_0, \dots, w'_{n-1})| = W^n H_n$, and this completes the proof.

3. Matrix transformations on the space $\mathfrak{A}_r$. If $r > 0$, we denote by $\mathfrak{A}_r$ the complex vector space of functions analytic in the disc $D_r = \{z: |z| < r\}$; $\mathfrak{A}_r$, given the topology of uniform convergence on compact subsets of D_r , is known to be a Fréchet-Montel space [8].

Let $\lambda \in c_0$ have terms in D_r and define $\sigma_0(z) = 1$, $\sigma_n(z) = (z - \lambda_1)(z - \lambda_2) \dots (z - \lambda_n)$, $n = 1, 2, 3, \dots$. Then $\{\sigma_n\}_{n=0}^\infty$ is a basis for $\mathfrak{A}_r$ [10]; in fact, $f = \sum_{n=0}^\infty a_n \sigma_n$ is the Newton series expansion for $f \in \mathfrak{A}_r$. It is not difficult to show that $f = \sum_{n=0}^\infty a_n \sigma_n$ belongs to $\mathfrak{A}_r$ if and only if $\limsup_{n \rightarrow \infty} |a_n|^{1/n} \leq r^{-1}$. Thus we may identify $\mathfrak{A}_r$ with the space of complex sequences $x = (x_0, x_1, \dots)$ such that $\limsup_{n \rightarrow \infty} |x_n|^{1/n} \leq 1/r$.

Let M be an infinite complex matrix and let $f = (f_0, f_1, \dots) \in \mathfrak{A}_r$. Then Mf is the sequence whose n th term is given by $(Mf)_n = \sum_{k=0}^\infty M_{nk} f_k$. We say that M maps $\mathfrak{A}_r$ to $\mathfrak{A}_r$ provided that $f \in \mathfrak{A}_r$ implies $Mf \in \mathfrak{A}_r$. The following theorem of M. M. Dragilev ([5], [6]) characterizes a large class of matrix transformations on $\mathfrak{A}_r$.

Theorem 3. *Let M be an infinite upper triangular matrix such that $M_{kk} = 1$, $0 \leq k < \infty$, let N denote the unique upper triangular inverse of M , and suppose*

$R > 0$. A necessary and sufficient condition that either M or N map $\mathfrak{A}_r$ one-to-one and onto $\mathfrak{A}_r$ for each $r > R$ is that

$$(3.1) \quad \limsup_{n \rightarrow \infty} \left[\max_{0 \leq j \leq n} |M_{jn}| R^{j-n} \right]^{1/n} \leq 1$$

and

$$(3.2) \quad \limsup_{n \rightarrow \infty} \left[\max_{0 \leq j \leq n} |N_{jn}| R^{j-n} \right]^{1/n} \leq 1.$$

Furthermore, it is not difficult to show that either M and N simultaneously map $\mathfrak{A}_r$ one-to-one onto itself or neither does.

Lemma 4. Suppose $R > 0$ and let $\{\lambda_k\}_{k=1}^\infty$ and $\{z_k\}_{k=0}^\infty$ be sequences in D_R such that $\lambda_k \rightarrow 0$ and $|z_k| \leq WR$, $0 \leq k < \infty$. Then the matrices A and B of (2.10) satisfy (3.1) and (3.2).

Proof. Consider the matrix B . If $r_n = |\lambda_n|$, $1 \leq n < \infty$, and $j < k$, then

$$\begin{aligned} |B_{jk}| &= |(z_j - \lambda_{j+1})(z_j - \lambda_{j+2}) \cdots (z_j - \lambda_k)| \\ &\leq (WR + r_{j+1})(WR + r_{j+2}) \cdots (WR + r_k) \\ &= (WR)^{k-j} [(1 + r_{j+1}/WR) \cdots (1 + r_k/WR)] \\ &\leq (WR)^{k-j} (1 + \epsilon_1)(1 + \epsilon_2) \cdots (1 + \epsilon_k), \end{aligned}$$

where $\epsilon_n = r_n/(WR)$. Now $\epsilon_n \rightarrow 0$ and therefore

$$\lim_{n \rightarrow \infty} [(1 + \epsilon_1)(1 + \epsilon_2) \cdots (1 + \epsilon_n)]^{1/n} = 1.$$

Since $W < 1$, we have $|B_{jk}| R^{j-k} \leq W^{k-j} (1 + \epsilon_1)(1 + \epsilon_2) \cdots (1 + \epsilon_k) < (1 + \epsilon_1)(1 + \epsilon_2) \cdots (1 + \epsilon_k)$ and it follows that

$$\limsup_{k \rightarrow \infty} \left[\max_{0 \leq j \leq k} |B_{jk}| R^{j-k} \right]^{1/k} \leq 1.$$

For the matrix A , (2.7) implies

$$|A_{jk}| R^{j-k} = |C_{k-j}(\lambda_{j+1}/R; z_j/R, \dots, z_{k-1}/R; \lambda_{j+1}/R, \dots, \lambda_k/R)|,$$

for $j < k$. By the maximum principle, there exist points $w_j, w_{j+1}, \dots, w_k$ on the circle $|z| = W$ such that $|A_{jk}| R^{j-k} \leq |C_{k-j}(\mu_{j+1}; w_j, \dots, w_{k-1}; \mu_{j+1}, \dots, \mu_k)|$, where $\mu_i = \lambda_i/R$, $1 \leq i < \infty$. By (2.12) and Lemma 3,

$$|A_{jk}|R^{j-k} \leq \left| \frac{w_j - \mu_{j+1}}{w_j} \right| \sum_{\substack{j+2 \leq p_1 < \dots < p_n \leq k-1 \\ 0 \leq n \leq k-j-2}} |\mu_{p_1}| |\mu_{p_2}| \dots |\mu_{p_n}| \\ \leq \frac{W+1}{W} (1 + |\mu_1|)(1 + |\mu_2|) \dots (1 + |\mu_{k-1}|).$$

Since $|\mu_n| \rightarrow 0$, it follows that $\limsup_{k \rightarrow \infty} [\max_{0 \leq j \leq k} |A_{jk}|R^{j-k}]^{1/k} \leq 1$, and this completes the proof.

Theorem 4. Let f be analytic in D , let $0 < R < 1$, and suppose that $\{z_k\}_{k=0}^\infty$ and $\{\lambda_k\}_{k=1}^\infty$ are sequences in D such that $|z_k| \leq WR$, $0 \leq k < \infty$, and $\lambda_k \rightarrow 0$. Then

$$(3.3) \quad f(z) = \sum_{k=0}^{\infty} \Delta^k f(z_k) C_k(z; z_0, \dots, z_{k-1}; \lambda_1, \dots, \lambda_k),$$

with uniform convergence on compact subsets of D .

Proof. By Theorem 3 and Lemma 4, the matrices A and B map $\bar{U}_r$ one-to-one and onto $\bar{U}_r$ for each $r > R$, where $0 < R < 1$. Thus A and B map $\bar{U}_1$ one-to-one onto itself. By a theorem of Köthe and Toeplitz [9], A and B are weakly continuous and, hence, [11, p. 158] are continuous. It follows that $\{A(\sigma_n)\}_{n=0}^\infty$ is a basis for $\bar{U}_1$, since $\{\sigma_n\}_{n=0}^\infty$ is a basis. But (2.8) implies that $A(\sigma_n)(z) = C_n(z; z_0, \dots, z_{n-1}; \lambda_1, \dots, \lambda_n)$ ($0 \leq n < \infty$), and therefore functions $f \in \bar{U}_1$ admit expansions $f(z) = \sum_{k=0}^\infty \alpha_k C_k(z; z_0, \dots, z_{k-1}; \lambda_1, \dots, \lambda_k)$ with uniform convergence on compact subsets of D . The linear functionals $g \mapsto \Delta^j g(z_j)$, $0 \leq j < \infty$, are readily shown to be continuous. This fact, together with (2.6) implies $\alpha_k = \Delta^k f(z_k)$, $0 \leq k < \infty$, which completes the proof.

Since $R < 1$ is arbitrary in Theorem 4, (3.3) implies that $C_\lambda \geq W$ for all $\lambda \in c_0$, which proves the first part of Theorem 1. To complete the proof of Theorem 1, we will prove that for each $\lambda \in l_1$, there exists an extremal function $F \in \bar{U}_1$ such that each of $\Delta_\lambda^k F(z)$ has a zero in $|z| \leq W$ but $F \neq 0$. With the expansion (3.3), this will imply $C_\lambda \leq W$ for all $\lambda \in l_1$.

Thus suppose $\lambda \in l_1$ and let $r_k = |\lambda_k|$, $1 \leq k < \infty$. Then the sequence $s_n = (1 + r_1)(1 + r_2) \dots (1 + r_n)$ converges [7, p. 223] and so there exists a constant $K > 1$ such that $(1 + r_1)(1 + r_2) \dots (1 + r_n) < K$ for all n .

Let N be a positive integer such that $r_n < W/2$, for $n \geq N$, and such that $\sum_{n=N}^\infty r_n < \beta/(\beta + 2)$ (see (1.7)). If $n \geq N$, there exist, by Lemma 3, points $z_0^{(n)}$, $z_1^{(n)}, \dots, z_{n-1}^{(n)}$ on $|z| = W$ such that

$$(3.4) \quad |B_{n-N}(0; z_N^{(n)}, z_{N+1}^{(n)}, \dots, z_{n-1}^{(n)})| = W^{n-H} H_{n-N}.$$

For $n \geq N$, define $F_n(z) = C_n(z; z_0^{(n)}, \dots, z_{n-1}^{(n)}; \lambda_1, \dots, \lambda_n)$. By (2.8), we have

$$(3.5) \quad F_n(z) = \sum_{k=0}^n C_{n-k}(\lambda_{k+1}; z_k^{(n)}, \dots, z_{n-1}^{(n)}; \lambda_{k+1}, \dots, \lambda_n) \sigma_k(z).$$

As in the proof of Lemma 4,

$$|C_{n-k}(\lambda_{k+1}; z_k^{(n)}, \dots, z_{n-1}^{(n)}; \lambda_{k+1}, \dots, \lambda_n)| \\ \leq ((W+1)/W)(1+r_1)(1+r_2) \cdots (1+r_{n-1}) \leq ((W+1)/W)K,$$

for $0 \leq k \leq n$. Therefore for $n \geq N$ and $z \in D$, $|F_n(z)| \leq ((W+1)/W)K \sum_{k=0}^n |\sigma_k(z)|$. Noting that $\{\sigma_k\}$ is a basis, the sequence $\{F_n\}_{n=N}^\infty$ is uniformly bounded on compact subsets of D . Therefore, there exists a set S of positive integers such that the sequence $\{F_n\}_{n \in S}$ converges uniformly on compact subsets of D to a function $F \in \mathcal{U}_1$. For each nonnegative integer k , the sequence $\{\Delta^k F_n\}_{n \in S}$ converges uniformly on compact subsets to $\Delta^k F$, by continuity of the map $f \mapsto \Delta^k f$. Since $\Delta^k F_n(z_k^{(n)}) = 0$, $n \in S$, uniform convergence implies that $\Delta^k F$ has a zero on the circle $|z| = W$.

To complete the construction, there remains to show $F \neq 0$. Since $F(z) = F(\lambda_1) + \sum_{k=1}^\infty \Delta^k F(\lambda_{k+1}) \sigma_k(z)$, it suffices to show that $\Delta^N F(\lambda_{N+1}) \neq 0$. By (3.5),

$$\Delta^N F(\lambda_{N+1}) = \lim_{n \in S} C_{n-N}(\lambda_{N+1}; z_N^{(n)}, \dots, z_{n-1}^{(n)}; \lambda_{N+1}, \dots, \lambda_n).$$

For each $n \geq N+2$, (3.4), (1.7) and (2.12) imply that

$$|C_{n-N}(\lambda_{N+1}; z_N^{(n)}, \dots, z_{n-1}^{(n)}; \lambda_{N+1}, \dots, \lambda_n)| \\ \geq \left| \frac{z_N^{(n)} - \lambda_{N+1}}{z_N^{(n)}} \right| \left\{ \beta - \sum_{\substack{N+2 \leq p_1 < \dots < p_m \leq n-1 \\ 1 \leq m \leq n-N-2}} r_{p_1} r_{p_2} \cdots r_{p_m} \right\}.$$

Writing $\tau_n = r_{N+2} + r_{N+3} + \cdots + r_{n-1}$, the previous inequality implies

$$|C_{n-N}(\lambda_{N+1}; z_N^{(n)}, \dots, z_{n-1}^{(n)}; \lambda_{N+1}, \dots, \lambda_n)| \\ \geq \frac{W-W/2}{W} \{\beta - \tau_n - \tau_n^2 - \cdots - \tau_n^{n-N-2}\} \\ \geq \frac{1}{2} \left\{ \beta - \sum_{j=1}^\infty \left(\sum_{n=N}^\infty r_n \right)^j \right\} \geq \frac{1}{2} \left\{ \beta - \sum_{j=1}^\infty \left(\frac{\beta}{2+\beta} \right)^j \right\} = \frac{\beta}{4}$$

for each $n \in S$ with $n \geq N+2$, and it follows that $|\Delta^N F(\lambda_{N+1})| \geq \beta/4 > 0$.

Remark. Fix $\rho > 0$ and define $C_{\lambda, \rho}$ to be the supremum of numbers $c > 0$

such that if $f \in \mathfrak{U}_\rho$ and each of $\Delta_\lambda^k f(z)$, $0 \leq k < \infty$, has a zero in $|z| \leq c$, then $f \equiv 0$. By taking $0 < R < \rho$ in Theorem 4, we see that $C_{\lambda, \rho} \geq \rho W$ for all $\lambda \in c_0$. If $\lambda \in l_1$, let $\mu = \rho^{-1}\lambda = (\rho^{-1}\lambda_1, \rho^{-1}\lambda_2, \dots)$ and construct the extremal function $F \in \mathfrak{U}_1$ with respect to the sequence μ . Define $G(z) = F(z/\rho)$. Then $G \in \mathfrak{U}_\rho$, $G \not\equiv 0$, and each of $\Delta_\lambda^k G(z)$ has a zero on $|z| = \rho W$. It follows that $C_{\lambda, \rho} = \rho W$ for $\lambda \in l_1$.

Added in proof. The authors wish to thank Professor J. L. Franks for pointing out this problem.

REFERENCES

1. J. D. Buckholtz, *Zeros of partial sums of power series*. II, Michigan Math. J. 17 (1970), 5–14. MR 41 #3718.
2. J. D. Buckholtz and J. L. Frank, *Whittaker constants*, Proc. London Math. Soc. 23 (1971), 348–370.
3. J. D. Buckholtz and J. K. Shaw, *Zeros of partial sums and remainders of power series*, Trans. Amer. Math. Soc. 166 (1972), 269–284.
4. P. J. Davis, *Interpolation and approximation*, Blaisdell, New York, 1963. MR 28 #393.
5. M. M. Dragilev, *On the convergence of the Abel-Gončarov interpolation series*, Uspehi Mat. Nauk 15 (1960), no. 3 (93), 151–155. (Russian) MR 22 #4839.
6. J. L. Frank and J. K. Shaw, *Abel-Gončarov polynomial expansions*, J. Approximation Theory (to appear).
7. E. Hille, *Analytic function theory*. Vol. 1, Introduction to Higher Math., Ginn, Boston, Mass., 1959. MR 21 #6415.
8. G. Köthe, *Topological vector spaces*. I, Die Grundlehren der math. Wissenschaften, Band 159, Springer-Verlag, New York, 1959. MR 40 #1750.
9. A. L. Perissini, *Concerning the order structure of Köthe sequence spaces*. II, Michigan Math. J. 11 (1964), 357–364.
10. M. Pommiez, *Sur les différences divisées successives et les restes des séries de Newton généralisées*, Ann. Fac. Sci. Univ. Toulouse (4) 28 (1964), 101–110. MR 33 #7559b.
11. H. H. Schaefer, *Topological vector spaces*, Macmillan, New York, 1966. MR 33 #1689.

DEPARTMENT OF MATHEMATICS, VIRGINIA POLYTECHNIC INSTITUTE AND STATE UNIVERSITY, BLACKSBURG, VIRGINIA 24061