A WEDDERBURN THEOREM FOR ALTERNATIVE ALGEBRAS WITH IDENTITY OVER COMMUTATIVE RINGS

BY

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ABSTRACT. In this paper, we study alternative algebras Λ over a commutative, associative ring R with identity. When Λ is finitely generated as an R -module, we define the radical J of Λ . We show that matrix units and split Cayley algebras can be lifted from Λ/J to Λ when R is a Hensel ring. We also prove the following Wedderburn theorem: *Let Λ be an alternative algebra over a complete local ring R of equal characteristic. Suppose Λ is finitely generated as an R -module, and Λ/J is separable over $\bar{R}$ ($\bar{R}$ the residue class field of R). Then there exists an $\bar{R}$ -subalgebra S of Λ such that $S + J = \Lambda$ and $S \cap J = 0$.*

Introduction. Let Λ denote an alternative algebra over a field R . Let J denote the radical of Λ . It is well known that the Wedderburn theorem holds for alternative algebras. That is, if Λ is finite dimensional over R , and Λ/J is separable over R , then there exists a separable subalgebra S of Λ such that $S \oplus J = \Lambda$. If Λ is associative, the author in [4] has generalized this result to the case where R is a split Hensel ring. The purpose of this paper is to obtain similar results for alternative algebras over complete local rings R of equal characteristic. The precise result is as follows: Let R be a complete local ring of equal characteristic. Let $\bar{R}$ denote the residue class field of R . Let Λ be an alternative algebra over R such that Λ is finitely generated as an R -module and, Λ/J (J the radical of Λ) is separable over $\bar{R}$. Then there exists an $\bar{R}$ -subalgebra S of Λ such that $S + J = \Lambda$ and $S \cap J = 0$.

In order to prove this result, we must carefully define what we mean by the radical J of Λ . Once this has been done, the results follow much as in the associative case.

Preliminaries. Throughout the rest of this paper, R will denote an associative, commutative ring with identity 1. Λ will always denote an alternative ring with identity. Thus, Λ is a not necessarily associative or commutative ring which satisfies the following two identities:

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$$(1) \quad x^2y = x(xy), \quad yx^2 = (yx)x \quad \text{for all } x, y \in \Lambda.$$

It is not absolutely necessary to assume Λ has an identity under multiplication. In the process of proving the main results of this paper, we could always first adjoin an identity to Λ . But for convenience we shall always assume Λ contains an identity.

By the center $C(\Lambda)$ of Λ , we shall, as usual, mean the collection of all elements c in Λ which both commute and associate with all elements of Λ . We are now ready for the following definition: Λ is an R -algebra if there exists a ring homomorphism $\theta: R \rightarrow C(\Lambda)$ such that $\theta(1)$ is the identity of Λ . From now on we shall suppress θ and simply write $r\lambda$ instead of $\theta(r)\lambda$, $r \in R$ and $\lambda \in \Lambda$. If Λ is an R -algebra, then Λ is naturally an R -module. In particular, an ideal $J \subset \Lambda$ is an R -submodule of Λ , i.e. $RJ \subset J$. We say Λ is free, finitely generated, flat etc. over R if Λ is free, finitely generated, flat etc. as an R -module. Throughout this paper, we assume that all ring homomorphisms of any rings in question which have identities take the identity to the identity. By an R -algebra homomorphism from Λ to another alternative algebra Λ' over R , we shall mean an algebra homomorphism which is also an R -module homomorphism.

Many of the results which hold for alternative algebras over fields pass over to similar results in our setting. In particular, the Moufang identities [9, p. 28], Artin's theorem [9, Theorem 3.1] and the Peirce decomposition [9, pp. 32-37] hold for any alternative algebra Λ over R . We shall use these results freely whenever needed.

Now let R be a local ring, i.e. R is a Noetherian ring with exactly one maximal ideal m . Let π_0 denote the natural projection of R onto $\bar{R} = R/m$. We shall say that R is split if there exists a ring homomorphism $\epsilon_0: \bar{R} \rightarrow R$ such that $\pi_0\epsilon_0$ is the identity map on $\bar{R}$. It follows from [6, Theorem 9] that any complete local ring of equal characteristic is split. If R is split, then via ϵ_0 we may identify $\bar{R}$ with a subring of R containing 1. If Λ is an alternative algebra over a split local ring R , and J is an ideal in Λ containing $m\Lambda$, then $0 \rightarrow J \rightarrow \Lambda \rightarrow \Lambda/J \rightarrow 0$ can naturally be viewed as a short exact sequence of $\bar{R}$ -algebras. We shall say that this sequence splits if there exists an $\bar{R}$ -algebra homomorphism $\epsilon: \Lambda/J \rightarrow \Lambda$ such that $\pi\epsilon$ is the identity map on Λ/J . Here π is of course the natural projection of Λ onto Λ/J .

Finally, we need the definition of a Hensel ring. Suppose R is a local ring (R need not be Noetherian here) with maximal ideal m . Let X be an indeterminate over R and consider the polynomial ring $R[X]$. We have a natural ring homomorphism $\sigma: R[X] \rightarrow \bar{R}[X]$ induced by π_0 . Namely if $f(X) = \sum r_i X^i \in R[X]$, then $\sigma(f) = \sum \pi_0(r_i) X^i \in \bar{R}[X]$. Let us write $\bar{f}$ for $\sigma(f)$. R is called a Hensel ring if

every monic polynomial $f(X) \in R[X]$ satisfies the following condition: If there exist two relatively prime polynomials $g_1(X)$ and $g_2(X) \in \bar{R}[X]$ such that $\bar{f} = g_1 g_2$ and g_1 is monic, then there exist two polynomials $b_1(X)$ and $b_2(X) \in R[X]$ such that $b_1 b_2 = f$, $\bar{b}_1 = g_1$, $\bar{b}_2 = g_2$ and b_1 is monic. It is well known that complete local rings are Hensel rings [8, Theorem 30.4].

The author assumes the reader is familiar with the theory of associative algebras over Hensel rings [3].

1. The radical of an alternative algebra over a commutative ring. Let Λ be an alternative algebra over R which is finitely generated. Let $\Omega(R)$ denote the collection of all maximal ideals of R . Then for each $m \in \Omega(R)$, $\Lambda/m\Lambda$ is an alternative algebra over the field R/m . Since Λ is finitely generated as an R -module, $\Lambda/m\Lambda$ is finite dimensional over R/m . Thus, the radical of $\Lambda/m\Lambda$ is well defined and can be taken to be the collection of all properly nilpotent elements in $\Lambda/m\Lambda$ [9, Theorem 3.7]. Let $J(m)$ be the ideal in Λ containing $m\Lambda$ and such that $J(m)/m\Lambda$ is the radical of $\Lambda/m\Lambda$. Thus, $J(m)$ is the full inverse image of the radical of $\Lambda/m\Lambda$ under the natural projection $\Lambda \rightarrow \Lambda/m\Lambda$. We now define the radical J of Λ as follows:

$$(2) \quad J = \bigcap_{m \in \Omega(R)} J(m).$$

We note that if R is a field, then our definition of the radical of Λ agrees with the classical definition for alternative algebras. If we assume Λ is an associative algebra, then J is just the Jacobson radical of Λ [3, Corollary, p. 125]. In any case, we note that we have defined the radical of Λ for only those alternative algebras which are finitely generated over R . We shall need the following facts about J .

Proposition 1. *Let Λ be an alternative algebra which is finitely generated over R . Let J be the radical of Λ . Then if $x \in J$, $1 - x$ is a unit in Λ .*

Proof. If Λ has a generating set of cardinality n over R , then for every $m \in \Omega(R)$ the dimension of $\Lambda/m\Lambda$ over R/m is less than or equal to n . Hence by [9, Theorem 3.7], $J(\Lambda/m\Lambda)^n = 0$. Thus, if $x \in J$, the image $\bar{x}$ of x in $\Lambda/m\Lambda$ is nilpotent, i.e. $\bar{x}^n = 0$ for any m . So,

$$(\bar{1} - \bar{x})(\bar{1} + \bar{x} + \dots + \bar{x}^{n-1}) = \bar{1} \quad \text{in } \Lambda/m\Lambda.$$

Thus,

$$(3) \quad 1 - (1 - x)(1 + x + \dots + x^{n-1}) \in \bigcap_{m \in \Omega(R)} m\Lambda.$$

So, there is a $z \in \bigcap m\Lambda$ such that $1 = (1 - x)(1 + x + \dots + x^{n-1}) + z$. Now if $\lambda \in \Lambda$, then $\lambda = [(1 - x)(1 + x + \dots + x^{n-1})]\lambda + z\lambda$. By Artin's theorem, $R[\lambda, x]$ is associative. Thus, $[(1 - x)(1 + x + \dots + x^{n-1})]\lambda = (1 - x)(\lambda + \dots + x^{n-1}\lambda)$. So as R -modules, we have $\Lambda = (1 - x)\Lambda + m\Lambda$ for every $m \in \Omega(R)$. It now follows

from Nakayama's lemma [3, Corollary, p. 124] that $\Lambda = (1 - x)\Lambda$. So there exists a $\lambda^1 \in \Lambda$ such that $(1 - x)\lambda^1 = 1$. In a similar manner, we show $(1 - x)$ has a left inverse. Therefore, $1 - x$ has a unique two sided inverse. $\square$

An element $x \in \Lambda$ will be called quasi-regular if there exists an element $y \in \Lambda$ such that $x + y - xy = x + y - yx = 0$. The element y is called the quasi-inverse of x and is easily seen to be unique. It follows from [10, Lemma 2] that x is a quasi-regular if and only if $1 - x$ is a unit in Λ . As in the associative theory, we say an ideal $I \subset \Lambda$ is quasi-regular if every element of I is quasi-regular. It follows from Proposition 1 that J is a quasi-regular ideal. We shall need the following proposition:

Proposition 2 (McCrimmon). *Let Λ be an alternative algebra, finitely generated over R . Let J be the radical of Λ and e an idempotent in Λ . Then $J \cap e\Lambda e = eJe$ is a quasi-regular ideal in the subalgebra $e\Lambda e$.*

Proof. It follows easily from the Moufang identities that $e\Lambda e$ is a subalgebra of Λ . Since J is an ideal in Λ , we have $eJe \subset J \cap e\Lambda e$. Let $x \in J \cap e\Lambda e$. Then $x = e\lambda e$ for some $\lambda \in \Lambda$. Now $ex = e(e\lambda e) = e\lambda e = x$. Similarly $xe = x$. Hence, $x = exe \in eJe$. So, $eJe = J \cap e\Lambda e$.

Suppose $x \in J \cap e\Lambda e$. Then $x \in J$. So, x is a quasi-regular element in Λ . Let y be the quasi-inverse of x and consider $eye \in e\Lambda e$. Using the Moufang identities, we have $x(eye) = ((xe)y)e = (xy)e = (x + y)e = xe + ye = x + ye$. If we now multiply by e , we get $e[x(eye)] = x + eye$. Now x and $eye \in e\Lambda e$. Therefore, $x(eye) \in e\Lambda e$, and $e[x(eye)] = x(eye)$. Thus, eye is a right quasi-inverse for x . A similar argument shows eye is a left quasi-inverse for x . Thus, x has a quasi-inverse eye in $e\Lambda e$. Since x was an arbitrary element of eJe , we have shown eJe is a quasi-regular ideal in $e\Lambda e$. $\square$

If Λ is a finitely generated, associative algebra over R , and x is a quasi-regular element of Λ , then the quasi-inverse of x is of the form $r_1x + r_2x^2 + \cdots + r_nx^n$ for $r_i \in R$. We shall need this same result for alternative algebras.

Proposition 3. *Let Λ be an alternative algebra, finitely generated over R . Let x be a quasi-regular element of Λ with quasi-inverse y . Then $y = r_1x + \cdots + r_nx^n$ for some n and $r_i \in R$.*

Proof. Consider the subalgebra of Λ generated by x and y , i.e. $R[x, y]$. Since y is the quasi-inverse of x , $yx = xy = x + y$. Thus, $R[x, y]$ is a commutative, associative ring. Since Λ is finitely generated over R , it follows from [3, Theorem 8] that both x and y satisfy monic polynomials with coefficients in R . Thus, $R[x, y]$ is finitely generated as an R -module. The result now follows from applying [3, Theorem 9] to $R[x, y]$. $\square$

Finally we need to know how the radical behaves under homomorphic images.

Proposition 4. *Let Λ_1 and Λ_2 be two alternative algebras over R which are finitely generated. Let J_1 and J_2 be the radicals of Λ_1 and Λ_2 respectively. Suppose $\sigma: \Lambda_1 \rightarrow \Lambda_2$ is an R -algebra epimorphism, then $\sigma(J_1) \subset J_2$.*

Proof. For each $m \in \Omega(R)$, let $J_i(m)$ denote the pull back of the radical of $\Lambda_i/m\Lambda_i$ under the natural projection $\Lambda_i \rightarrow \Lambda_i/m\Lambda_i$. The map σ induces an R/m -algebra homomorphism $\sigma_m: \Lambda_1/m\Lambda_1 \rightarrow \Lambda_2/m\Lambda_2$ which is onto. So, we have the following commutative square of epimorphisms:

$$\begin{array}{ccc} \Lambda_1 & \xrightarrow{\sigma} & \Lambda_2 \\ \downarrow & & \downarrow \\ \Lambda_1/m\Lambda_1 & \xrightarrow{\sigma_m} & \Lambda_2/m\Lambda_2 \end{array}$$

Since the radical of $\Lambda_1/m\Lambda_1$ consists of all properly nilpotent elements and σ_m is onto, σ_m takes the radical of $\Lambda_1/m\Lambda_1$ into the radical of $\Lambda_2/m\Lambda_2$. Since $J_1(m)$ is mapped onto the radical of $\Lambda_1/m\Lambda_1$, it follows that $\sigma(J_1(m)) \subset J_2(m)$. Since this holds for every $m \in \Omega(R)$, we have $\sigma(\bigcap_{m \in \Omega(R)} J_1(m)) \subset \bigcap_{m \in \Omega(R)} J_2(m)$. Thus $\sigma(J_1) \subset J_2$. $\square$

II. Alternative algebras over Hensel rings. In this section, we shall assume that R is a Hensel ring. This type of assumption is inevitable in trying to prove a Wedderburn type theorem because Hensel rings are the only commutative rings which permit idempotents and matrix units to be lifted from Λ/J to Λ .

Theorem 1. *Let Λ be a finitely generated alternative algebra over a Hensel ring R . Let I be an ideal in Λ . If $\bar{e}_1, \dots, \bar{e}_n$ are pairwise orthogonal idempotents in Λ/I , then there exist pairwise orthogonal idempotents $e_1, \dots, e_n$ in Λ such that $\pi(e_i) = \bar{e}_i$. Here π is the natural projection of Λ onto Λ/I .*

Proof. Suppose $\bar{e}$ is an idempotent in Λ/I . Let $c \in \Lambda$ such that $\pi(c) = \bar{e}$. Set $S = R[c]$. Then S is a commutative extension of R which by [3, Theorem 8] is finitely generated as an R -module. $I \cap S$ is an ideal in S , and $S/I \cap S$ contains $\bar{e}$. Thus by [3, Theorem 24], S contains an idempotent e such that $\pi(e) = \bar{e}$. Hence, any idempotent in Λ/I can be lifted to Λ .

Now suppose e is an idempotent in Λ , and $\bar{e}_1$ is an idempotent in Λ/I such that $\bar{e}\bar{e}_1 = \bar{e}_1\bar{e} = 0$ ($\bar{e} = \pi(e)$). Then there exists an idempotent $e_1 \in \Lambda$ such that $e_1e = ee_1 = 0$, and $\pi(e_1) = \bar{e}_1$. The proof of this is as follows: Let $T(e) = \{x \in \Lambda \mid xe = ex = 0\}$, the set of two sided annihilators of e in Λ . Suppose $x, y \in T(e)$. Then $(xy)e = (x, y, e) + x(ye) = (x, y, e) = -(x, e, y) = 0$. Here $(x, y, z) = (xy)z - x(yz)$ is the associator of three elements. Thus, $T(e)$ is a subalgebra of Λ and clearly consists of all elements of the form $\lambda - e\lambda - \lambda e + e\lambda e$, $\lambda \in \Lambda$. Hence

$\pi(T(e)) = \{\bar{x} \in \Lambda/I \mid \bar{x}\bar{e} = \bar{e}\bar{x} = 0\} = T(\bar{e})$. $\bar{e}_1 \in T(\bar{e})$ by hypothesis. Thus, there exists an element $c \in T(e)$ such that $\pi(c) = \bar{e}_1$. Now consider $R\langle c \rangle$ the subalgebra of Λ consisting of all polynomials in c without constant term. Then $R\langle c \rangle \subset T(e)$. It follows from [3, Theorem 21] that $R\langle c \rangle$ contains an idempotent e_1 such that $\pi(e_1) = \bar{e}_1$. Thus, Λ contains an idempotent e_1 with $\pi(e_1) = \bar{e}_1$ and $e_1e = 0 = ee_1$.

Now suppose $\bar{e}_1, \dots, \bar{e}_n$ are pairwise orthogonal idempotents in Λ/I . Lift $\bar{e}_1$ to an idempotent $e_1 \in \Lambda$. By the second paragraph of this proof, lift $\bar{e}_2$ to an idempotent e_2 in Λ orthogonal to e_1 . Suppose we have lifted $\bar{e}_1, \dots, \bar{e}_m$, $1 \leq m < n$, to pairwise orthogonal idempotents $e_1, \dots, e_m$ in Λ . We can lift $\bar{e}_{m+1}$ to an idempotent e_{m+1} in Λ which is orthogonal to $e = \sum_{i=1}^m e_i$. Then for $i = 1, \dots, m$, we have $e_i e_{m+1} = (ee_i e) e_{m+1} = e[e_i (ee_{m+1})] = 0$. Similarly, $e_{m+1} e_i = 0$. Thus, $e_1, \dots, e_{m+1}$ are pairwise orthogonal, and the proof is completed by induction. $\square$

For future reference, we note that a slight modification of the proof of Theorem 1 yields the following result: Let Λ be an alternative algebra over R which is not necessarily finitely generated as an R -module. Suppose I is an ideal in Λ for which $I^2 = 0$. Then if $\bar{e}_1, \dots, \bar{e}_n$ are pairwise orthogonal idempotents in Λ/I , there exist pairwise orthogonal idempotents $e_1, \dots, e_n \in \Lambda$ such that $\pi(e_i) = \bar{e}_i$. In this case, using the same S and $R\langle c \rangle$ as appears in the proof of Theorem 1, we note $I \cap S$ and $I \cap R\langle c \rangle$ are nilpotent ideals in S and $R\langle c \rangle$ respectively. Thus, we may use the results in [7, Proposition 3.4, p. 42] to lift idempotents from $S/I \cap S$ to S and from $R\langle c \rangle / I \cap R\langle c \rangle$ to $R\langle c \rangle$. Hence, Theorem 1 holds in general if $I^2 = 0$.

For the next proposition, we do not require R to be a Hensel ring.

Proposition 5. *Let Λ be an alternative algebra over R . Let I be a quasi-regular ideal in Λ . Let e and f be two pairwise orthogonal idempotents in Λ whose images in Λ/I we denote by $\bar{e}$ and $\bar{f}$. Suppose there exist elements $\bar{a} \in \bar{e}(\Lambda/I)\bar{f}$ and $\bar{b} \in \bar{f}(\Lambda/I)\bar{e}$ such that $\bar{a}\bar{b} = \bar{e}$ and $\bar{b}\bar{a} = \bar{f}$; then there exist elements $a \in e\Lambda f$ and $b \in f\Lambda e$ such that $ab = e$, $ba = f$, $\pi(a) = \bar{a}$, and $\pi(b) = \bar{b}$.*

Here π as usual denotes the natural projection of Λ onto Λ/I .

Proof. It is well known that $(x, e, f) = 0$ if e and f are pairwise orthogonal idempotents. Thus, $e\Lambda f$ and $f\Lambda e$ are unambiguous. Let $a_1 \in e\Lambda f$ and $b_1 \in f\Lambda e$ such that $\pi(a_1) = \bar{a}$, $\pi(b_1) = \bar{b}$. Then $a_1 = e\lambda_1 f$ and $b_1 = f\lambda_2 e$ for some $\lambda_1, \lambda_2 \in \Lambda$. Thus, $a_1 b_1 = (e\lambda_1 f)(f\lambda_2 e) = e \cdot (\lambda_1 f)(f\lambda_2) \cdot e \in e\Lambda e$. Also, $a_1 b_1 - e \in I$. Thus, $e - a_1 b_1 \in I \cap e\Lambda e$ which by the proof of Proposition 2 is a quasi-regular ideal in $e\Lambda e$. Since $e - a_1 b_1$ is quasi-regular in $e\Lambda e$, $e - (e - a_1 b_1) = a_1 b_1$ is a unit in $e\Lambda e$. Thus, there exists an $x \in e\Lambda e$ such that $(a_1 b_1)x = e = x(a_1 b_1)$. Similarly, there exists a $y \in f\Lambda f$ such that $y(b_1 a_1) = f = (b_1 a_1)y$.

Set $a = a_1$ and $b = b_1x$. Then $a \in e\Lambda f$. Since $x = e\lambda_3e$, for some $\lambda_3 \in \Lambda$, to show $b \in f\Lambda e$ we must show $(f\lambda_2e)(e\lambda_3e) \in f\Lambda e$. Now $f[(f\lambda_2e)(e\lambda_3e)] = -(f, f\lambda_2e, e\lambda_3e) + [f(f\lambda_2e)](e\lambda_3e)$. By the Moufang identities, $(f, f\lambda_2e, e\lambda_3e) = (f^2, f\lambda_2e, e\lambda_3e) = (f, f\lambda_2e, f(e\lambda_3e) + (e\lambda_3e)f) = 0$ since $f(e\lambda_3e) = (fe) \cdot \lambda_3 \cdot e = 0 = e \cdot \lambda_3 \cdot (ef) = (e\lambda_3e)f$. Also $f(f\lambda_2e) = f\lambda_2e$ by Artin's theorem. Hence, $f[(f\lambda_2e)(e\lambda_3e)] = (f\lambda_2e)(e\lambda_3e)$. Using the same techniques, we can show $[(f\lambda_2e)(e\lambda_3e)]e = (f\lambda_2e)(e\lambda_3e)$. Hence $b_1x = (f\lambda_2e)(e\lambda_3e) \in f\Lambda e$.

Now $ab = a_1(b_1x) = -(a_1, b_1, x) + e$. But $-(a_1, b_1, x) = (e\lambda_1f, e\lambda_3e, f\lambda_2e) = (e\lambda_1f)(e\lambda_3e) \cdot (f\lambda_2e) - (e_1\lambda_1f) \cdot (e\lambda_3e)(f\lambda_2e) = [e \cdot (\lambda_1f)(e\lambda_3e) \cdot e] \cdot (f\lambda_2e) - (e\lambda_1f) \cdot [e \cdot (\lambda_3e)(f\lambda_2e) \cdot e] = e \cdot [(\lambda_1f)(e\lambda_3e) \cdot e(f\lambda_2e)] - [(e\lambda_1f)e \cdot (\lambda_3e)(f\lambda_2e)] \cdot e = 0$. Thus $ab = e$.

In order to show $ba = f$, we first need $yb_1 = b_1x$. Now the proof above shows $b_1a_1b_1 \in f\Lambda e$. From this fact, one can easily argue that $(y, b_1a_1b_1, x) = 0$ and $(y, b_1, a_1) = 0$. These are routine arguments like the one above and will be omitted. We then have $yb_1 = y(b_1e) = y \cdot [b_1 \cdot (a_1b_1)x]$. But we have shown that $(a_1, b_1, x) = 0$. So, $y(b_1e) = y \cdot [b_1 \cdot (a_1b_1)x] = y \cdot (b_1 \cdot a_1(b_1x)) = y \cdot (b_1a_1b_1)x = y(b_1a_1b_1) \cdot x = ((yb_1)a_1 \cdot b_1) \cdot x = (y(b_1a_1) \cdot b_1) \cdot x = (fb_1)x = b_1x$. Thus, $yb_1 = b_1x$. It now follows that $ba = (b_1x)a = (yb_1)a = y(b_1a) = f$.

To finish the proof, we note that $\pi(a) = \bar{a}$ and $\pi(b) = \bar{b}$. $\square$

We can now prove the main result of this section.

Theorem 2. Let Λ be an alternative algebra over a Hensel ring R . Assume Λ is finitely generated over R , and I is a quasi-regular ideal in Λ . Suppose $\{\bar{e}_{ij} | i, j = 1, \dots, n\}$ is a system of matrix units in Λ/I , then there exists a system of matrix units $\{e_{ij} | i, j = 1, \dots, n\}$ in Λ such that $\pi(e_{ij}) = \bar{e}_{ij}$.

Proof. Since $\bar{e}_{11}, \dots, \bar{e}_{nn}$ are pairwise orthogonal idempotents in Λ/I , there exist, by Theorem 1, pairwise orthogonal idempotents $e_1, \dots, e_n \in \Lambda$ such that $\pi(e_i) = \bar{e}_{ii}$. Now for each $i = 1, \dots, n$, we have $\bar{e}_{11}\bar{e}_{1i} = \bar{e}_{1i} = \bar{e}_{1i}\bar{e}_{ii}$, $\bar{e}_{ii}\bar{e}_{i1} = \bar{e}_{i1} = \bar{e}_{i1}\bar{e}_{11}$. Thus, $\bar{e}_{1i} \in \bar{e}_{11}(\Lambda/I)\bar{e}_{ii}$, $\bar{e}_{i1} \in \bar{e}_{ii}(\Lambda/I)\bar{e}_{11}$ and $\bar{e}_{1i}\bar{e}_{i1} = \bar{e}_{11}$, $\bar{e}_{i1}\bar{e}_{1i} = \bar{e}_{ii}$. Applying Proposition 5, we get elements $e_{i1} \in e_i\Lambda e_1$ and $e_{1i} \in e_1\Lambda e_i$ such that $e_{i1}e_{1i} = e_i$, $e_{1i}e_{i1} = e_1$, $\pi(e_{1i}) = \bar{e}_{1i}$ and $\pi(e_{i1}) = \bar{e}_{i1}$. Put $e_{11} = e_1$ and $e_{ij} = e_{i1}e_{1j}$ for $i \neq 1, j \neq 1$. We shall show that $\{e_{ij} | i, j = 1, \dots, n\}$ are the required matrix units of Λ .

We first note that $e_{ij} \in e_i\Lambda e_j$ because $e_{ij} = e_{i1}e_{1j} = (e_i\lambda_1e_1)(e_1\lambda_2e_j)$ for elements $\lambda_1, \lambda_2 \in \Lambda$ [9, 3.17]. A simple argument involving the Moufang identities shows $e_i[(e_i\lambda_1e_1)(e_1\lambda_2e_j)] = [(e_i\lambda_1e_1)(e_1\lambda_2e_j)]e_j = e_{ij}$. Thus, $e_{ij} \in e_i\Lambda e_j$. We need to show that $e_{ij}e_{jk} = e_{ik}$ for all $i, j, k = 1, \dots, n$.

Let us first show $e_{ij}e_{kl} = 0$ if $k \neq j$. A linearized form of the Moufang identity [9, 3.5] is as follows:

$$(4) \quad y \cdot (xa)z + y \cdot (za)x = (yx)a \cdot z + (yz)a \cdot x$$

for all x, y, z and $a \in \Lambda$. If we set $y = e_{ij}$, $x = e_k$, $a = e_{kl}$ and $z = e_l$ and use the fact that $e_{ij} \in e_i \Lambda e_j$, (4) becomes

$$(5) \quad e_{ij} e_{kl} + e_{ij} \cdot (e_l e_{kl}) e_k = (e_{ij} e_l) e_{kl} \cdot e_k.$$

If $k = l$, then $e_{ij} e_{kl} = e_{ij} e_k = 0$. Thus, without loss of generality we may assume $k \neq l$. In this case, (5) becomes

$$(6) \quad e_{ij} e_{kl} = [(e_{ij} e_l) e_{kl}] e_k.$$

If $l \neq j$, then (6) implies $e_{ij} e_{kl} = 0$. Thus, we may assume $l = j$. In this case, (6) becomes

$$(7) \quad e_{ij} e_{kj} = (e_{ij} e_{kj}) e_k.$$

If $k = i$, (7) implies $e_{ij} e_{ij} = (e_{ij} e_{ij}) e_i = e_{ij} (e_{ij} e_i) = 0$. If $i = j$, $e_{ij} e_{kj} = e_i e_{kj} = 0$. Thus, we have reduced the proof to showing that $e_{ij} e_{kj} = 0$ when i, j and k are all distinct.

Since i, j and k are all distinct, we get $e_i(e_{ij} e_{kj}) = (e_{ij}, e_i, e_{kj}) + e_{ij} e_{kj} = e_{ij} e_{kj}$. Thus, $e_{ij} e_{kj} \in e_i \Lambda$. Now $0 = (e_j e_{ij}) e_{kj} = (e_j, e_{ij}, e_{kj}) + e_j(e_{ij} e_{kj})$. Since $e_{ij} e_{kj} \in e_i \Lambda$, $e_j(e_{ij} e_{kj}) = 0$. Thus, $0 = (e_j, e_{ij}, e_{kj})$. So, $0 = (e_{ij}, e_j, e_{kj}) = (e_{ij} e_j) e_{kj} - e_{ij} (e_j e_{kj}) = e_{ij} e_{kj}$. Therefore, in all cases $e_{ij} e_{kl} = 0$ if $k \neq j$.

It remains to show that $e_{ij} e_{jk} = e_{ik}$. If $i = j$, we have $e_{ij} e_{jk} = e_i e_{ik} = e_{ik}$. If $k = j$, we have $e_{ij} e_{jk} = e_{ij} e_j = e_{ij} = e_{ik}$. Thus, we may assume $i \neq j$ and $j \neq k$.

Now $e_{ij} e_{jk} = e_{ij} (e_{j1} e_{1k}) = -(e_{ij}, e_{j1}, e_{1k}) + (e_{ij} e_{j1}) e_{1k}$. Using the previous result, one easily argues that $(e_{ij}, e_{j1}, e_{1k}) = 0$. So $e_{ij} e_{jk} = (e_{ij} e_{j1}) e_{1k}$.

Now $e_{ij} e_{j1} = (e_{i1} e_{1j}) e_{j1} = (e_{i1}, e_{1j}, e_{j1}) + e_{i1} (e_{1j} e_{j1})$. Again one may argue that $(e_{i1}, e_{1j}, e_{j1}) = 0$. We thus get $e_{ij} e_{j1} = e_{i1}$. Thus, $e_{ij} e_{jk} = (e_{ij} e_{j1}) e_{1k} = e_{i1} e_{1k} = e_{ik}$. Since $e_{ii} = e_i$ and $\pi(e_{ij}) = \bar{e}_{ij}$, the proof is complete. $\square$

The proof of Theorem 2 requires Theorem 1, Proposition 5 and well known identities for alternative rings. The proof of Proposition 5 does not require Λ to be finitely generated over R . Any ideal I in Λ whose square is zero is clearly a quasi-regular ideal. Thus, using the remark following Theorem 1, we could prove the following analogue of Theorem 2: Let Λ be an alternative algebra over R which is not necessarily finitely generated. Let I be an ideal in Λ such that $I^2 = 0$. Then if $\{\bar{e}_{ij} \mid i, j = 1, \dots, n\}$ is a system of matrix units in Λ/I , there exists a system of matrix units $\{e_{ij}\} \in \Lambda$ such that $\pi(e_{ij}) = \bar{e}_{ij}$. We shall have use of this result in §III of this paper.

Let R_2 denote the ring of all 2×2 matrices with coefficients in R . R_2 is naturally equipped with an involution $x \rightarrow \bar{x}$ satisfying $x + \bar{x} \in R$ and $x\bar{x} \in R$. Namely, if $x = \begin{pmatrix} a & b \\ -c & d \end{pmatrix}$, then $\bar{x} = \begin{pmatrix} d & -b \\ c & a \end{pmatrix}$. We shall say that an alternative algebra Λ over R is a split Cayley algebra if $\Lambda \cong R_2 \oplus vR_2$ where v is an element such that $v^2 = 1$, and $xv = v\bar{x}$ for all $x \in R_2$.

Suppose Λ is an alternative algebra over a local ring R . Let m be the unique maximal ideal of R . If Λ is finitely generated over R , then the radical J of Λ is well defined. Since $J \supset m\Lambda$, Λ/J is naturally an R/m -algebra.

Theorem 3. *Let Λ be an alternative algebra over a Hensel ring R . Suppose Λ is finitely generated over R , and Λ/J is a split Cayley algebra over R/m . Then there exists a split Cayley algebra Λ_0 over R contained in Λ such that $\pi(\Lambda_0) = \Lambda/J$.*

Proof. In this proof, we denote $\pi(x)$ by $[x]$ for $x \in \Lambda$. By hypothesis, $\Lambda/J = (R/m)_2 \oplus [\omega](R/m)_2$ is a split Cayley algebra over the field R/m . Now by Theorem 2, there exist matrix units $\{e_{ij} \mid i, j = 1, 2\}$ in Λ such that $(R/m)_2 = \sum_{i,j=1}^2 (R/m)[e_{ij}]$. Set $e = e_{11} + e_{22}$. Then $[e] = 1$ the identity element of Λ/J . Thus, $1 - e \in J$. By Proposition 1, e is a unit in Λ . So, there exists a $z \in \Lambda$ such that $ez = ze = 1$. Since $R[z, e]$ is an associative subalgebra of Λ , we have $0 = [(1 - e)e]z = (1 - e)(ez) = 1 - e$. Thus $e_{11} + e_{22} = 1$.

The proof from this point on is similar to [9, 3.21]. We shall borrow freely from this result. Find f_{ij} , $(i \neq j) \in \Lambda$ such that $[f_{ij}] = [\omega][e_{ij}]$. We may assume $f_{ij} \in e_{ii}\Lambda e_{jj}$. Also $e_{ji}f_{ij} = c_j \in J \cap \Lambda_{jj}$. Here $\Lambda_{11}, \Lambda_{12}, \Lambda_{21}$ and Λ_{22} are the Peirce spaces arising from the idempotents e_{11} and e_{22} in Λ . Set $b_{ij} = f_{ij} - e_{ij}c_j$ for $i \neq j$. Then $b_{ij} \in \Lambda_{ij}$, $[b_{ij}] = [f_{ij}]$ and $e_{ji}b_{ij} = b_{ij}e_{ji} = 0$ for $i \neq j$. We also have $[b_{ij}][b_{ji}] = [f_{ij}][f_{ji}] = [e_{ii}]$. Thus, $b_{ij}b_{ji} = e_{ii} - a_i$ for some element $a_i \in J \cap \Lambda_{ii}$. Now $\Lambda_{ii} = e_{ii}\Lambda e_{ii}$; and, by Proposition 2, $J \cap \Lambda_{ii}$ is a quasi-regular ideal in Λ_{ii} . Thus, there exists an element $a'_i \in \Lambda_{ii}$ such that $(e_{ii} - a_i)(e_{ii} - a'_i) = e_{ii} = (e_{ii} - a'_i)(e_{ii} - a_i)$. Furthermore, Proposition 3 implies that a'_i is a polynomial in a_i without constant term. Now set

$$(8) \quad p_{12} = (e_{11} - a'_1)b_{12} \quad \text{and} \quad p_{21} = b_{21}.$$

We note that $p_{12} \in \Lambda_{12}$, $p_{21} \in \Lambda_{21}$, $[p_{12}] = [b_{12}] = [f_{12}]$ and $[p_{21}] = [b_{21}] = [f_{21}]$. We wish to show that $p_{ij}p_{ji} = e_{ii}$ for $i \neq j$. Now $p_{12}p_{21} = [(e_{11} - a'_1)b_{12}]b_{21}$. These elements have the form $e_{11} - a'_1 = e_{11}\lambda_1 e_{11}$, $b_{12} = e_{11}\lambda_2 e_{22}$ and $b_{21} = e_{22}\lambda_3 e_{11}$ for some λ_1, λ_2 and $\lambda_3 \in \Lambda$. From this, one can easily argue that $(e_{11} - a'_1, b_{12}, b_{21}) = 0$. Thus, $p_{12}p_{21} = (e_{11} - a'_1)(b_{12}b_{21}) = (e_{11} - a'_1)(e_{11} - a_1) = e_{11}$. To show that $p_{21}p_{12} = e_{22}$, we need the following identities,

$$(9) \quad a_i b_{ij} = b_{ij} a'_j, \quad (i \neq j).$$

$$(10) \quad a'_i b_{ij} = b_{ij} a'_j,$$

For (9), we have $a_i b_{ij} = (e_{ii} - b_{ij}b_{ji})b_{ij} = b_{ij} - (b_{ij}b_{ji})b_{ij} = b_{ij} - b_{ij}(b_{ji}b_{ij}) = b_{ij} - b_{ij}(e_{jj} - a_j) = b_{ij} - b_{ij} + b_{ij}a_j = b_{ij}a_j$. Therefore, (9) is proven.

(10) follows from (9) and the fact that a'_i is a polynomial in a_i . We first note that $(a_i, b_{ij}, a_j) = 0$. Thus using (9), we have

$$(11) \quad (b_{ij}a_j)(e_{jj} - a_j) = (e_{ii} - a_i)(b_{ij}a_j).$$

Now $e_{ii} - a_i' \in Re_{ii}[a_i] \subset \Lambda_{ii}$. Thus, $e_{ii} - a_i'$ associates with $e_{ii} - a_i$ and $(b_{ij}a_j)$. Thus from (11), we get

$$(12) \quad (e_{ii} - a_i')[(b_{ij}a_j)(e_{jj} - a_j)] = b_{ij}a_j.$$

One can now directly argue that $(e_{ii} - a_i', b_{ij}a_j, e_{jj} - a_j) = 0$. Thus, multiplying both sides of (12) by $(e_{ii} - a_i')$, we get

$$(13) \quad (e_{ii} - a_i')(b_{ij}a_j) = (b_{ij}a_j)(e_{jj} - a_j').$$

Since $a_i + a_i' - a_i'a_i = 0$, we get $a_i' = (-a_i)(e_{ii} - a_i')$. Thus using (9), (13) and the fact that $(e_{ii} - a_i')$ is a polynomial in a_i with coefficients in Re_{ii} , we get

$$\begin{aligned} a_i'b_{ij} &= [(-a_i)(e_{ii} - a_i')]b_{ij} = (e_{ii} - a_i')(-a_i b_{ij}) = (e_{ii} - a_i')(-b_{ij}a_j) \\ &= (-b_{ij}a_j)(e_{jj} - a_j') = b_{ij}[(-a_j)(e_{jj} - a_j')] = b_{ij}a_j'. \end{aligned}$$

Therefore (10) is proven.

We can now prove $p_{21}p_{12} = e_{22}$. We first note that (9) and (10) imply $p_{12} = (e_{11} - a_1')b_{12} = b_{12} - a_1'b_{12} = b_{12} - b_{12}a_2' = b_{12}(e_{22} - a_2')$. Thus, $p_{21}p_{12} = b_{21}[b_{12}(e_{22} - a_2')] = (b_{21}b_{12})(e_{22} - a_2') = (e_{22} - a_2)(e_{22} - a_2') = e_{22}$.

We next note that $e_{ij}p_{ji} = p_{ji}e_{ij} = 0$ if $i \neq j$. For $e_{12}p_{21} = e_{12}b_{21} = 0$ and $e_{21}p_{12} = e_{21} \cdot (e_{11} - a_1')b_{12} = e_{21}b_{12} \cdot (e_{22} - a_2') = 0$. Now set $v = p_{12} + p_{21}$. We note that $p_{ij} \in \Lambda_{ij}$, and thus $p_{ij}^2 = 0$ by [9, 3.20]. So $v^2 = (p_{12} + p_{21})^2 = p_{12}p_{21} + p_{21}p_{12} = e_{11} + e_{22} = 1$. We also have that $[v] = [w]$. Set $R_2 = \sum_{i,j=1}^2 Re_{ij} \subset \Lambda$. Then, $v \notin R_2$. One can easily show $vR_2 \cap R_2 = 0$. Thus, we can consider $\Lambda_0 = R_2 \oplus vR_2 \subset \Lambda$. Clearly $\pi(\Lambda_0) = \Lambda/J$. So the theorem will be complete if we show that $xv = v\bar{x}$ for all $x \in R_2$.

Let $x \in R_2$. Then $x = ae_{11} + be_{12} + ce_{21} + de_{22}$ for constants a, b, c and $d \in R$. Then $\bar{x} = de_{11} - be_{12} - ce_{21} + ae_{22}$, $xv = ap_{12} + be_{12}p_{12} + ce_{21}p_{21} + dp_{21}$, and $v\bar{x} = ap_{12} - bp_{12}e_{12} - cp_{21}e_{21} + dp_{21}$. But if $x_{ij}, y_{ij} \in \Lambda_{ij}$ ($i \neq j$), then $x_{ij}y_{ij} = -y_{ij}x_{ij}$ by [9, 3.21]. Thus, $xv = v\bar{x}$. $\square$

In Theorem 3, if we replace J by an ideal I whose square is zero, then we could drop the hypothesis that Λ is finitely generated over R . We would then have the following result: Suppose Λ is an alternative algebra over R , and I is an ideal in Λ such that $I^2 = 0$. Suppose Λ/I is a split Cayley algebra over $R/I \cap R$. Then there exists a split Cayley algebra Λ_0 over R contained in Λ such that $\pi(\Lambda_0) = \Lambda/I$. A proof of this remark would follow from the remark after Theorem 2 and the proof of [9, Lemma 3.21]. $\square$

III. A Wedderburn theorem for alternative algebras over complete local rings of equal characteristic. Throughout this section, we assume R is a split local ring. Thus, $\bar{R} = R/m$ is imbedded in R via some ring monomorphism ϵ . We shall drop ϵ and consider $\bar{R}$ as contained in R .

Proposition 6. *Let Λ be an alternative algebra (not necessarily finitely generated) over a split local ring R . Suppose I is an ideal in Λ which contains $m\Lambda$ and has square zero. Then if Λ/I is central separable over $\bar{R}$, $0 \rightarrow I \rightarrow \Lambda \rightarrow \Lambda/I \rightarrow 0$ splits as $\bar{R}$ -algebras.*

Proof. It is clearly sufficient to show that there exists an $\bar{R}$ -subalgebra S of Λ such that $I \cap S = 0$ and $I + S = \Lambda$. Since Λ/I is central separable over the field $\bar{R}$, Λ/I is either a Cayley algebra of dimension eight over $\bar{R}$, or Λ/I is $n \times n$ matrices over an associative division algebra D . In the latter case, D has center $\bar{R}$ and is finite dimensional over $\bar{R}$. In either case, let K be a field which contains $\bar{R}$, is finite dimensional over $\bar{R}$ and splits Λ/I , i.e. $\Lambda/I \otimes_{\bar{R}} K$ is either a split Cayley algebra over K or $n' \times n'$ matrices $K_{n'}$ over K . Since Λ/I is finite dimensional over $\bar{R}$, such a field K exists. Write $K = \bar{R}(\epsilon_1, \dots, \epsilon_r)$ with $\epsilon_1 = 1$.

Now consider $\Lambda \otimes_{\bar{R}} K$. This is an alternative algebra over the field K , and $I \otimes_{\bar{R}} K$ is an ideal nilpotent of index two in $\Lambda \otimes_{\bar{R}} K$. Furthermore, $(\Lambda \otimes_{\bar{R}} K)/(I \otimes_{\bar{R}} K) \cong (\Lambda/I) \otimes_{\bar{R}} K$. Thus, the remarks made after Theorems 2 and 3 when applied to $\Lambda \otimes_{\bar{R}} K$ imply that there exists a K -subalgebra S' of $\Lambda \otimes_{\bar{R}} K$ such that $S' \oplus (I \otimes_{\bar{R}} K) = \Lambda \otimes_{\bar{R}} K$. The proof now proceeds exactly as in the associative case [1, pp. 47–48] to obtain an $\bar{R}$ -subalgebra S of Λ with $S \oplus I = \Lambda$. $\square$

Proposition 7. *Suppose Λ is an alternative algebra (not necessarily finitely generated) over a split local ring R . Suppose I is an ideal in Λ which contains $m\Lambda$ and has square zero. Then if Λ/I is separable over $\bar{R}$, $0 \rightarrow I \rightarrow \Lambda \rightarrow \Lambda/I \rightarrow 0$ splits as $\bar{R}$ -algebras.*

Proof. Λ/I separable over $\bar{R}$ means Λ/I is finite-dimensional over $\bar{R}$, and Λ/I decomposes into a finite direct sum of ideals $S_1 \oplus \dots \oplus S_n$. Each S_i is central simple over its center Z_i which in turn is a separable field over $\bar{R}$. Let the identities of the subalgebras S_i be $\bar{e}_1, \dots, \bar{e}_n$. Then we can lift these pairwise orthogonal idempotents to pairwise orthogonal idempotents $e_1, \dots, e_n$ in Λ . Since $\bar{e}_1 + \dots + \bar{e}_n = 1$, it follows that $e_1 + \dots + e_n = 1$ in Λ . We also know that $0 \rightarrow e_i I e_i \rightarrow e_i \Lambda e_i \rightarrow S_i \rightarrow 0$ is exact. Thus, for the purposes of proving the proposition, we may assume Λ/I is central simple over its center Z , and that Z is a finite separable field extension of $\bar{R}$.

Hence, we may assume that Λ/I is either $n \times n$ -matrices over an associative division algebra D having center Z , or that Λ/I is a Cayley algebra over Z . In either case, since Z is a finite separable extension of $\bar{R}$, there exists a field F such that F is a finite separable extension of $\bar{R}$, and $Z \otimes_{\bar{R}} F \cong \bigoplus_{i=1}^l \bar{e}_i F$ for pairwise orthogonal idempotents $\bar{e}_1, \dots, \bar{e}_l$ in $Z \otimes_{\bar{R}} F$. Consider the exact sequence of F -algebras: $0 \rightarrow I \otimes_{\bar{R}} F \rightarrow \Lambda \otimes_{\bar{R}} F \xrightarrow{R} \Lambda/I \otimes_{\bar{R}} F \rightarrow 0$. If Λ/I is as-

sociative, then the center of $\Lambda/I \otimes_{\bar{R}} F$ is $Z \otimes_{\bar{R}} F$ [2, Corollary 1.6]. If Λ/I is nonassociative, then Λ/I is a Cayley algebra over Z . Thus, $\Lambda/I = Q \oplus \nu Q$ for some quaternion algebra Q over Z . So $Q \otimes_{\bar{R}} F$ is an associative subalgebra of $\Lambda/I \otimes_{\bar{R}} F$. Hence the center $C(\Lambda/I \otimes_{\bar{R}} F)$ of $\Lambda/I \otimes_{\bar{R}} F$ is contained in the center $C(Q \otimes_{\bar{R}} F)$ of $Q \otimes_{\bar{R}} F$. Now Q is central separable over Z , and Z is separable over $\bar{R}$. Thus, Q is separable over $\bar{R}$. It again follows from [2, Corollary 1.6] that $C(Q \otimes_{\bar{R}} F) = Z \otimes_{\bar{R}} F$. Thus, in either case the center of $\Lambda/I \otimes_{\bar{R}} F$ is $Z \otimes_{\bar{R}} F = \bigoplus_{i=1}^l F \bar{e}_i$.

Now from the remark after Theorem 1, we may lift $\{\bar{e}_1, \dots, \bar{e}_l\}$ to pairwise orthogonal idempotents $e_1, \dots, e_l$ in $\Lambda \otimes_{\bar{R}} F$ whose sum is 1. For each $i = 1, \dots, l$, $0 \rightarrow e_i(I \otimes_{\bar{R}} F)e_i \rightarrow e_i(\Lambda \otimes_{\bar{R}} F)e_i \rightarrow (\Lambda/I \otimes_{\bar{R}} F)\bar{e}_i \rightarrow 0$ is an exact sequence of F -algebras. $e_i(I \otimes_{\bar{R}} F)e_i$ is a nilpotent ideal of index two, and the center of $(\Lambda/I \otimes_{\bar{R}} F)\bar{e}_i$ is isomorphic to F . Since $\Lambda/I \otimes_{\bar{R}} F$ is separable over F , $(\Lambda/I \otimes_{\bar{R}} F)\bar{e}_i$ is central separable over F . Thus, Proposition 6 implies there exists an F -subalgebra S_i of $e_i(\Lambda \otimes_{\bar{R}} F)e_i$ such that $S_i \oplus e_i(I \otimes_{\bar{R}} F)e_i = e_i(\Lambda \otimes_{\bar{R}} F)e_i$. Setting $S' = S_1 + \dots + S_l$, we get an F -subalgebra of $\Lambda \otimes_{\bar{R}} F$ such that $(I \otimes_{\bar{R}} F) \oplus S' = \Lambda \otimes_{\bar{R}} F$. Following the proof of Wedderburn's theorem in [1, pp. 47–48], we can then find an $\bar{R}$ -subalgebra S of Λ such that $S \oplus I = \Lambda$. This immediately implies $0 \rightarrow I \rightarrow \Lambda \rightarrow \Lambda/I \rightarrow 0$ splits. $\square$

We can now prove the main theorem.

Theorem 4. *Let Λ be an alternative algebra over a complete local ring R of equal characteristic. Suppose Λ is finitely generated as an R -module, and Λ/J is separable over $\bar{R} = R/m$. Then there exists an $\bar{R}$ -subalgebra S of Λ such that $S + J = \Lambda$ and $S \cap J = 0$.*

Proof. We had noted in the preliminaries that a complete local ring R of equal characteristic contains a copy of its residue class field $\bar{R}$. We choose a copy of $\bar{R}$ in R and regard Λ as an $\bar{R}$ -algebra via this copy.

For each $n \geq 1$, define $J^{2^n} = J^{2^{n-1}} J^{2^{n-1}}$. We first prove that Λ is a complete Hausdorff space in its J^{2^n} -adic topology. By definition, J is the complete inverse image of the radical of $\Lambda/m\Lambda$. Since Λ is finitely generated as an R -module, $\Lambda/m\Lambda$ is a finite-dimensional alternative algebra. Hence, the radical of $\Lambda/m\Lambda$ is nilpotent. Therefore, there exists an $n_0 > 0$ such that $J^{2^{n_0}} \subset m\Lambda$. This implies $\bigcap_{n=1}^{\infty} J^{2^n} \subset \bigcap_{n=1}^{\infty} m^n \Lambda$. But by [11, Theorem 9, p. 262], $\bigcap m^n \Lambda = 0$. Thus, Λ is a Hausdorff space in its J^{2^n} -adic topology. By [11, Theorem 5, p. 256], Λ is a complete Hausdorff space in its $m\Lambda$ -adic topology. Thus, Λ is complete in its J^{2^n} -adic topology.

Now for each $n \geq 1$, we have

$$(14) \quad 0 \rightarrow J/J^{2^n} \rightarrow \Lambda/J^{2^n} \rightarrow \Lambda/J \rightarrow 0$$

is an exact sequence of alternative algebras. Since Λ/J is separable over $\bar{R}$, one can easily argue, using Proposition 4, that J/J^{2^n} is the radical of Λ/J^{2^n} . If $n = 1$, Proposition 7 implies (14) splits as $\bar{R}$ -algebras. Hence, there exists an $\bar{R}$ -subalgebra S_2 of Λ/J^2 such that $S_2 \oplus J/J^2 = \Lambda/J^2$. In particular, S_2 is isomorphic to Λ/J . If $n = 2$, we have the following commutative diagram with exact rows:

$$(15) \quad \begin{array}{ccccccc} 0 & \rightarrow & J/J^2 & \rightarrow & \Lambda/J^2 & \xrightarrow{\pi_2} & \Lambda/J \rightarrow 0 \\ & & \uparrow x_2 & & \uparrow x_2 & & \parallel \\ 0 & \rightarrow & J/J^4 & \rightarrow & \Lambda/J^4 & \xrightarrow{\pi_4} & \Lambda/J \rightarrow 0 \end{array}$$

Here x_2 is the natural projection of Λ/J^4 onto Λ/J^2 . Since x_2 is an R -algebra homomorphism, x_2 is also an $\bar{R}$ -algebra homomorphism. Set $S'_4 = x_2^{-1}(S_2)$. Then S'_4 is an $\bar{R}$ -subalgebra of Λ/J^4 . We note that

$$(16) \quad 0 \rightarrow J^2/J^4 \rightarrow S'_4 \rightarrow S_2 \cong \Lambda/J \rightarrow 0$$

is an exact sequence of $\bar{R}$ -algebras. Thus, by Proposition 7, (16) splits. Hence, there exists an $\bar{R}$ -subalgebra S_4 of Λ/J^4 such that $S_4 \oplus J^2/J^4 = S'_4$. Since $x_2(S_4) = S_2$, we have $\pi_4(S_4) = \Lambda/J$. From this it follows that $S_4 \oplus J/J^4 = \Lambda/J^4$. Thus (14) splits when $n = 2$.

By induction, we can show that for each $n \geq 1$ there exists an $\bar{R}$ -subalgebra S_{2^n} of Λ/J^{2^n} such that $S_{2^n} \oplus J/J^{2^n} = \Lambda/J^{2^n}$ and $x_{2^{n+1}}(S_{2^{n+1}}) = S_{2^n}$. Since Λ is complete in its J^{2^n} -adic topology, the inverse limit $\varprojlim \Lambda/J^{2^n}$ is isomorphic to Λ . If we set $S = \varprojlim S_{2^n}$ (see [11, pp. 305–306]), then S is an $\bar{R}$ -subalgebra of Λ for which $S \oplus J = \Lambda$. $\square$

IV. A conjecture concerning Hensel rings. One may ask if there is a broader class of rings for which Theorem 4 holds. The completeness of R was used heavily in the proof of Theorem 4, but Theorems 1, 2 and 3 seem to suggest that Theorem 4 may hold if R is a split Hensel ring. The author conjectures that Theorem 4 holds if we assume R is a split Hensel ring instead of R being a complete local ring. In view of [5, Theorem], this would be the broadest class of local rings for which one could hope to prove Theorem 4.

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