OVERRINGS OF COMMUTATIVE RINGS. III: NORMAL PAIRS

BY

EDWARD D. DAVIS

ABSTRACT. A pair of integral domains (A, B) is a normal (resp., QR-) pair provided that A is a subring of B and all intermediate rings are normal in B (resp., rings of quotients of A). The special case of B the field of fractions of A (e.g., Prüfer domains and Dedekind domains with torsion class group) has been studied in detail. It is shown that any domain A possesses a unique overring B maximal with respect to forming a normal (resp., QR-) pair with A. An explicit description of this overring and all the intermediate rings in terms of localizations A is obtained, and further details are provided in the presence of a noetherian-like condition on A. In addition, the "overring" characterizations of Prüfer domains are extended to "intermediate ring" characterizations of normal pairs.

1. **Introduction, notation and preliminaries.** Recall that a Prüfer domain—in the noetherian case, Dedekind domain—is an integral domain for which all localizations are valuation rings. A number of papers have studied these rings in terms of the overrings contained in the field of fractions. The result in this direction which motivates the present paper is that Prüfer domains are characterized by the normality of all such overrings [1]. Gilmer, in his survey address on Prüfer-like conditions on the overrings of a domain [5], introduced the concept of an *integrally closed pair*: a pair of domains (A, B) with A a subring of B and all intermediate rings normal (e.g., a Prüfer domain and its field of fractions); and Kaplansky, in the discussion following Gilmer's address, conjectured a possible characterization of such pairs. We subsequently established the noetherian case of this conjecture [2], and in that note promised a more general treatment in a future paper. This is the promised paper.

Although our results have extensions to the context of rings with divisors of zero, we limit our attention here to integral domains. Given a domain A, $\mathcal{F}(A)$ will denote its field of fractions. By a *pair* of domains (A, B) we understand that B is an A-subalgebra of $\mathcal{F}(A)$, and by an *intermediate ring* of such a pair we mean an A-subalgebra of B. The assumption that $\mathcal{F}(A) \supset B$ is superfluous for most of our purposes (see the remark following the proof of Proposition 1 below), its inclusion in the notion of "pair" is merely for the convenience of exposition. The

pair (A, B) will be said to be *normal* provided that each intermediate ring is normal in B. (In [2] we used the more cumbersome "relatively integrally closed pair". Furthermore, we observed there that, but for the minor exception of A a field and B an algebraic extension, integrally closed pairs are special cases of normal pairs.) Observe that for S a multiplicative system in A, every intermediate ring of (A_S, B_S) is of the form C_S, for C an intermediate ring of (A, B). Hence (A_S, B_S) is normal if (A, B) is. Conversely, (A, B) is normal if, for every maximal ideal M of A, (A_M, BA_M) is normal. Moreover, to check the normality of (A, B) it suffices to check the normality of (A, C) for each finitely generated A-subalgebra C of B. We shall use these points freely and without explicit comment. Call a domain A *trivial* if there is no overring B, other than A itself, such that (A, B) is normal. Call a prime ideal P of a domain A *trivial* if A_P is trivial, *normal* if (A, A_P) is normal. For $\mathcal{P}$ a set of prime ideals of the domain A, let $A_{\mathcal{P}}$, denote the intersection of the localizations of A at the primes of $\mathcal{P}$.

In §2 we see that (A, B) is normal if, and only if, B is an intermediate ring of the pair $(A, A_{\mathcal{P}})$, where $\mathcal{P}$ is the set of trivial primes of A. In that section we also show how to translate the "overring" characterizations of Prüfer domains due to Richman [12], Storrer [13] and Davis [3] into "intermediate ring" characterizations of normal pairs. In §3 we examine the condition that each intermediate ring of a pair be a ring of quotients of the subring. In §4 we develop further the ideas of §2 and §3 for the case of pairs with the first member noetherian.

2. **Characterizations of normal pairs.** Our first proposition—and main lemma of this paper—says that "locally" the relationship between the subring and overring of a normal pair is the same as that obtaining between a valuation ring and its field of fractions (cf. Theorem 1 of [1]).

Proposition 1. *The pair (A, B) is normal if for each $x \in B$, x or $1/x \in A$. Conversely, if (A, B) is normal and A local, then for each $x \in B$, x or $1/x \in A$.*

Proof. Observe that if x is integrally dependent on a ring containing $1/x$, then x lies in that ring too. This remark suffices to prove the first assertion of the proposition. As for the second, let $x \in B - A$. Since x is integrally dependent on $A[x^2]$, we have

(i) $$x = a + bx^2 + \cdots + cx^{2n}, \qquad a, b, \cdots, c \in A.$$

Dividing (i) by x shows that $a/x \in B$; multiplying by a^{2n-1}/x^{2n} yields an equation of integral dependence of a/x on A. So $a/x \in A$, and it must be a nonunit because $x \notin A$. Now divide (i) by x^2 and rearrange the terms to obtain

(ii) $$(1 - a/x)/x = b + \cdots + cx^{2n-2}.$$

Since $1 - a/x$ is a unit of A, this equation shows that $1/x \in B$. Now divide (ii) by

x^{2n-2} to obtain an equation of integral dependence of $1/x$ on A, and thereby to conclude $1/x \in A$.

Observe that the proof of the second assertion did not employ the fact that $B \subset \mathcal{F}(A)$; indeed it is a consequence of the condition that every ring between A and B is normal in B.

> **Proposition 2.** *Assume* (A, B) *normal and* A *local. Then*
> (1) *For* P *a prime ideal of* B, $B_P = A_{P \cap A}$.
> (2) *For* P *a prime ideal of* B, $(A/P \cap A, B/P)$ *is a normal pair.*
> (3) *For* I *a proper ideal of* B, $I \cap A = I$.
> (4) $B = A_M$ *for* M *a certain prime ideal of* A.
> (5) *For* P *a prime ideal of* A *contained in* M, $PB = P$.
> (6) *Each ideal of* A *contains or is contained in* M.

Proof. Proposition 1 implies that B is a ring of quotients of A. (1) is an immediate consequence of this fact. (2) follows from the "x or $1/x$" criterion of Proposition 1. (3) is an immediate consequence of Proposition 1. To prove (4) it is enough, in view of (1), to show that B is local; and this fact follows from (3) which implies that the sum of proper ideals of B is contained in the maximal ideal of A. (5) is a consequence of (4) and (3). As for (6), note that for $y \in M$ and $x \in A - M$, y/x is by (4) a nonunit of B. By Proposition 1, $y/x \in A$. This establishes (6).

> **Theorem 1.** *Assume* A *local. Then* (A, B) *is normal if, and only if,* $B = A_M$, *where* A/M *is a valuation ring and* $MB = M$.

Proof. The necessity of the two conditions is given by (2), (4) and (5) of the preceding proposition. As for the sufficiency, we need only show, in view of Proposition 1, that for each unit x of B, x or $1/x \in A$. Since A/M is a valuation ring, of the field B/MB, x or $1/x \in A + MB = A$.

Remark. Theorem 1 says that all "local" normal pairs are obtained by "composing" a local domain with a valuation ring. Specifically, given a local domain B, let A be the preimage with respect to the canonical homomorphism from B to its residue field of a valuation ring of the residue field. Then (A, B) is normal by Theorem 1. So if the residue field of B is not algebraic over a finite field, B occurs as the second member of infinitely many local normal pairs. If the valuation ring is taken minimal among the set of valuation rings of the residue field of B, then the resulting normal pair is maximal as far as the first member is concerned— i.e., there is no local subring of A forming a normal pair with B. We next show that a local normal pair can be maximized with respect to the second member, and in a unique way. This last fact is valid even in the nonlocal case (Theorem 2 below).

Proposition 3. *Assume A local. Then*

(1) *If (A, R) and (R, B) are normal, then so is (A, B).*

(2) *A contains a unique prime which is both trivial and normal.*

Proof. The R of (1) is local by Proposition 2. Hence for $x \in B$, x or $1/x \in R$ by Proposition 1; then x or $1/x \in A$ (again Proposition 1). So (A, B) is normal by Proposition 1. Now let B be the compositum of all rings R such that (A, R) is normal. Each such R is of the form A_P for a normal prime P by (4) of Proposition 2. The linear ordering of the normal primes—a consequence of (6) of Proposition 2—implies that every finite set of such R's is contained in one of them. If follows that (A, C) is normal for every finitely generated A-subalgebra C of B, and so that (A, B) is normal. Now $B = A_P$ for some normal prime P, which must be trivial because of (1). That this P is uniquely determined by "trivial" and "normal" is clear from the linear ordering of the normal primes.

Remark. Localization shows that (1) is valid without the local assumption on A. Exercise: Can one prove either version of (1) directly from the definition of "normal pair"?

Proposition 4. *Suppose (A, B) normal. Then*

(1) *For P a prime ideal of B, $A_{P \cap A} = B_P$ and $(A/P \cap A, B/P)$ is normal.*

(2) *For P a prime ideal of A with $PB \neq B$, PB is prime and $P = PB \cap A$. Every prime of B is the extension of its contraction to A.*

(3) *The trivial primes of B are precisely the extensions of the trivial primes of A.*

(4) *$B = A_{\mathscr{P}}$, where $\mathscr{P}$ is the set of contractions of the prime (respectively, maximal) ideals of B.*

(5) *$B = A_{\mathscr{P}}$, where $\mathscr{P}$ is the set of prime ideals of A which extend to proper (respectively, maximal) ideals of B.*

Proof. (1) follows from Proposition 2 by localization. As for (2), first note that $BA_P = A_P$ by Proposition 2. Let $Q = PA_P \cap B$. Then one readily checks via Proposition 2 and (1) that $QB_M = PB_M$ for every maximal ideal M of B. So $Q = PB$, and the rest of (2) then follows. That every trivial prime of B is an extension of a trivial prime of A is a consequence of (1) and (2). Conversely, for P a trivial prime of A, $A_P = BA_P$, whence $PB \neq B$, and PB is then a trivial prime by (1) and (2). (4) and (5) follow from (1), (2) and (3) and the fact that $B = B_{\mathfrak{M}}$, for $\mathfrak{M}$ the set of prime (respectively, maximal) ideals of B.

Theorem 2. *Let $B = A_{\mathscr{P}}$, $\mathscr{P}$ the set of trivial primes of A. Then*

(1) *(A, B) is normal.*

(2) *(A, C) is normal $\Leftrightarrow A \subset C \subset B \Leftrightarrow C = A_{\mathfrak{M}}$, $\mathfrak{M} \supset \mathscr{P}$.*

(3) B *is trivial.*

(4) B *is uniquely determined by* (1) *and* (3).

Proof. For M a maximal ideal of A, there is a trivial prime $P \subset M$ with (A_M, A_P) normal (Proposition 3). Since BA_M is an intermediate ring of (A_M, A_P), (A_M, BA_M) is normal. Consequently (A, B) is normal. Suppose (A, C) is normal. Then by (3) and (5) of Proposition 4, $C = A_{\mathfrak{N}}$, $\mathfrak{M} \supset \mathfrak{P}$. This remark suffices to prove (2). B is trivial by (1), the remark following the proof of Proposition 3 and (2). (4) is an immediate consequence of (2).

Remarks. An immediate corollary of (2) is that for C the compositum of a collection or rings R with (A, R) normal, (A, C) is normal. Exercise: Can one see this directly from the definition of "normal pair"? Observe that we have not given a criterion for determining whether or not A is trivial–i.e., whether or not $A = B$. Clearly if every maximal ideal of A is trivial, then so is A. But it is by no means clear that a trivial domain cannot possess nontrivial maximal ideals. In the noetherian case (see §4 below) everything is clear: a domain is trivial only if each of its maximal ideals is.

An immediate corollary to Proposition 1 is that every intermediate ring of a normal pair (A, B) is A-flat. This fact is well known in the case of $B = \mathfrak{F}(A)$–i.e. for A a Prüfer domain–and the property is indeed characteristic of this class of rings (Richman [12]). This result is a consequence of either of the following characterizations of Prüfer domains: A is Prüfer $\Leftrightarrow$ for each $x \in \mathfrak{F}(A)$, $A[x] \otimes A[x]$ has no A-torsion (Storrer [13]) $\Leftrightarrow$ for each $0 \neq x \in \mathfrak{F}(A)$, $A[x] \otimes A[1/x]$ has no A-torsion [3]. All three extend to characterizations of normal pairs; both Richman's and Storrer's proofs apply in the more general setting, but the other requires a bit of extra work. The generalizations are as follows:

Theorem 3. *The following are equivalent for the pair* (A, B).

(1) (A, B) *is normal.*

(2) *Each intermediate ring is* A-*flat.*

(3) *For each* $x \in B$, $A[x] \otimes A[x]$ *has no* A-*torsion.*

(4) *For each* $0 \neq x \in B$, $A[x] \otimes A[1/x]$ *has no* A-*torsion.*

Proof. (1) $\Rightarrow$ (2) has already been noted. That (2) $\Rightarrow$ (3) and (4) is clear. (3) $\Rightarrow$ (1) is Lemma 3 of [13]. It remains to prove (4) $\Rightarrow$ (1). As in [3], (4) implies:

(5) *For* $x \in B$ *and* $y = 1/x$, $XY - 1$ *lies in the ideal of the polynomial ring* $A[X, Y]$ *generated by the kernels of the* A-*algebra homomorphisms* $A[X] \rightarrow A[x]$ *and* $A[Y] \rightarrow A[y]$.

To prove (5) $\Rightarrow$ (1) it suffices to take A local. In that case, as in [3], (5) implies that, for each $x \in B$, x or $1/x$ is integrally dependent on A. Proposition 1 then says that (N, B) is normal, where N is the normalization of A in B.

If we knew N to be necessarily local, we could then argue exactly as in [3]—specifically (i), (ii), and (iii) on pp. 236–237—to conclude that $A = N$.

We prove that for A local (5) implies that N is local. Let x be a nonunit of N, and let $XY - 1 = f + g$ with f and g in the extensions of the kernels of $A[Y] \rightarrow A[y]$ and $A[X] \rightarrow A[x]$ respectively. That y cannot be integrally dependent on A implies that the constant term of g cannot be a unit, and so that the constant term of f is a unit. Let $f = a + bX + \cdots$, with $a, b, \cdots \in A[Y]$. Since $xY - 1 = a + bx + \cdots$, we see that every coefficient of a other than the constant term must lie in $A \cap xN \subset M$, the maximal ideal of A. Observe now that because $f(X, y) = 0$, $a(y) = 0$. Multiplying this last equation by a suitably high power of x shows that that power of x lies in MN. So the nonunits of N form, an ideal, namely the radical of MN.

3. **Normal pairs and rings of quotients.** A number of papers have studied domains with the QR-*property*: domains A with the property that each A-subalgebra of $\mathcal{F}(A)$ is a ring of quotients of A. Such rings are necessarily Prüfer domains, and in the noetherian case, exactly those Dedekind domains with torsion class group [1], [6], [7]. More generally, Pendleton proves [11]: A Prüfer domain has the QR-property if, and only if, the radical of each of its finitely generated ideals is the radical of a principal ideal. Both results have their formulations in the context of normal pairs. We say that a pair has the QR-*property* or is a QR-*pair* if each of its intermediate rings is a ring of quotients of the subring of the pair. Such a pair is necessarily normal—a fact contained implicitly in the proof of Proposition 1.

Proposition 5. *Let* (A, B) *be a normal pair and* I *a finitely generated* A-*submodule of* B. *Then* $IB = B$ *if, and only if,* I *is invertible and* $I^{-1} \subset B$.

Proof. That $IB = B$ is implied by the other two conditions is clear. Conversely, suppose $IB = B$. To prove I is invertible it is enough to take A local and to prove I is principal. Let $x \in I$ be a unit of B. Then for $y \in I$, $y/x \in B$, and so by Proposition 1, y/x or $x/y \in A$. By induction then, I is singly generated. To show that $I^{-1} \subset B: B = I^{-1}IB = I^{-1}B \supset I^{-1}$.

Proposition 5 indicates how to reformulate Pendleton's theorem in the context of normal pairs. This having been done, for a proof one need only repeat the argument given in [11]. We remark, however, that in the course of the proof one must translate yet another characterization of Prüfer domains into a characterization of normal pairs: (A, B) is normal $\Leftrightarrow$ for each $x \in B$, the denominator and numerator ideals of x in A are comaximal. Modulo the definitions (see [11]), this fact is an immediate consequence of Proposition 1.

Theorem 4 (Pendleton's criterion). *The normal pair* (A, B) *has the QR-prop-*

erty if, and only if, the radical of each finitely generated ideal of A which extends to the unit ideal of B is the radical of a principal ideal.

Given a pair (A, B), the A-submodules of B form a monoid with respect to multiplication, and the group of invertible elements of this monoid is the *group of fractional ideals of the pair*. The quotient of this group by the subgroup consisting of those principal fractional ideals generated by units of B is the *class group of the pair* or Pic (A, B). Observe that for $B = \mathcal{F}(A)$, this reduces to the usual definition of Pic (A), and that Pic (A, B) is a subgroup of Pic (A).

Corollary. *If* Pic (A, B) *is a torsion group, then the pair* (A, B) *has the QR-property.*

Proof. Suppose I is a finitely generated ideal of A with $IB = B$. Then, according to Proposition 5, I belongs to the group of fractional ideals of the pairs. Some power of I is then principal, and since the radical of I is the radical of this principal ideal, we conclude from Pendleton's criterion that (A, B) is a QR-pair.

A partial converse to this result—essentially the noetherian case—will be recorded in §4. The converse is not in general true, even for $B = \mathcal{F}(A)$ (Heinzer [9]). The next theorem shows that for any domain A, there is a unique A-subalgebra of $\mathcal{F}(A)$ maximal with respect to forming a QR-pair with A. Unfortunately we do not have a criterion for determining in general whether or not this overring actually differs from A. (Cf. the analogous question in §2 concerning the unique maximal overring B such that (A, B) is normal.) In §4 we do give an explicit criterion for the noetherian case.

Theorem 5. *Given a domain A, let B be the compositum of all rings R such that (A, R) is a QR-pair. Then (A, B) is a QR-pair.*

Proof. That (A, B) is normal is a consequence of Theorem 2. To prove that (A, B) has the QR-property it suffices to prove: if both (A, A_x) and (A, A_y) have the QR-property, then so does (A, A_{xy}). We shall make several applications of Pendleton's criterion. Let I be a finitely generated ideal of A with $IA_{xy} = A_{xy}$. Since $(I + xA)A_x = A_x$, there is, by Pendleton's criterion, $u \in I + xA$ such that rad $(uA) = $ rad $(I + xA)$. Likewise there is $v \in I + yA$ such that rad $(vA) = $ rad $(I + yA)$. Since $IA_{xy} = A_{xy}$, $xy \in $ rad (I), whence $uv \in $ rad (I). Now rad $(uvA) = $ rad $(uA) \cap $ rad $(vA) \supset I$, so rad $(I) = $ rad (uvA). Thus (A, A_{xy}) has the QR-property by Pendleton's criterion.

The following provides further examples of QR-pairs.

Theorem 6. *Assume that $\mathcal{F}(A)$ is of transcendence degree 0 or 1 over its prime field according to whether the characteristic is 0 or not. Then the following are equivalent for the pair (A, B).*

(1) A is normal in B.

(2) (A, B) is normal.

(3) (A, B) has the QR-property.

Proof. It suffices to prove $(1) \Rightarrow (3)$. Define a subring R of A as follows. In characteristic 0, R is the normalization of $\mathbf{Z}$ in A; in nonzero characteristic, R is the normalization in A of $F[t]$, where F is the prime field and $t \in A$ is transcendental over F. It is clear that $\mathcal{F}(A) = \mathcal{F}(R)$ and that R is normal in every ring between R and B. It follows that each such ring is a ring of quotients of R [4].

4. The noetherian case. Somewhat more generally we will deal with a class of rings we designate with Krull's name—this because his principal ideal theorem is valid in them. A domain is *krullian* provided that the radical of each of its principal ideals is the intersection of a finite number of height 1 primes.

Examples. Any domain with normalization a Krull domain (in particular, any noetherian domain) and any 1-dimensional domain with the property that each nonzero element lies in but finitely many maximal ideals. Notice that in certain krullian domains the principal ideals have primary decompositions in the usual sense: noetherian domains and krullian domains A such that $A = A_{\mathcal{P}}$, for $\mathcal{P}$ the set of height 1 primes of A (e.g., Krull domains and 1-dimensional krullian domains).

Proposition 6. *A local krullian domain is nontrivial if, and only if, it is a 1-dimensional valuation ring.*

Proof. Suppose P is a nonmaximal normal prime ideal of the local krullian domain A, and let $x \notin P$ be a nonunit. Then $xA \supset P$ by Proposition 2. So $P = 0$, for it must be contained properly in a height 1 prime. Thus $(A, \mathcal{F}(A))$ is normal—i.e., A is a valuation ring—and since every prime of a valuation ring is normal, the only prime of A apart from 0 is the maximal ideal.

Proposition 6 gives us the "noetherian" version of Theorem 2:

Theorem 7. *Assume A krullian. Then (A, B) is normal if, and only if, $B = A_{\mathfrak{M}}$, where $\mathfrak{M}$ is a set of maximal ideals excluding at most such for which the corresponding localization is a valuation ring. The excluded set may be taken to be those maximal M for which $MB = B$; it is unique if the primary decomposition theorem holds for the principal ideals of A.*

Corollary (Kaplansky's conjecture). *Assume A noetherian. Then (A, B) is normal if, and only if, $B = A_{\mathfrak{M}}$, for $\mathfrak{M}$ a set of maximal ideals excluding at most invertible such. The excluded set is unique, consisting of those M for which $MB = B$.*

Proof. By Proposition 6, every nonzero, nonmaximal prime of A is contained

in a trivial maximal ideal, and the localizations at the nontrivial maximal ideals are valuation rings. That being the case, we may substitute "maximal" for "prime" in Theorem 2 and thereby prove the first assertion of Theorem 7. As for the uniqueness, it suffices to show that $A \neq A_{\mathfrak{M}}$, where $\mathfrak{M}$ consists of all the maximal ideals of A with the exception of a single height 1 maximal ideal M. Let $0 \neq x \in M$ and select $y \notin M$ lying in all the primary components of xA other than that associated to M. Then $y/x \in A_P$ for every prime $P \neq M$, and $y/x \notin A_M$. The corollary is an immediate consequence of the theorem, for in the noetherian case the nontrivial maximal ideals are locally principal (i.e., invertible).

A special case of normal pairs has been studied by Grell [8] and subsequently reworked by Krull [10]. The main lemma of that circle of ideas is: If a 1-dimensional, noetherian local domain A is normal in an A-subalgebra B of $\mathcal{F}(A)$, then $B = A$ or $\mathcal{F}(A)$. The proof requires "noetherian" only to the extent of using the fact that the normalization of A is a semilocal Prüfer domain (see the proof of Satz 1 of [10]). Having noted that, we may reformulate Grell's Fundamental Struktursatz as follows: Assume that R is a 1-dimensional domain with normalization a krullian Prüfer domain, and let B be an R-subalgebra of $\mathcal{F}(R)$. Then for A the normalization of R in B, the pair (A, B) is normal.

Our final two theorems give further information on QR-pairs in the event that the first member of the pair is krullian—and especially noetherian.

Theorem 8. *Assume A krullian and (A, B) normal. Then the following are equivalent.*

(1) (A, B) *is a QR-pair.*

(2) *Each maximal ideal of A which extends to the unit ideal of B is the radical of a principal ideal.*

(3) $B = A_{\mathfrak{M}}$, *where $\mathfrak{M}$ is a set of maximal ideals excluding at most such which are radicals of principal ideals and for which the corresponding localizations are valuation rings.*

(4) $B = A_S$, *where S is a multiplicative system in A generated by elements having for radicals maximal ideals with corresponding localizations valuation rings.*

Proof. $(1) \Rightarrow (2)$. Suppose that M is maximal with $MB = B$. Then there is $x \in M$ such that $1/x \in B$. Now every prime ideal containing x extends to the unit ideal of B, and so is maximal and of height 1 by Theorem 7. Therefore x lies in but finitely many prime ideals, and all of them are maximal. Select $y \in M$ lying in no other prime containing x. Then $\mathrm{rad}(xA + yA) = M$. So M is the radical of a principal ideal by Theorem 4.

$(2) \Rightarrow (3)$. By Theorem 7, $B = A_{\mathfrak{M}}$, where $\mathfrak{M}$ is the set of maximal ideals of A

with the exception of those M for which $MB = B$. The excluded set has the desired properties by Theorem 7 and (2).

(3) $\Rightarrow$ (4). For each $M \in \mathfrak{M}$, select $x \in M$ with $\mathrm{rad}\,(xA) = M$. Then $B = A_S$, where S is the multiplicative system generated by the chosen x's.

(4) $\Rightarrow$ (1). If $\mathrm{rad}\,(xA) = M$, a maximal ideal with A_M a valuation ring, then (A, A_x) is a QR-pair, for the pair has no intermediate rings other than A and A_x by Theorem 7. Now (4) says that B is the compositum of such A_x's, whence (A, B) is a QR-pair by Theorem 5.

Theorem 9. *Assume:* A *krullian*; (A, B) *normal; for each maximal ideal* M *of* A *with* $MB = B$, *the valuation ring* A_M *rational. Then* (A, B) *is a QR-pair if, and only if,* $\mathrm{Pic}(A, B)$ *is a torsion group.*

Corollary. *Assume* A *noetherian and* (A, B) *normal. Then* (A, B) *is a QR-pair if, and only if,* $\mathrm{Pic}\,(A, B)$ *is a torsion group.*

Proof. That (A, B) is a QR-pair if $\mathrm{Pic}\,(A, B)$ is a torsion group has been noted above (Corollary to Theorem 4). Conversely, assume the QR-property. One easily checks that each invertible fractional ideal of the pair is of the form I/x, where $x \in A$ is a unit in B and I is an invertible integral ideal of the pair, according to Proposition 5, a finitely generated ideal of A with $IB = B$. So to prove $\mathrm{Pic}\,(A, B)$ is torsion it suffices to show that some power of each such I is principal. Since $IB = B$, the only primes containing I extend to the unit ideal of B, and so by Theorem 7 and the fact that A is krullian, they are maximal, of height 1 and finite in number. It follows that I is the product of ideals primary for such maximal ideals. Now each of the primary components of I is invertible—it is a factor of an invertible ideal—and so finitely generated. Consequently we lose nothing by assuming that I itself is primary for a maximal ideal M with $MB = B$. M is the radical of a principal ideal xA by Theorem 8, whence since the valuation ring A_M is rational, $I^n A_M = x^m A_M$ for suitably chosen positive integers m and n. Because both I^n and $x^m A$ are M-primary, it follows that $I^n = x^m A$. The corollary is clearly a special case of the theorem.

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DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF NEW YORK AT ALBANY, ALBANY, NEW YORK 12222