INVERSE SEMIGROUPS WHICH ARE SEPARATED OVER A SUBSEMIGROUP

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ABSTRACT. An inverse semigroup T is separated over a subsemigroup S if T is generated, as an inverse semigroup, by S and for each $a, b \in S$ there exists $x \in Sa \cap Sb$ such that $a^{-1}ab^{-1}b = x^{-1}x$ and dually for right ideals. For example, if T is generated as an inverse semigroup by a semigroup S whose principal left and right ideals form chains under inclusion, then T is separated over S . In this paper we investigate the structure of inverse semigroups T which are separated over subsemigroups S .

The structure theory of inverse semigroups has been the object of much study over recent years with particular attention being paid to 0-bisimple and 0-simple inverse semigroups ([2], [9], [10], [11], [13], for example). These papers attempted to determine the structure of various 0-bisimple or 0-simple inverse semigroups directly in terms of groups and semilattices. However the degree of complication involved even in these cases leads one to suspect that this is, in general, a futile task although it is possible in some cases.

In a general sense, the structure of inverse semigroups is determined by its semilattice of idempotents and a semilattice of groups. This is a consequence of a theorem of Munn [11] which shows that the maximal fundamental homomorphic image S/μ of an inverse semigroup S is a full subsemigroup of the semigroup T_E of isomorphisms between the principal ideals of the semilattice E of idempotents of S . The canonical homomorphism $\mu: S \rightarrow S/\mu$ is idempotent separating so its kernel is a semilattice of groups. The problem of constructing idempotent separating extensions of semilattices of groups by inverse semigroups has been solved, theoretically at least, by D'Alarcao [4] and Coudron [3] so that one could, in principle, construct all inverse semigroups if one could construct all fundamental inverse semigroups; the latter, however, remain a mystery.

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In this paper, we shall adopt a more internal approach to describing inverse semigroups. Suppose that θ is a homomorphism of a semigroup S into an inverse semigroup T . Then we shall say that T is separated over S , by θ , if T is generated as an inverse semigroup by $S\theta$ and, for each $a, b \in S$,

$$a\theta(a\theta)^{-1}b\theta(b\theta)^{-1} = x\theta(x\theta)^{-1} \quad \text{for some } x \in aS \cap bS,$$

$$(a\theta)^{-1}a\theta(b\theta)^{-1}b\theta = (y\theta)^{-1}y\theta \quad \text{for some } y \in Sa \cap Sb.$$

The main aim of this paper is to investigate the structure of an inverse semigroup T , which is separated over a semigroup S , in terms of S . Special cases of this concept have been considered before. For example, let T be a bisimple monoid and let S be the right unit subsemigroup of T ; if S is right reflexive then T is separated over S . Clifford [1] has described the structure of T in terms of S . On the other hand, Eberhart and Selden [5] have described the structure of all one parameter inverse semigroups. Any such semigroup T is separated over a sub-semigroup S of the multiplicative semigroup of the positive reals.

Theorem 3.5 gives an explicit method of construction for all fundamental inverse semigroups which are separated over an arbitrary semigroup S . Thus, by using D'Alarcao's extension theorem [4] one could, in principle, construct all inverse semigroups which are separated over S . We have not been able to do this explicitly without imposing conditions on S . A semigroup S is *naturally quasisemilatticed* if the sets of principal left and right ideals of S form semilattices under inclusion; thus an inverse semigroup is *naturally quasisemilatticed*. If S is naturally semilatticed and T is separated over S by θ then, for $a, b \in S$,

$$a\theta(a\theta)^{-1}b\theta(b\theta)^{-1} = (a \wedge_r b)\theta[(a \wedge_r b)\theta]^{-1},$$

$$(a\theta)^{-1}a\theta(b\theta)^{-1}b\theta = [(a \wedge_l b)\theta]^{-1}(a \wedge_l b)\theta,$$

where, for example, $a \wedge_r b$ in S is such that $aS^1 \cap bS^1 = (a \wedge_r b)S^1$. There is thus a universal inverse semigroup $E(S)$ in the category of inverse semigroups which are separated over S . An explicit construction and several coordinatisations for $E(S)$ are given in §4 while the congruences and ideal structure form the subject matter of §5.

Whenever the sets of principal left and right ideals of a semigroup S are chains under inclusion, every inverse semigroup generated, as an inverse semigroup, by a homomorphic image of S is separated over S . Hence $E(S)$ is the free inverse semigroup on S and so S can be embedded in an inverse semigroup if and only if it can be embedded in $E(S)$. The last result remains true if S is naturally quasisemilatticed (Theorem 4.6) so that we can use $E(S)$ to obtain a set of necessary and sufficient conditions for the embeddability of such semigroups in inverse semigroups.

The main tools used in this paper are what we term *shift representations* of S by one-to-one partial transformations. These representations generalise both the Vagner-Preston representations of inverse semigroups and the regular representations of cancellative semigroups. They are described in §2.

The theory undergoes considerable simplification when the semigroup S under consideration is cancellative. It is applied in §6 to give necessary and sufficient conditions on a cancellative semigroup so that each element of $I(S)$ should be of the form $ab^{-1}c$ with $a, b, c \in S$; the precise conditions are that the sets of principal left and right ideals of S should be chains under inclusion. The theory is also applied to give a characterisation of the positive cone of a right ordered group.

The final section consists of several examples of inverse semigroups which arise from the general theory. In particular the theory gives a method for constructing 0-simple inverse semigroups in which $\mathcal{D} \neq \mathcal{J}$. The $\mathcal{D}$ -classes in these semigroups are traversed by a semigroup but no $\mathcal{D}$ -class is a subsemigroup so that the 0-simple inverse semigroups obtained here are, in a sense, dual to those considered by Munn [12].

1. **Embedding a semigroup in an inverse semigroup.** If S is any semigroup, it follows from general categorical considerations, or from [8], that there is an inverse semigroup $I(S)$ and a homomorphism $\eta: S \rightarrow I(S)$ with the following property: given any homomorphism θ of S into an inverse semigroup T , there is a unique homomorphism $\psi: I(S) \rightarrow T$ such that the diagram

$$\begin{array}{ccc} S & & T \\ \eta \downarrow & \searrow \theta & \\ I(S) & \xrightarrow{\psi} & \end{array}$$

commutes. The semigroup $I(S)$ is called the *free inverse semigroup* on S . One of the aims of this paper is to investigate the structure of $I(S)$ and some related semigroups when the ideal structure of S has certain special properties; in particular, when the sets of principal left and right ideals of S form chains under inclusion.

It follows easily from the functorial properties of S^1 , S^0 and $I(S)$ that $I(S^1)$ and $I(S)^1$ and $I(S^0)$ and $I(S)^0$ are naturally isomorphic. Hence, in studying the relationships between S and $I(S)$ we may, without loss of generality, assume that S has a zero and an identity. We shall assume the latter throughout this paper.

Because any homomorphism of S into an inverse semigroup can be uniquely factored through η , S can be embedded in an inverse semigroup if and only if η is one-to-one. We can use this to give a short proof of Schein's theorem [16] which gives necessary and sufficient conditions for embedding semigroups in inverse semigroups.

Let $S = S^1$ be a semigroup. Then a nonempty subset H of S is *strong* if

$ax, bx, ay \in H$ together imply $by \in H$. Clearly, if nonvoid, the intersection of strong subsets is strong.

Let $H \neq \square$ be a strong subset of $S = S^1$ and define

$$x \equiv y \quad (\mathcal{R}_H) \quad \text{if and only if } H \cdot x = H \cdot y$$

where, for example, $H \cdot x = \{u \in S : x u \in H\}$. Then $\mathcal{R}_H$ is a right congruence on S [2, § 10.2] and can be used to construct a representation of S by one-to-one partial transformations in the following way [2, § 11.4]. Set $W_H = \{x \in S : H \cdot x = \square\}$. W_H is clearly an $\mathcal{R}_H$ -class of S , and let $\mathcal{X}_H$ be the set of $\mathcal{R}_H$ -classes different from W_H . For each $a \in S$, define

$$\bar{x}\rho_a^H = \overline{xa} \quad \text{for each } \bar{x} \in \mathcal{X}_H \text{ such that } \overline{xa} \in \mathcal{X}_H.$$

Then the mapping $\rho^H: a \rightarrow \rho_a^H$ is a representation of S by one-to-one partial transformations of $\mathcal{X}_H$; thus ρ^H is a homomorphism of S into the symmetric inverse semigroup $\mathcal{I}(\mathcal{X}_H)$ on $\mathcal{X}_H$.

Recall that, if T is an inverse semigroup, the natural partial order on T is defined by

$$x \leq y \quad \text{if and only if } x = ey \text{ for some } e = e^2 \in T \text{ [2, § 7.1].}$$

Lemma 1.1. *Let θ be a homomorphism of a semigroup $S = S^1$ into an inverse semigroup T and let $a \in S$. Then $K = \{x \in S : a\theta \leq x\theta\}$ is a strong subset of S which contains a .*

Proof. Suppose $bx, by, cx \in K$. Then $a\theta \leq (bx)\theta$, $a\theta \leq (by)\theta$, $a\theta \leq (cx)\theta$ and so, also, $(a\theta)^{-1} \leq (bx)\theta^{-1}$. Thus

$$a\theta = a\theta(a\theta)^{-1}a\theta \leq (cx)\theta(bx)\theta^{-1}(by)\theta = c\theta(x\theta x\theta^{-1}b\theta^{-1}b\theta)y\theta \leq (cy)\theta.$$

Hence $cy \in K$. This shows that K is strong and, clearly, $a \in K$.

Lemma 1.2. *Let $S = S^1$ be a semigroup and let $a \in S$. Then $\hat{a} = \{x \in S : a\eta \leq x\eta\}$ is the smallest strong subset of S which contains a .*

Proof. By Lemma 1.1, $\hat{a}$ is a strong subset of S which contains a . On the other hand, suppose that H is a strong subset of S and $a \in H$. Let $\rho^H: S \rightarrow \mathcal{I}(\mathcal{X}_H)$ be the representation of S obtained from H and suppose that $x \in \hat{a}$. Since ρ^H can be factored through η , it follows that $a\rho^H \leq x\rho^H$ and so, in particular, the domain $\Delta\rho_a^H$ of ρ_a^H is contained in $\Delta\rho_x^H$. Now $\bar{a} = \overline{1a} \in \mathcal{X}_H$ so $\bar{1} \in \Delta\rho_a^H$; hence $\bar{1} \in \Delta\rho_x^H$. Further, since $\rho_a^H \leq \rho_x^H$,

$$\bar{a} = \bar{1}\rho_a^H = \bar{1}\rho_x^H = \bar{x}.$$

Hence $H \cdot x = H \cdot a$ and so, since $1 \in H \cdot a$, $x \in H$. This shows that $\hat{a} \subseteq H$.

Theorem 1.3 (Schein [16]). *Let $S = S^1$ be a semigroup. Then S can be embedded in an inverse semigroup if and only if for each pair of distinct elements of S there is a strong subset of S which contains one of the pair but not the other.*

Proof. Suppose that η is one-to-one and that $a \neq b$ in S . Then $a\eta \neq b\eta$ and so $a\eta \not\leq b\eta$ or $b\eta \not\leq a\eta$; thus $b \notin \hat{a}$ or $a \notin \hat{b}$.

Conversely, if H is strong and $a \in H$, $b \notin H$ then, since $\hat{a} \subseteq H$, $b \notin \hat{a}$ and so $a\eta \not\leq b\eta$; in particular, $a\eta \neq b\eta$.

The method of proof of Theorem 1.3 can be used to give the relationship between the ideal structure of S and that of $I(S)$.

Proposition 1.4. *Let $S = S^1$ be a semigroup and let $\eta: S \rightarrow I(S)$ be the canonical homomorphism of S into the free inverse semigroup on S . Then $a\eta(a\eta)^{-1} \leq b\eta(b\eta)^{-1}$ if and only if $\hat{a} \cap bS \neq \emptyset$.*

Proof. Suppose $\hat{a} \cap bS \neq \emptyset$. Then $bx \in \hat{a}$ for some $x \in S$ and so $a\eta \leq (bx)\eta$. Hence $a\eta = b\eta(b\eta)^{-1}a\eta$; that is $a\eta(a\eta)^{-1} \leq b\eta(b\eta)^{-1}$.

Conversely, suppose that $a\eta(a\eta)^{-1} \leq b\eta(b\eta)^{-1}$ and let ρ be the representation of S by one-to-one partial transformations obtained from the strong subset $\hat{a}$. Then, since ρ can be factored through η , $a\rho(a\rho)^{-1} \leq b\rho(b\rho)^{-1}$; that is $\Delta a\rho \subseteq \Delta b\rho$. Since $\bar{1} \in \Delta a\rho$, this implies $\bar{1} \in \Delta b\rho$, so that $\bar{b} \in \mathcal{X}_{\hat{a}}$; that is $bS \cap \hat{a} \neq \emptyset$.

Corollary 1.5. *The mapping α defined by $(aS)\alpha = (a\eta)I(S)$ is an order isomorphism of the set of principal right ideals of S into the set of principal right ideals of $I(S)$ if and only if $\hat{a} \cap bS \neq \emptyset$ implies $a \in bS$.*

If T is an inverse semigroup, then the intersection of principal right (left) ideals is again principal and, indeed, if $aT \cap bT = cT$ then $xaT \cap xbT = xcT$ for each $x \in T$. Thus, when one considers the relationships between S and $I(S)$ it is of interest to suppose that S is naturally quasisemilatticed in the sense of the following definition.

Definition. Let $S = S^1$ be a semigroup. Then S is naturally quasisemilatticed if, for each $a, b \in S$, there exists $a \wedge_r b \in S$ such that $aS \cap bS = (a \wedge_r b)S$ and, for each $x \in S$, $(xa \wedge_r xb)S = x(a \wedge_r b)S$ and dually for left ideals.

If $S = S^1$ is a semigroup in which $\mathcal{D}$ is trivial then S is naturally quasisemilatticed if and only if it is a left semilatticed semigroup under the partial ordering $a \leq_r b$ if and only if $a \in bS$ and dually. Any semigroup in which the sets of principal left and right ideals form chains under inclusion is naturally quasisemilatticed as is the positive cone of an l -group and the multiplicative semigroup of a principal ideal domain. The free monoid on a set X is not naturally quasisemilatticed; however if a zero is adjoined, the resulting monoid is naturally quasisemilatticed.

In §6 we shall give necessary and sufficient conditions for embedding a naturally quasiseamilatticed semigroup into an inverse semigroup. These conditions, unlike those in Theorem 1.3, do not involve strong subsets; the latter are hard to find in general.

2. Shift representations of semigroups. Let $S = S^1$ be a semigroup and let σ be an equivalence on $S \times S$ which obeys the following condition:

$$(1) \quad (a, xb) \sigma (c, xd) \text{ if and only if } (ax, b) \sigma (cx, d)$$

for all $a, b, c, d, x \in S$ and, for each $x \in S$, define a partial transformation ρ_x^σ on the set $(S \times S)/\sigma$ of σ -classes by

$$(a, xb) \sigma \rho_x^\sigma = (ax, b) \sigma.$$

Then ρ_x^σ is clearly a one-to-one partial transformation of $(S \times S)/\sigma$.

Lemma 2.1. *Let σ be an equivalence, which obeys (1), on a semigroup $S = S^1$. Then the mapping $\rho^\sigma: S \rightarrow I((S \times S)/\sigma)$ defined by $x\rho^\sigma = \rho_x^\sigma$ is a representation of S by one-to-one partial transformations $(S \times S)/\sigma$ if and only if*

$$(2) \quad (a, b) \sigma (c, d) \text{ implies } (a, b) \sigma (xa, dy) \text{ for some } x, y \in S.$$

Proof. For any $a, b \in S$, $\Delta\rho_{ab}^\sigma \subseteq \Delta\rho_a^\sigma \rho_b^\sigma$ and further, if $(x, aby)\sigma \in \Delta\rho_{ab}^\sigma$,

$$(x, aby) \sigma \rho_{ab}^\sigma = (xab, y)\sigma = (xa, by) \sigma \rho_b^\sigma = (x, aby) \sigma \rho_a^\sigma \rho_b^\sigma.$$

Hence ρ^σ is a representation if and only if $\Delta\rho_a^\sigma \rho_b^\sigma \subseteq \Delta\rho_{ab}^\sigma$ for all $a, b \in S$.

Suppose that (2) holds. Then $(x, ay)\sigma \in \Delta\rho_a^\sigma \rho_b^\sigma$ implies $(xa, y) \sigma (u, bv)$ for some $u, v \in S$. Hence, by (2), $(xa, y) \sigma (rxa, bvs)$ for some $r, s \in S$. Thus, by (1), $(x, ay) \sigma (rx, abvs)$ so that $(x, ay)\sigma \in \Delta\rho_{ab}^\sigma$.

Conversely, suppose that $\Delta\rho_a^\sigma \rho_b^\sigma \subseteq \Delta\rho_{ab}^\sigma$ and let $(a, b) \sigma (c, d)$. Then $(1, ab) \sigma \rho_a^\sigma = (a, b)\sigma = (c, d)\sigma$ implies $(1, ab)\sigma \in \Delta\rho_a^\sigma \rho_d^\sigma = \Delta\rho_{ad}^\sigma$. Hence $(1, ab) \sigma (x, ady)$ for some $x, y \in S$ and so, by (1), $(a, b) \sigma (xa, dy)$.

Definition. If $S = S^1$ is a semigroup then an equivalence σ on $S \times S$ is called a *shift equivalence* if (1) and (2) are satisfied. If σ is a shift equivalence on $S \times S$ then the corresponding representation ρ^σ of S by one-to-one partial transformations of $(S \times S)/\sigma$ is called a *shift representation* of S .

Equivalence relations on $S \times S$ which obey (1) arise naturally when one considers homomorphisms of S into inverse semigroups as the following examples show.

Proposition 2.2. *Let θ be a homomorphism of a semigroup $S = S^1$ into an inverse semigroup T and define equivalences $\sigma_L, \sigma_R, \sigma_E$ on $S \times S$ as follows:*

$$(a, b) \sigma_L (c, d) \Leftrightarrow b\theta(ab)\theta^{-1} = d\theta(cd)\theta^{-1},$$

$$(a, b) \sigma_R (c, d) \Leftrightarrow (ab)\theta^{-1}a\theta = (cd)\theta^{-1}c\theta,$$

$$(a, b) \sigma_E (c, d) \Leftrightarrow a\theta^{-1}a\theta b\theta b\theta^{-1} = c\theta^{-1}c\theta d\theta d\theta^{-1}.$$

Then each of these equivalences obeys (1).

Proof. We show σ_E obeys (1).

$$\begin{aligned} (a, xb) \sigma_E (c, xd) &\Leftrightarrow a\theta^{-1}a\theta(xb)(xb)\theta^{-1} = c\theta^{-1}c\theta(xd)\theta(xd)\theta^{-1} \\ &\Leftrightarrow a\theta^{-1}(ax)\theta b\theta b\theta^{-1}x\theta^{-1} = c\theta^{-1}(cx)\theta d\theta d\theta^{-1}x\theta^{-1} \\ &\Leftrightarrow x\theta^{-1}a\theta^{-1}(ax)\theta b\theta b\theta^{-1} = x\theta^{-1}c\theta^{-1}(cx)\theta d\theta d\theta^{-1} \\ &\Leftrightarrow (ax)\theta^{-1}(ax)\theta b\theta b\theta^{-1} = (cx)\theta^{-1}(cx)\theta d\theta d\theta^{-1} \\ &\Leftrightarrow (ax, b) \sigma_E (cx, d) \end{aligned}$$

since idempotents commute.

The other two are proved similarly.

There is clearly a smallest equivalence on $S \times S$ which obeys (1). In some important cases, this can easily be described and is a shift equivalence.

Lemma 2.3. Let $S = S^1$ be a semigroup and define a relation τ_0 on $S \times S$ by $(a, b) \tau_0 (c, d) \Leftrightarrow$ there exist $x_0, \dots, x_n, y_0, \dots, y_n$ such that $a = x_0, c = x_n, b = y_0, d = y_n$ and $x_{i-1}y_{i-1} = x_i y_{i-1} = x_i y_i, 1 \leq i \leq n$. Then τ_0 is an equivalence and is contained in the smallest equivalence on $S \times S$ which obeys (1).

Proof. τ_0 is clearly an equivalence on $S \times S$. Further, if σ is an equivalence on $S \times S$ which obeys (1) then $x_{i-1}y_{i-1} = x_i y_{i-1} = x_i y_i$ implies

$$(x_{i-1}y_{i-1}, 1) \sigma (x_i y_{i-1}, 1) \quad \text{and} \quad (1, x_i y_{i-1}) \sigma (1, x_i y_i).$$

Thus, by (1), $(x_{i-1}, y_{i-1}) \sigma (x_i, y_{i-1}) \sigma (x_i, y_i)$ so that, from the definition of τ_0 , $\tau_0 \subseteq \sigma$.

Propositions 2.6, 2.7, 2.9 give examples of types of semigroups on which τ_0 is a shift and thus is the finest shift on $S \times S$. Under these circumstances we can use τ_0 to give necessary and sufficient conditions for embeddability in inverse semigroups.

Lemma 2.4. Let $S = S^1$ be a semigroup such that τ_0 is a shift and let ρ be the shift representation associated with τ_0 . Then $\rho_a = \rho_b$ if and only if $\hat{a} = \hat{b}$.

Proof. If τ_0 is a shift, then ρ can be factored through η and so $\hat{a} = \hat{b}$ implies $\rho_a = \rho_b$.

On the other hand, $\rho_a = \rho_b$ implies $(1, a) \tau_0(x, by)$ and $(a, 1) \tau_0(xb, y)$ for some $x, y \in S$. The first of these equivalences implies the existence of $u_0, \dots, u_n, v_0, \dots, v_n$ in S such that $u_0 = 1, u_n = x, v_0 = a, v_n = by$ and $u_{i-1}v_{i-1} = u_i v_{i-1} = u_i v_i, 1 \leq i \leq n$. Then $v_0 = a \in \hat{a}$. Suppose $v_{i-1} \in \hat{a}$; then $u_{i-1}v_{i-1} = u_i v_{i-1} = u_i v_i = a \in \hat{a}$ implies $u_{i-1}v_i \in \hat{a}$ and so $u_{i-1} \in \hat{a} \cdot v_i \cap \hat{a} \cdot v_{i-1}$. Since $\hat{a}$ is strong and $1 \in \hat{a} \cdot v_{i-1}$, this implies $1 \in \hat{a} \cdot v_i$ so that $v_i \in \hat{a}$. Hence, by induction, $by \in \hat{a}$. Dually, the second equivalence implies $xb \in \hat{a}$.

Since $xb y = a \in \hat{a}$ and $by \in \hat{a}$ we have $y \in \hat{a} \cdot xb \cap \hat{a} \cdot b$ and so, since $\hat{a}$ is strong and $1 \in \hat{a} \cdot xb, 1 \in \hat{a} \cdot b$; thus $b \in \hat{a}$. Finally, by duality, we also get $a \in \hat{b}$. Hence $\hat{a} = \hat{b}$.

Theorem 2.5. *Let $S = S^1$ be a semigroup on which τ_0 is a shift and let $\rho: S \rightarrow \mathcal{H}((S \times S)/\tau_0)$ be the corresponding shift representation. Then S can be embedded in an inverse semigroup if and only if ρ is one-to-one.*

We now give some examples of semigroups in which τ_0 obeys (1) and (2).

Proposition 2.6. *Let $S = S^1$ be a left cancellative semigroup. Then τ_0 is a shift equivalence on $S \times S$.*

Proof. Suppose S is left cancellative and let $(a, b) \tau_0(c, d)$. Then $a = x_0, c = x_n, b = y_0, d = y_n$ and $x_{i-1}y_{i-1} = x_i y_{i-1} = x_i y_i, 1 \leq i \leq n$, for some $x_i, y_i \in S$. Since S is left cancellative, this implies $y_{i-1} = y_i, 1 \leq i \leq n$; hence each y_i is b and so $(a, b) \tau_0(c, d)$ implies $b = d$ and $ab = cb$. On the other hand, $b = d, ab = cb$ clearly implies $(a, b) \tau_0(c, d)$. Hence

$$(a, b) \tau_0(c, d) \iff b = d, \quad ab = cb.$$

It follows from this characterisation of τ_0 that $(a, xb) \tau_0(c, xd)$ if and only if $axb = cxd, xb = xd$. Since S is left cancellative, the last two equations hold if and only if $axb = cxd$ and $b = d$. Hence (1) holds. Finally, from the characterisation of τ_0 , $(a, b) \tau_0(c, d)$ implies $(a, b) \tau_0(a, d)$ so that (2) holds trivially.

Proposition 2.7. *Let $S = S^1$ be an inverse semigroup. Then τ_0 is a shift equivalence on $S \times S$.*

Proof. Suppose $(a, xb) \tau_0(c, xd)$; then $a = u_0, c = u_n, xb = v_0, xd = v_n$ and $u_{i-1}v_{i-1} = u_i v_{i-1} = u_i v_i, 1 \leq i \leq n$, for some $u_i, v_i \in S$. Set $p_0 = ax, p_n = cx, q_0 = b, q_n = d$ and $p_i = u_i x, q_i = x^{-1} v_i, 1 \leq i \leq n$. We show that $p_{i-1}q_{i-1} = p_i q_i, 1 \leq i \leq n$. This proves that $(ax, b) \tau_0(cx, d)$ and, together with its dual, gives (1).

Since $u_{i-1}v_{i-1} = u_i v_{i-1}$, it follows that $u_{i-1}v_{i-1}v_{i-1}^{-1}xx^{-1} = u_i v_{i-1}v_{i-1}^{-1}xx^{-1}$ and so, since idempotents commute, $(u_{i-1}x)(x^{-1}v_{i-1}) = (u_i x)(x^{-1}v_{i-1})$;

similarly $(u_i x)(x^{-1} v_{i-1}) = (u_i x)(x^{-1} v_i)$, $1 \leq i \leq n$. Hence, for $1 < i < n$, $p_{i-1} q_{i-1} = p_i q_{i-1} = p_i q_i$. Further

$$p_0 q_0 = axb = u_0 v_0 = u_1 v_0 = u_1 x b = p_1 q_0$$

and, as above, $u_1 x x^{-1} v_0 = u_1 x x^{-1} v_1 = p_1 q_1$ so that, since $v_0 = xb$, $p_1 q_0 = u_1 v_0 = u_1 x x^{-1} v_0 = p_1 q_1$. Similarly $p_{n-1} q_{n-1} = p_n q_{n-1} = p_n q_n$. Thus $p_{i-1} q_{i-1} = p_i q_{i-1} = p_i q_i$, $1 \leq i \leq n$.

Finally, suppose that $(a, b) \tau_0 (c, d)$; then $a = x_0$, $c = x_n$, $b = y_0$, $d = y_n$ and $x_{i-1} y_{i-1} = x_i y_{i-1} = x_i y_i$, $1 \leq i \leq n$, for some $x_i, y_i \in S$ and some positive integer n . As in the immediately preceding paragraph, this implies $(x_0 a^{-1} a, y_0) \tau_0 (x_n a^{-1} a, y_n)$; that is $(a, b) \tau_0 (c a^{-1} a, d)$. Hence (2) holds.

Corollary 2.8. *Let $S = S^1$ be an inverse semigroup and let ρ be the shift representation associated with τ_0 . Then ρ is faithful.*

Proposition 2.9. *Let $S = S^1$ be a naturally quasiorordered semigroup on which $\mathcal{D}$ is trivial. Then τ_0 is a shift equivalence on $S \times S$.*

Proof. This is a special case of Theorem 3.9 so we omit a proof.

3. Fundamental inverse semigroups separated over a semigroup S .

Lemma 3.1. *Let θ be a homomorphism of a semigroup S into an inverse semigroup T . Let $a, b, c \in S$ and suppose that*

$$a\theta a\theta^{-1} b\theta b\theta^{-1} = x\theta x\theta^{-1}, \quad b\theta^{-1} b\theta c\theta^{-1} c\theta = u\theta^{-1} u\theta$$

where $x = ay = bz$, $u = vb = wc$. Then

$$a\theta^{-1} b\theta c\theta^{-1} = y\theta(vbz)\theta^{-1} u\theta.$$

Proof. For convenience of notation, let us identify S with its image in T . Then

$$\begin{aligned} a^{-1} b c^{-1} &= a^{-1} a a^{-1} b b^{-1} b c^{-1} = a^{-1} (ay)(ay)^{-1} b c^{-1} = a^{-1} a y y^{-1} a^{-1} b c^{-1} \\ &= y y^{-1} a^{-1} b c^{-1} = y x^{-1} b c^{-1} = y x^{-1} b b^{-1} b c^{-1} c c^{-1} = y x^{-1} b (wc)^{-1} w c c^{-1} \\ &= y x^{-1} b (wc)^{-1} w = y (bz)^{-1} b (vb)^{-1} w = y (vbz)^{-1} w \end{aligned}$$

since idempotents in T commute.

Lemma 3.1 is similar to Lemma 3.4 in [5].

Theorem 3.2. *Let θ be a homomorphism of $S = S^1$ into an inverse semigroup T . If T is separated over S by θ then $T = \{a\theta b\theta^{-1} c\theta : b \in Sa \cap cS, a, c \in S\}$.*

Proof. As in Lemma 3.1, we identify S and $S\theta$. Let $ab^{-1}c, de^{-1}f \in K$, where K denotes the right side of the equation for T , and suppose that $b = ua = cv$, $e = pd = fq$.

By Lemma 2.1, if $bb^{-1}cd(cd)^{-1} = bb^{-1}$ and $(cd)^{-1}cde^{-1}e = k^{-1}k$ with $b = by = cdz$ and $k = xcd = we$, then

$$b^{-1}cde^{-1} = y(xcdz)^{-1}w$$

so that $ab^{-1}cde^{-1}f = ay(xcdz)^{-1}wf$. Further $xcdz = xby = xuy \in Say$ and $xcdz = wez = wqz \in wfS$ so that $ab^{-1}cde^{-1}f \in K$. Since, by Lemma 3.1, K is closed under inverses, it follows that $K = T$.

Definition. Let T be an inverse semigroup and let $S = S^1$ be a subsemigroup of T . Then T is an *inverse semigroup of strong quotients* of S if each element of T is of the form $ab^{-1}c$ where $b \in Sa \cap cS$.

In the light of this definition, we have

Corollary 3.3. *Let T be an inverse semigroup which is separated over a subsemigroup S . Then T is an inverse semigroup of strong quotients of S .*

The inverse semigroups which are separated over a semigroup $S = S^1$ appear to be closely related to the shift representations of S . We have not been able to determine this relationship in general; however we have been able to characterise fundamental inverse semigroups which are separated over S .

Lemma 3.4. *Let θ be a homomorphism of a semigroup $S = S^1$ into an inverse semigroup T . Suppose that T is separated over S by θ and define σ_E on $S \times S$ by*

$$(a, b) \sigma_E (c, d) \Leftrightarrow a\theta^{-1}a\theta b\theta b\theta^{-1} = c\theta^{-1}c\theta d\theta d\theta^{-1}$$

for all $a, b, c, d \in S$. Then σ_E is a shift equivalence on $S \times S$ and $S \times S / \sigma_E$ is a semilattice, isomorphic to the semilattice of idempotents of T , under the partial ordering

$$(a, b) \sigma_E \leq (c, d) \sigma_E \Leftrightarrow (a, b) \sigma_E (u, v) \text{ for some } u \in Sa \cap cS, v \in bS \cap dS.$$

Proof. Since T is separated over S , Theorem 3.2 shows that each element of T is of the form $a\theta b\theta^{-1}c\theta$ where $b \in Sa \cap cS$. For such an element of T ,

$$\begin{aligned} a\theta b\theta^{-1}c\theta(a\theta b\theta^{-1}c\theta)^{-1} &= a\theta b\theta^{-1}c\theta c\theta^{-1}b\theta a\theta^{-1} \\ &= a\theta b\theta^{-1}b\theta a\theta^{-1} \quad \text{since } b \in cS \\ &= u\theta^{-1}u\theta a\theta a\theta^{-1} \quad \text{if } b = ua. \end{aligned}$$

Hence the mapping defined by $(u, a) \sigma_E \rightarrow u\theta^{-1}u\theta a\theta a\theta^{-1}$ is a bijection of $(S \times S) / \sigma_E$ onto the semilattice of idempotents of T . Further, since

$a\theta^{-1}a\theta b\theta b\theta^{-1} \leq c\theta^{-1}c\theta d\theta d\theta^{-1}$ if and only if $a\theta^{-1}a\theta b\theta b\theta^{-1} = a\theta^{-1}a\theta c\theta^{-1}c\theta b\theta b\theta^{-1}d\theta d\theta^{-1}$ and since T is separated over S , $a\theta^{-1}a\theta b\theta b\theta^{-1} \leq c\theta^{-1}c\theta d\theta d\theta^{-1}$ if and only if $(a, b) \sigma_E(u, v)$ for some $u \in Sa \cap Sc$, $v \in bS \cap dS$. Hence $(S \times S)/\sigma_E$ is a semilattice under

$$(a, b)\sigma_E \leq (c, d)\sigma_E \Leftrightarrow (a, b) \sigma_E(u, v) \text{ for some } u \in Sa \cap Sc, v \in bS \cap dS.$$

Finally, Proposition 2.2 shows that σ_E obeys (1) while, since $(S \times S)/\sigma_E$ is a semilattice under the partial order described above, σ_E clearly obeys (2). Hence σ_E is a shift.

Lemma 3.5. *Let $S = S^1$ be a semigroup and let σ be an equivalence on $S \times S$. Suppose that $(S \times S)/\sigma$ is a semilattice under*

$$(a, b)\sigma \leq (c, d)\sigma \Leftrightarrow (a, b) \sigma(u, v) \text{ for some } u \in Sa \cap Sc, v \in bS \cap dS,$$

Then,

- (i) $(1, a)\sigma \wedge (1, b)\sigma = (1, v)\sigma$ for some $v \in aS \cap bS$,
- (ii) $(a, 1)\sigma \wedge (b, 1)\sigma = (u, 1)\sigma$ for some $u \in Sa \cap Sb$,
- (iii) $(a, 1)\sigma \wedge (1, b)\sigma = (a, b)\sigma$

for $a, b \in S$.

Proof. (i) Suppose $(1, a)\sigma \wedge (1, b)\sigma = (x, y)\sigma$. Then, because $(x, y)\sigma \leq (1, a)\sigma$, there exist $x_1 \in S$, $y_1 \in yS \cap aS$ such that $(x_1, y_1)\sigma = (x, y)$. Since $(x_1, y_1)\sigma \leq (1, b)\sigma$, there exist $u \in S$, $v \in y_1S \cap bS \subseteq aS \cap bS$ such that $(x_1, y_1)\sigma = (u, v)$. Thus $(1, a)\sigma \wedge (1, b)\sigma = (u, v)\sigma$. But $(u, v)\sigma \leq (1, v)\sigma \leq (1, a)\sigma, (1, b)\sigma$ from the definition of $\leq$ since $v \in aS \cap bS$. Hence we must have $(1, a)\sigma \wedge (1, b)\sigma = (1, v)\sigma$.

(ii) This is dual to (i).

(iii) From the definition of the partial order on $(S \times S)/\sigma$, $(a, b)\sigma \leq (a, 1)\sigma, (1, b)\sigma$. On the other hand, if $(x, y)\sigma \leq (a, 1)\sigma, (1, b)\sigma$, then $(x, y) \sigma (x_1, y_1)$ for some $x_1 \in Sa \cap Sx$ and then, since $(x_1, y_1)\sigma \leq (1, b)\sigma$, $(x_1, y_1) \sigma (x_2, y_2)$ for some $x_2 \in Sx_1 \cap Sa$ and $y_2 \in y_1S \cap bS \subseteq bS$. Thus $(x, y)\sigma = (x_2, y_2)\sigma \leq (a, b)\sigma$. Hence $(a, 1)\sigma \wedge (1, b)\sigma = (a, b)\sigma$.

Suppose that T is an inverse semigroup with semilattice of idempotents E and for each $a \in T$ define a partial transformation μ_a of E by $x\mu_a = a^{-1}xa$ for each $x \in Eaa^{-1}$. Then Munn [11] shows that $\mu: T \rightarrow \mathcal{A}(E)$ defined by $a\mu = \mu_a$ is a representation of T by partial one-to-one transformations of E and that T/μ "is" the maximum fundamental homomorphic image of T .

Theorem 3.6. *Let $S = S^1$ be a semigroup and let θ be a homeomorphism of S into a fundamental inverse semigroup T which is separated over S by θ . Define σ_E on $S \times S$ by*

$$(a, b) \sigma_E (c, d) \Leftrightarrow a\theta^{-1}a\theta b\theta b\theta^{-1} = c\theta^{-1}c\theta d\theta d\theta^{-1}$$

and let $\rho: S \rightarrow \mathcal{G}((S \times S)/\sigma_E)$ be the shift representation associated with σ_E . Then T is isomorphic to the inverse hull of $S\rho$ in $\mathcal{G}((S \times S)/\sigma_E)$.

Conversely, let σ be an equivalence on $S \times S$ which obeys (1) and is such that $(S \times S)/\sigma$ is a semilattice under

$$(a, b)\sigma \leq (c, d)\sigma \Leftrightarrow (a, b)\sigma (u, v) \text{ for some } u \in Sa \cap Sc, \quad v \in bS \cap dS$$

and let ρ be the shift representation associated with σ . Then the inverse hull of $S\rho$ in $\mathcal{G}((S \times S)/\sigma)$ is fundamental and $\sigma = \sigma_E$.

Proof. Let θ be as in the statement of the theorem. Then, by Lemma 3.4, the mapping ϕ defined by $\alpha\phi = (a, b)\sigma_E$ if $\alpha = a\theta^{-1}a\theta b\theta b\theta^{-1}$ is an isomorphism from the set E of idempotents of T onto $(S \times S)/\sigma_E$. Thus we can use $(S \times S)/\sigma_E$ to obtain a representation ψ of T equivalent to μ and hence to obtain an isomorphic copy of T/μ . For each $\alpha \in T$, since ψ is equivalent to μ ,

$$\Delta\psi_\alpha = \{e\phi \in (S \times S)/\sigma_E : e \in \Delta\mu_\alpha\} = \{e\phi \in (S \times S)/\sigma_E : e \leq \alpha\alpha^{-1}\}.$$

Hence, if $\alpha = a\theta(b\theta)^{-1}c\theta$, where $b = ua = cv$,

$$\begin{aligned} \Delta\psi_\alpha &= \{e\phi : e \leq a\theta(b\theta)^{-1}c\theta c\theta^{-1}b\theta a\theta^{-1}\} \\ &= \{e\phi : e \leq u\theta^{-1}u\theta a\theta a\theta^{-1}\} = \{e\phi : e\phi \leq (u, a)\sigma_E\} \\ &= \{(xu, ay)\sigma_E : x, y \in S\} \text{ by Lemma 3.4.} \end{aligned}$$

This is independent of the particular choice of $a, b, c, u, v \in S$, with $b = ua = cv$, such that $\alpha = a\theta(b\theta)^{-1}c\theta$. Further, using the fact that ψ is equivalent to μ , direct calculation shows that $(xu, ay)\sigma_E \psi_\alpha = (xc, vy)\sigma_E$.

Consider the diagram

$$\begin{array}{ccc} S & & \\ \theta \downarrow & \nearrow \rho & \\ & & \mathcal{G}((S \times S)/\sigma_E) \\ & \nwarrow \psi & \\ T & & \end{array}$$

Let $a \in S$; then, since $a\theta = a\theta(a\theta)^{-1}a\theta$ where $a = 1 \cdot a = a \cdot 1$,

$$\Delta a\theta\psi = \{(x, ay)\sigma_E : x, y \in S\} = \Delta a\rho$$

and, for $(x, ay)\sigma_E \in \Delta a\theta\psi$,

$$(x, ay)\sigma_E a\theta\psi = (xa, y)\sigma_E = (x, ay)\sigma_E \rho_a$$

from the calculations in the preceding paragraph. Hence $\rho = \theta\psi$ and the diagram commutes. Since $T\psi \approx T/\mu$ is generated, as an inverse semigroup, by $S\theta\psi = S\rho$,

it follows that T/μ is isomorphic to the inverse hull of $S\rho$ in $\mathcal{H}((S \times S)/\sigma_E)$. In particular, if T is fundamental, so that μ is an isomorphism [11], T is isomorphic to the inverse hull of $S\rho$ in $\mathcal{H}((S \times S)/\sigma_E)$.

Conversely, suppose that σ is an equivalence on $S \times S$ which obeys (1) and is such that $(S \times S)/\sigma$ is a semilattice under

$$(a, b)\sigma \leq (c, d)\sigma \Leftrightarrow (a, b)\sigma (u, v) \text{ for some } u \in Sa \cap Sc, v \in bS \cap dS.$$

Then, clearly, σ obeys (2) and so gives rise to a shift representation ρ of S by one-to-one partial transformations of $(S \times S)/\sigma$. For each $a \in S$,

$$\Delta\rho_a = \{(x, ay)\sigma: x, y \in S\} = \{(u, v)\sigma: (u, v)\sigma \leq (1, a)\sigma\}.$$

Hence, by Lemma 3.5 (i), since $(S \times S)/\sigma$ is a semilattice

$$\begin{aligned} \Delta\rho_a \cap \Delta\rho_b &= \{(u, v)\sigma: (u, v)\sigma \leq (1, a)\sigma \wedge (1, b)\sigma\} \\ &= \{(u, v)\sigma: (u, v)\sigma \leq (1, y)\sigma\} \\ &= \Delta\rho_y \text{ for some } y \in aS \cap bS. \end{aligned}$$

Thus $\rho_a \rho_a^{-1} \rho_b \rho_b^{-1} = \rho_y \rho_y^{-1}$ for some $y \in aS \cap bS$ and, dually, $\rho_a^{-1} \rho_a \rho_b^{-1} \rho_b = \rho_x^{-1} \rho_x$ for some $x \in Sa \cap Sb$. Hence the inverse hull K of $S\rho$ is separated over S by ρ and so, by Corollary 3.3, is an inverse semigroup of strong quotients of $S\rho$. In particular, the idempotents of K are all of the form $\rho_a^{-1} \rho_a \rho_b \rho_b^{-1}$. Further,

$$\rho_a^{-1} \rho_a \rho_b \rho_b^{-1} \leq \rho_c^{-1} \rho_c \rho_d \rho_d^{-1} \Leftrightarrow (a, b)\sigma \leq (c, d)\sigma,$$

by Lemma 3.5 (iii). Hence the semilattice of idempotents of K is isomorphic to $(S \times S)/\sigma$ and $\sigma = \sigma_E$. From the proof of the first part of the theorem, K/μ , the maximum fundamental homomorphic image of K , is isomorphic to the inverse hull of $S\rho$ in $\mathcal{H}((S \times S)/\sigma)$; that is, to K itself. Hence K is fundamental.

Remark. The proof of the first part of Theorem 3.6 shows the following: if T is separated by θ over S then T/μ is isomorphic to the inverse hull of $S\rho$ in $\mathcal{H}((S \times S)/\sigma_E)$.

The second part of the theorem shows that if σ is an equivalence on $S \times S$ which obeys (1) and is such that $(S \times S)/\sigma$ is a semilattice under the relation

$$(a, b)\sigma \leq (c, d)\sigma \Leftrightarrow (a, b)\sigma (u, v) \text{ for some } u \in Sa \cap Sc, v \in bS \cap dS,$$

then there is a homomorphism of S into an inverse semigroup T with semilattice $(S \times S)/\sigma$.

Theorem 3.6 characterises fundamental inverse semigroups which are separated over S in terms of equivalences on $S \times S$. To end this section, we show how such equivalences can be obtained from equivalences on S .

If π is a right congruence on $S = S^1$ then there is a natural action of S on the set S/π of equivalence classes as follows:

$$a\pi \cdot x = (ax)\pi \quad \text{for all } a, x \in S.$$

Dually, if π is a left congruence on S , then S acts naturally on the left of S/π .

Let π be a right congruence on S such that S/π is a semilattice. We say that S acts *naturally on the semilattice S/π* if

$$(\bar{a} \wedge \bar{b}) \cdot x = \bar{a} \cdot x \wedge \bar{b} \cdot x$$

for all $\bar{a}, \bar{b} \in S/\pi$, $x \in S$.

A dual definition holds for left congruences.

Lemma 3.7. *Let σ be an equivalence on $S \times S$ which obeys (1) and is such that $(S \times S)/\sigma$ is a semilattice under the partial ordering*

$$(a, b)\sigma \leq (c, d)\sigma \iff (a, b)\sigma(u, v) \quad \text{for some } u \in Sa \cap Sc, v \in bS \cap dS$$

and define

$$a L b \iff (a, 1)\sigma(b, 1), \quad a R b \iff (1, a)\sigma(1, b).$$

Then L is a right congruence on S , S/L is a semilattice (with operation $\wedge_l$) under

$$aL \leq bL \iff a L u \quad \text{for some } u \in Sa \cap Sb$$

and S acts naturally on S/L . Dual results hold for R . Further

$$(a, b)\sigma(c, d) \iff ab L (a \wedge_l c)b R (a \wedge_l c)(b \wedge_r d) L c(b \wedge_r d) R cd$$

where, for example, $a \wedge_l c$ denotes any element of S such that $(a \wedge_l c)L = (aL \wedge_l cL)$.

Proof. Let ρ be the shift representation associated with σ . Then $(a, b)\sigma(c, d)$ if and only if $\rho_a^{-1}\rho_a\rho_b\rho_b^{-1} = \rho_c^{-1}\rho_c\rho_d\rho_d^{-1}$. Hence $a L b$ implies $a\rho^{-1}a\rho = b\rho^{-1}b\rho$ which, in turn, implies $(ax)\rho^{-1}(ax)\rho = (bx)\rho^{-1}(bx)\rho$; that is, $ax L bx$. Thus L is a right congruence on S .

Let $a, b \in S$ and pick $u \in Sa \cap Sb$ such that $(a, 1)\sigma \wedge (b, 1)\sigma = (u, 1)\sigma$; by Lemma 3.5 (iii) such an element exists. Then, from the definition of the partial order on S/L , $uL \leq aL, bL$. On the other hand, if $vL \leq aL, bL$ then $vL = yL$ for some $y \in Sa \cap Sb$ and so $(v, 1)\sigma = (y, 1)\sigma \leq (a, 1)\sigma, (b, 1)\sigma$; thus $(v, 1)\sigma \leq (u, 1)\sigma$. This implies $(v, 1)\sigma = (v, 1)\sigma \wedge (u, 1)\sigma$ and so, by Lemma 3.5 (iii), $(v, 1)\sigma = (z, 1)\sigma$ for some $z \in Sv \cap Su \subseteq Su$. Hence $yL = zL \leq uL$. It follows that S/L is a semilattice with $aL \wedge bL = uL$ where $u \in Sa \cap Sb$ is such that $(a, 1)\sigma \wedge (b, 1)\sigma = (u, 1)\sigma$. Further, $u\rho^{-1}u\rho = a\rho^{-1}a\rho b\rho b^{-1}b\rho$ implies

$$\begin{aligned} (ux)\rho^{-1}(ux)\rho &= x\rho^{-1}(a\rho^{-1}a\rho b\rho b^{-1}b\rho)x\rho \\ &= x\rho^{-1}a\rho^{-1}a\rho x\rho x\rho^{-1}b\rho^{-1}b\rho x\rho = (ax)\rho^{-1}(ax)\rho(bx)\rho^{-1}(bx)\rho. \end{aligned}$$

Hence $(ux)L = (ax)L \wedge_l (bx)L$ and so S acts naturally on S/L .

Next $(a, b) \sigma (c, d)$ if and only if

$$ap^{-1}apbpbp^{-1} = cp^{-1}cpdpdp^{-1}$$

$$\text{implies } ap^{-1}apbpbp^{-1} = (a \wedge_l c)\rho^{-1}(a \wedge_l c)\rho bpbp^{-1}$$

$$\text{implies } (a \wedge_l c)\rho^{-1}(a \wedge_l c)\rho bpbp^{-1} = (a \wedge_l c)\rho^{-1}(a \wedge_l c)\rho(b \wedge_r d)\rho(b \wedge_r d)\rho^{-1}$$

$$\text{implies } (a \wedge_l c)\rho^{-1}(a \wedge_l c)\rho(b \wedge_r d)\rho(b \wedge_r d)\rho^{-1} = cp^{-1}cp(b \wedge_r d)\rho(b \wedge_r d)\rho^{-1}$$

$$\text{implies } cp^{-1}cp(b \wedge_r d)\rho(b \wedge_r d)\rho^{-1} = cp^{-1}cpdpdp^{-1}$$

where, for example, $(a \wedge_l c)L(aL \wedge_l cL)$. These implications give in sequence

$$(ab)\rho^{-1}(ab)\rho = [(a \wedge_l c)b]\rho^{-1}[(a \wedge_l c)b]\rho \text{ so } abL(a \wedge_l c)b$$

$$[(a \wedge_l c)b]\rho[(a \wedge_l c)b]\rho^{-1} = [(a \wedge_l c)(b \wedge_r d)]\rho[(a \wedge_l c)(b \wedge_r d)]\rho^{-1}$$

$$\text{so } (a \wedge_l c)bR(a \wedge_l c)(b \wedge_r d)$$

$$[(a \wedge_l c)(b \wedge_r d)]\rho^{-1}[(a \wedge_l c)(b \wedge_r d)]\rho = [c(b \wedge_r d)]\rho^{-1}[c(b \wedge_r d)]\rho$$

$$\text{so } (a \wedge_l c)(b \wedge_r d)Lc(b \wedge_r d)$$

$$[c(b \wedge_r d)]\rho[c(b \wedge_r d)]\rho^{-1} = (cd)\rho(cd)\rho^{-1} \text{ so } c(b \wedge_r d)Rcd.$$

Hence $(a, b) \sigma (c, d)$ implies

$$abL(a \wedge_l c)bR(a \wedge_l c)(b \wedge_r d)Lc(b \wedge_r d)Rcd.$$

The converse follows, as in the proof of Theorem 3.8, because σ is a shift.

Lemma 3.7 shows that σ is determined by the equivalences L and R . The next theorem shows how, starting with a pair of equivalences L and R we can obtain a shift σ .

Theorem 3.8. *Let $S = S^1$ be a semigroup and let L and R be respectively right and left congruences on S such that S/L and S/R are semilattices under*

$$aL \leq bL \Leftrightarrow aLc \text{ for some } c \in Sa \cap Sb,$$

$$aR \leq bR \Leftrightarrow aRc \text{ for some } c \in aS \cap bS.$$

Suppose also that S acts naturally on the semilattices S/L and S/R . Define a relation $\sigma = \sigma(L, R)$ on $S \times S$ by $(a, b) \sigma (c, d) \Leftrightarrow$ there exist finite sets

$x_0, \dots, x_n, y_0, \dots, y_n$ in S such that $a = x_0, c = x_n, b = y_0, d = y_n$ and, for $1 \leq i \leq n,$

$$x_{i-1}y_{i-1}Lx_iy_{i-1}R x_iy_i.$$

Then σ is the finest equivalence on $S \times S$ with the following properties:

- (i) σ obeys (1),
- (ii) $(S \times S)/\sigma$ is a semilattice under

$$(a, b)\sigma \leq (c, d)\sigma \Leftrightarrow (a, b)\sigma(u, v) \text{ for some } u \in Sa \cap Sc, v \in bS \cap dS,$$
- (iii) $a L c, b R d$ implies $(a, b)\sigma(c, d)$.

Proof. First, it is easy to see that σ is an equivalence on $S \times S$. Suppose that $(a, b)\sigma(c, d)$ and let $u, v \in S$. Also let $x_0, \dots, x_n, y_0, \dots, y_n$ be as in the definition of σ . Then

$$x_{i-1}y_{i-1} L x_i y_{i-1} \text{ implies } x_{i-1}y_{i-1} \wedge_l u y_{i-1} L x_i y_{i-1} \wedge_l u y_{i-1}$$

where, for $b, k \in S$, $b \wedge_l k$ denotes any element of $Sb \cap Sk$ such that $(b \wedge_l k)L = bL \wedge_l kL$. Since S acts naturally on the semilattice S/L , it follows from this that $(x_{i-1} \wedge_l u)y_{i-1} L (x_i \wedge_l u)y_{i-1}$ and hence, because L is a right congruence, $(x_{i-1} \wedge_l u)(y_{i-1} \wedge_r v) L (x_i \wedge_l u)(y_{i-1} \wedge_r v)$. Similarly, $x_i y_{i-1} R x_i y_i$ implies $(x_i \wedge_l u)(y_{i-1} \wedge_r v) R (x_i \wedge_l u)(y_i \wedge_r v)$, $1 \leq i \leq n$. Thus $(a \wedge_l u, b \wedge_r v)\sigma(c \wedge_l u, d \wedge_r v)$.

This shows, in particular, that the mapping $S/L \times S/R \rightarrow (S \times S)/\sigma$ defined by $(aL, bR) \rightarrow (a, b)\sigma$ is a semilattice homomorphism so that $(S \times S)/\sigma$ is a semilattice. Further, because of the order on S/L and S/R ,

$$(a, b)\sigma \leq (c, d)\sigma \Leftrightarrow (a, b)\sigma(a \wedge_l c, b \wedge_r d) \\ \Leftrightarrow (a, b)\sigma(u, v) \text{ for some } u \in Sa \cap Sc, v \in bS \cap dS.$$

Suppose that $a = u_0 \dots, u_n = c, xb = v_0 \dots, v_n = d$ and $u_{i-1}v_{i-1} L u_i v_{i-1} R u_i v_i$, $1 \leq i \leq n$. Define $q_i = w_i$, $0 \leq i \leq n$, where w_i is such that $xw_i \in xS \cap v_i S$ and $xw_i R = xR \wedge_r v_i R$ with $w_0 = b$, $w_n = d$ and set $p_i = u_i x$, $0 \leq i \leq n$. Then

$$p_{i-1}q_{i-1} = u_{i-1}xw_{i-1} L u_i xw_{i-1} = p_i q_{i-1} \text{ for } 1 \leq i \leq n$$

since $xw_{i-1} \in v_{i-1}S$ and L is a right congruence, and $p_0 q_0 = u_0 xb = u_0 v_0 L u_1 v_0 = u_1 xb = p_1 q_0$. Further, since S acts naturally on the semilattice S/R ,

$$p_i q_{i-1} R = u_i xw_{i-1} R = u_i xR \wedge_r u_i v_{i-1} R \\ = u_i xR \wedge_r u_i v_i R = u_i (xR \wedge_r v_i R) \\ = u_i xw_i R = p_i q_i R, \quad 1 \leq i \leq n.$$

Hence $(ax, b)\sigma(cx, d)$. The dual also holds so that σ obeys (1).

Finally, $a L c, b R d$ implies $(a, 1)\sigma(c, 1)$ and $(1, b)\sigma(1, d)$ and so $(a \wedge_l 1, b \wedge_r 1)\sigma(c \wedge_l 1, d \wedge_r 1)$ by the first paragraph of the proof; thus $(a, c)\sigma(b, d)$ so that (iii) holds.

Conversely, suppose that π obeys (i), (ii), (iii). Then $x_{i-1}y_{i-1} L x_i y_{i-1} R x_i y_i$ implies $(x_{i-1}y_{i-1}, 1) \pi (x_i y_{i-1}, 1)$, $(1, x_i y_{i-1}) \pi (1, x_i y_i)$ and so, by (i), $(x_{i-1}, y_{i-1}) \pi (x_i, y_{i-1}) \pi (x_i, y_i)$. Hence $(a, b) \sigma (c, d)$ implies $(a, b) \pi (c, d)$. Thus σ is, in fact, the smallest equivalence on $S \times S$ which obeys (i) and (iii).

If L and R are right and left congruences on $S = S^1$, which obey the hypotheses of Theorem 3.7, it is easy to see that $\mathcal{L} \subseteq L$, $\mathcal{R} \subseteq R$ where $\mathcal{L}$ and $\mathcal{R}$ are the familiar Green's relations. Since $\mathcal{L}$ and $\mathcal{R}$ obey the hypotheses of the theorem when S is naturally quasisemilatticed we get, immediately, the following result which is of fundamental importance in later sections.

Theorem 3.9. *Let $S = S^1$ be a naturally quasisemilatticed semigroup and define a relation τ on $S \times S$ by*

$$(a, b) \tau (c, d) \Leftrightarrow \text{there exist finite sets } x_0, \dots, x_n, y_0, \dots, y_n \text{ in } S$$

such that $a = x_0$, $c = x_n$, $b = y_0$, $d = y_n$ and $x_{i-1}y_{i-1} \mathcal{L} x_i y_{i-1} \mathcal{R} x_i y_i$, $1 \leq i \leq n$. Then τ is the finest equivalence σ on $S \times S$ which obeys (1) and is such that $(S \times S)/\sigma$ is a semilattice under

$$(a, b)\sigma \leq (c, d)\sigma \Leftrightarrow (a, b) \sigma (u, v) \text{ for some } u \in Sa \cap Sc, v \in bS \cap dS.$$

Remark. If $S = S^1$ is naturally quasisemilatticed then $(S \times S)/\sigma$ is a semilattice under the partial order in Theorem 3.8 if and only if $(a, b) \sigma (c, d)$ implies $(a \wedge_l u, b \wedge_r v) \sigma (c \wedge_l u, d \wedge_r v)$ for all $u, v \in S$ where, for example $a \wedge_l u$ denotes any element of S such that $S(a \wedge_l u) = Sa \cap Su$.

4. Naturally quasisemilatticed semigroups. If $S = S^1$ is a naturally quasisemilatticed semigroup then it is easy to see that an inverse semigroup T is separated over S , by a homomorphism θ , if and only if T is generated as an inverse semigroup and, for each $a, b \in S$,

$$a\theta a\theta^{-1}b\theta b\theta^{-1} = (a \wedge_r b)\theta(a \wedge_r b)\theta^{-1} \text{ if } (a \wedge_r b)S = aS \cap bS,$$

$$a\theta^{-1}a\theta b\theta^{-1}b\theta = (a \wedge_l b)\theta^{-1}(a \wedge_l b)\theta \text{ if } S(a \wedge_l b) = Sa \cap Sb.$$

It follows that there is a universal inverse semigroup $E(S)$ which is separated over S ; $E(S)$ is the quotient of $I(S)$ under the relations

$$aa^{-1}bb^{-1} = (a \wedge_r b)(a \wedge_r b)^{-1} \text{ if } (a \wedge_r b)S = aS \cap bS,$$

$$a^{-1}ab^{-1}b = (a \wedge_l b)^{-1}(a \wedge_l b) \text{ if } S(a \wedge_l b) = Sa \cap Sb.$$

In this section we shall give an explicit construction for $E(S)$, as the inverse hull of $S\rho$ under a shift representation ρ of S , and several coordinatisations of $E(S)$.

Throughout this section and the following ones we shall suppose that a choice of representatives has been made from the generators of the principal left

and right ideals of the naturally quasisemilatticed semigroup being considered; if $a, b \in S$ then $a \wedge_r b$ will denote the representative of the principal right ideal $aS \cap bS$ and $a \wedge_l b$ will denote the representative of the principal left ideal $Sa \cap Sb$. For each $a, b \in S$ we also choose elements $a *_r b$ and $a *_l b$ in S such that $a(a *_r b) = a \wedge_r b$, $(a *_l b)b = a \wedge_l b$.

Definition. Let $S = S^1$ be a naturally quasisemilatticed semigroup and let σ be an equivalence on $S \times S$. Then we shall say that σ is a *semilattice congruence* on $S \times S$ if $(S \times S)/\sigma$ is a semilattice under

$$(a, b)\sigma \leq (c, d)\sigma \iff (a, b)\sigma(u, v) \text{ for some } u \in Sa \cap Sc, v \in bS \cap dS.$$

Thus σ is a semilattice congruence if and only if, for every choice function on the generators of the principal left ideals and right ideals of S ,

$$(a, b)\sigma(c, d), (u, v)\sigma(x, y) \text{ implies } (a \wedge_l u, b \wedge_r v)\sigma(c \wedge_l x, d \wedge_r y).$$

Lemma 4.1. Let $S = S^1$ be a naturally quasisemilatticed semigroup and let σ be a semilattice congruence on S which obeys (1). Define a relation σ^* on $S \times S$ by

$$(a, b)\sigma^*(c, d) \iff (a, b)\sigma(c, d)\sigma(u, v) \text{ for some } u, v \in S$$

such that $av = cv$, $ub = ud$. Then σ^* is an equivalence on $S \times S$ which obeys (1) and

$$(3) \quad (a, b)\sigma^*(c, d) \iff (a, b)\sigma^*(x, y) \text{ for some } x \in Sa \cap Sc, y \in bS \cap dS;$$

in particular, σ^* is a shift.

Proof. First of all, σ^* is clearly reflexive and symmetric. Suppose that $(a, b)\sigma^*(c, d)$ and $(c, d)\sigma^*(e, f)$. Then there exist $x, y, u, v \in S$ such that $(a, b)\sigma(c, d)\sigma(u, v)$ with $av = cv$, $ub = ud$ and $(c, d)\sigma(e, f)\sigma(x, y)$ with $cy = ey$, $xd = xf$. Since σ is a semilattice congruence, $(a, b)\sigma(e, f)\sigma(u \wedge_l x, v \wedge_r y)$. Further, since $v \wedge_r y = v(v *_r y)$, $a(v \wedge_r y) = av(v *_r y) = cv(v *_r y) = c(v \wedge_r y)$ and similarly $c(v \wedge_r y) = e(v \wedge_r y)$; likewise $(u \wedge_l x)b = (u \wedge_l x)f$. Hence $(a, b)\sigma^*(e, f)$ and so σ^* is transitive.

Suppose now that $(a, xb)\sigma^*(c, xd)$. Then $(a, xb)\sigma(c, xd)\sigma(u, v)$ for some $u, v \in S$ such that $av = cv$, $uxb = uxd$. Then, since σ is a semilattice congruence $(a, xb)\sigma(u, x \wedge_r v) = (u, x(x *_r v))$ so that $(ax, b)\sigma(cx, d)\sigma(ux, x *_r v)$ by (1). Further,

$$ax(x *_r v) = a(x \wedge_r v) = ax(v *_r x) = cv(v *_r x) = cx(x *_r v) \text{ and}$$

$$(ux)b = u(xb) = u(xd) = (ux)d.$$

Hence, $(ax, b)\sigma^*(cx, d)$. The dual holds by symmetry so we get (1).

Next suppose that $(a, b)\sigma^*(c, d)$. Then it is easy to see from the definition

of σ^* that there exist $e \in Sa$, $f \in bS$ such that $(a, b) \sigma (c, d) \sigma (e, f)$ and $eb = ed$, $af = cf$. Since S is naturally quasisemilatticed and $eb = ed \in ebS \cap edS$, $eb = e(b \wedge_r d)t$ for some $t \in S$, and, similarly $af = s(a \wedge_l c)f$ for some $s \in S$. Because $(e \wedge_l a) \mathcal{L} e$, $f \mathcal{R} (f \wedge_r b)$ and, by Theorem 3.9, $\tau \subseteq \sigma$, these equations imply

$$(a, b) \sigma (e, b) \sigma (e, (b \wedge_r d)t) \quad \text{and} \quad (a, b) \sigma (s(a \wedge_l c), f).$$

Set

$$u' = s(a \wedge_l c) \wedge_l e, \quad v' = f \wedge_r (b \wedge_r d)t.$$

Then, since σ is a semilattice congruence and $(a, b) \sigma (s(a \wedge_l c), f) \sigma (e, (b \wedge_r d)t)$,

$$(a, b) \sigma (s(a \wedge_l c) \wedge_l e, f \wedge_r (b \wedge_r d)t) = (u', v').$$

Further

$$s(a \wedge_l c)v' = s(a \wedge_l c)f(f *_r (b \wedge_r d)t) = af(f *_r (b \wedge_r d)t) = av'$$

and similarly $u'(b \wedge_r d)t = u'b$.

Finally, since $(u', v') \leq (s(a \wedge_l c), (b \wedge_r d)t) \leq (a, b)$ in the natural quasi-order on $S \times S$ and each σ class is convex, the fact that $(a, b) \sigma (u', v')$ implies $(a, b) \sigma (s(a \wedge_l c), (b \wedge_r d)t)$. Hence we have shown

$$(a, b) \sigma (s(a \wedge_l c), (b \wedge_r d)t) \sigma (u', v') \quad \text{and} \quad av' = s(a \wedge_l c)v', \quad u'b = u'(b \wedge_r d)t;$$

that is $(a, b) \sigma^* (s(a \wedge_l c), (b \wedge_r d)t)$. Thus (3) holds.

Lemma 4.2. *Let $S = S^1$ be a naturally quasisemilatticed semigroup and let σ be an equivalence on $S \times S$ which obeys (1) and (3). Suppose that ρ is the corresponding shift representation of S . Then the inverse hull of $S\rho$ in $\mathcal{H}(S \times S)/\sigma$ is separated over S by ρ .*

Further the semilattice congruence σ_E defined by

$$(a, b) \sigma_E (c, d) \Leftrightarrow \rho_a^{-1} \rho_a \rho_b \rho_b^{-1} = \rho_c^{-1} \rho_c \rho_d \rho_d^{-1}$$

is contained in every semilattice congruence which contains σ .

Proof. Let $a, b \in S$; then $\Delta\rho_a = \{(x, ay)\sigma: x, y \in S\}$ and so, since σ obeys (3), $\Delta\rho_a \cap \Delta\rho_b = \{(x, (a \wedge_r b)y)\sigma: x, y \in S\} = \Delta\rho_a \wedge_r b$. Hence $\rho_a \rho_a^{-1} \rho_b \rho_b^{-1} = \rho_a \wedge_r b \rho_a^{-1} \wedge_r b$ and dually. Thus the inverse hull of $S\rho$ is separated over S by ρ .

By Lemma 3.4, σ_E is a semilattice congruence on $S \times S$. Suppose that π is also a semilattice congruence and that $\sigma \subseteq \pi$. Then

$$(a, b) \sigma_E (c, d) \text{ implies } (a, b) \pi (xc, dy), (c, d) \pi (ua, bv) \quad \text{for some } x, y, u, v \in S$$

and so, since π is a semilattice congruence, $(a, b) \pi (c, d)$. Hence $\sigma_E \subseteq \pi$.

It follows from Lemma 4.2 that, if σ is a semilattice congruence on $S \times S$ which obeys (1), then $\sigma_E^* \subseteq \sigma$. However σ need not equal σ_E^* . (For example, if S is cancellative with trivial group of units σ_E^* is always the identity while σ could be $S \times S$). However, if we take $\sigma = \tau$ then, since, by Theorem 3.8, τ is the smallest semilattice congruence which obeys (1), $\tau = \tau_E^*$. We can use this to find $E(S)$.

The next lemma is rather technical. It can be applied, among other things, to give necessary and sufficient conditions for embedding naturally quasisemilatticed semigroups in inverse semigroups.

Lemma 4.3. *Let $S = S^1$ be a semigroup and define an equivalence τ on $S \times S$ by $(a, b) \tau (c, d)$ if and only if there exist finite sets $x_0, \dots, x_n, y_0, \dots, y_n$ in S with $a = x_0, c = x_n, b = y_0, d = y_n$ and $x_{i-1}y_{i-1} \mathcal{L} x_i y_{i-1} \mathcal{R} x_i y_i, 1 \leq i \leq n$. Let $b = ua = cv, e = pd = fq$ and suppose there exist $x, y, \alpha, \beta, \gamma, \delta$ in S such that*

$$(u, a) \tau (xp, dy) \tau (\alpha, \beta) \quad \text{with } u\beta = xp\beta, \alpha a = \alpha dy$$

and

$$(c, v) \tau (xf, qy) \tau (\gamma, \delta) \quad \text{with } c\delta = xf\delta, \gamma v = \gamma qy.$$

Then

$$ab^{-1}c \leq de^{-1}f \text{ in the free inverse semigroup } I(S) \text{ on } S.$$

Proof. Let σ be defined on $S \times S$ by $(a, b) \sigma (c, d)$ if and only if $a^{-1}abb^{-1} = c^{-1}cdd^{-1}$ in $I(S)$. Then σ obeys (1) and $a \mathcal{L} c, b \mathcal{R} d$ implies $(a, b) \sigma (c, d)$. As in the proof of Theorem 3.7, this implies $\tau \subseteq \sigma$.

In $I(S)$:

$$\begin{aligned} ab^{-1}c &= aa^{-1}u^{-1}c = aa^{-1}u^{-1}uu^{-1}c \\ &= dy(dy)^{-1}(xp)^{-1}(xp)\beta\beta^{-1}u^{-1}c \quad \text{since } (u, a) \tau (xp, dy) \tau (\alpha, \beta) \\ &= dy(xpdy)^{-1}u\beta(u\beta)^{-1}c = dy(xfqy)^{-1}u\beta(u\beta)^{-1}c \\ &\leq dy(xfqy)^{-1}c \quad \text{since } u\beta(u\beta)^{-1} \text{ is idempotent.} \end{aligned}$$

Now, since $(xf, qy) \tau (\gamma, \delta)$ and $\tau \subseteq \sigma$,

$$(xf)^{-1}xfqy(qy)^{-1} = \gamma^{-1}\gamma\delta\delta^{-1}$$

so that

$$(xf)^{-1}xfqy(qy)^{-1} = (xf)^{-1}xf\gamma^{-1}\gamma\delta\delta^{-1}qy(qy)^{-1}$$

which implies $xfqy = xf\gamma^{-1}\gamma\delta\delta^{-1}qy$. Thus

$$\begin{aligned}
 ab^{-1}c &\leq dy(xfy^{-1}\gamma\delta\delta^{-1}qy)^{-1}c = dy(xf\delta\delta^{-1}\gamma^{-1}\gamma qy)^{-1}c \\
 &= dy(c\delta\delta^{-1}\gamma^{-1}\gamma qy)^{-1}c = dyy^{-1}q^{-1}\gamma^{-1}\gamma\delta\delta^{-1}c^{-1}c \\
 &= dyy^{-1}q^{-1}(xf)^{-1}xfqy(qy)^{-1} \text{ since } (c, v) \tau (xf, qy) \tau (\gamma, \delta) \text{ and } \tau \subseteq \sigma \\
 &= dyy^{-1}q^{-1}(xf)^{-1}xf = dyy^{-1}(fq)^{-1}x^{-1}xf \\
 &\leq de^{-1}f \text{ since } e = fq.
 \end{aligned}$$

Theorem 4.4. Let $S = S^1$ be a naturally quasiseamilatticed semigroup and let $\rho: S \rightarrow \mathcal{G}((S \times S)/\tau^*)$ be the shift representation of S associated with τ^* . Then the inverse hull of $S\rho$ in $\mathcal{G}((S \times S)/\tau^*)$ is isomorphic to the quotient $E(S)$ of $I(S)$ modulo the relations

$$aa^{-1}bb^{-1} = (a \wedge_r b)(a \wedge_r b)^{-1}, \quad a^{-1}ab^{-1}b = (a \wedge_l b)^{-1}(a \wedge_l b)$$

for all $a, b \in S$.

Proof. The proof of Lemma 4.2 shows that, for $a, b \in S$,

$$\rho_a \rho_a^{-1} \rho_b \rho_b^{-1} = \rho_{(a \wedge_r b)} \rho_{(a \wedge_r b)}^{-1}, \quad \rho_a^{-1} \rho_a \rho_b^{-1} \rho_b = \rho_{(a \wedge_l b)}^{-1} \rho_{(a \wedge_l b)}$$

so that the inverse hull T of $S\rho$ is a quotient of $E(S)$. More precisely, there is a unique homomorphism $\psi: E(S) \rightarrow T$ such that $\rho = \mu\psi$ where μ denotes the canonical homomorphism $S \rightarrow E(S)$.

Let $b = ua = cv$, $e = pd = fq$ and suppose that $\rho_a \rho_b^{-1} \rho_c \leq \rho_d \rho_e^{-1} \rho_f$. Then since, for example, $\Delta \rho_a \rho_b^{-1} \rho_c = \{(xu, ay)\tau^*: x, y \in S\}$, there exist $x, y \in S$ such that $(u, a)\tau^*(xp, dy)$ and $(u, a)\tau^* \rho_a \rho_b^{-1} \rho_c = (xp, dy)\tau^* \rho_d \rho_e^{-1} \rho_f$; that is $(c, v)\tau^*(xf, qy)$. The first and third of these relations are precisely those in Lemma 4.3. Hence, in $I(S)$, $ab^{-1}c \leq de^{-1}f$. Since $E(S)$ is a quotient of $I(S)$, we have there $a\mu b\mu^{-1}c\mu \leq d\mu e\mu^{-1}f\mu$. Therefore $(a\mu b\mu^{-1}c\mu)\psi = (d\mu e\mu^{-1}f\mu)\psi$ implies $a\mu b\mu^{-1}c\mu = d\mu e\mu^{-1}f\mu$ and so ψ is one-to-one; thus an isomorphism.

If $S = S^1$ is a semigroup whose principal left and right ideals form chains then the relations

$$aa^{-1}bb^{-1} = (a \wedge_r b)(a \wedge_r b)^{-1}, \quad a^{-1}ab^{-1}b = (a \wedge_l b)^{-1}(a \wedge_l b)$$

hold in $I(S)$. Hence we have

Theorem 4.5. Let $S = S^1$ be a semigroup whose principal left and right ideals form chains under inclusion and let ρ be the shift representation of S associated with τ^* . Then $I(S)$ is isomorphic to the inverse hull of $S\rho$ in $\mathcal{G}((S \times S)/\tau^*)$.

As a consequence of its description as a subsemigroup of $\mathcal{G}((S \times S)/\tau^*)$, the semigroup $E(S)$ admits several natural coordinatisations. Before giving these,

we show how $E(S)$ can be used to give necessary and sufficient conditions for embedding a naturally quasisemilatticed semigroup in an inverse semigroup.

Theorem 4.6. *Let $S = S^1$ be a naturally quasisemilatticed semigroup. Then S can be embedded in an inverse semigroup if and only if the canonical homomorphism $\mu: S \rightarrow E(S)$ is one-to-one.*

Proof. Let η be the canonical homomorphism $S \rightarrow I(S)$. Then, since μ can be factored through η , $\eta \circ \eta^{-1} \subseteq \mu \circ \mu^{-1}$. On the other hand, $a\mu = b\mu$ implies $a\mu a\mu^{-1}a\mu = b\mu b\mu^{-1}b\mu$ in $E(S)$ and so, by Lemma 4.2, $aa^{-1}a = bb^{-1}b$ in $I(S)$. Thus $a\mu = b\mu$ implies $a\eta = b\eta$. Hence $\eta \circ \eta^{-1} = \mu \circ \mu^{-1}$.

Theorem 4.7. *Let $S = S^1$ be a naturally quasisemilatticed semigroup and let U be the set of all 4-tuples (a, v, u, c) of elements of S with $ua = cv$. Define a binary operation on U by*

$$(a, v, u, c)(d, q, p, f) = (a(v *_r d), q(d *_r v), (p *_l c)u, (c *_l p)f).$$

Further define

$$(a, v, u, c) \sim (d, q, p, f) \iff \text{there exist } x, y, z, w \in S$$

such that $(u, a) \tau^(xp, dy)$, $(c, v) \tau^*(xf, qy)$, $(p, d) \tau^*(zu, aw)$, $(f, q) \tau^*(zc, vw)$.*

Then $\sim$ is a congruence on U and $U/\sim$ is isomorphic to $E(S)$.

Proof. First of all, it is easy to see that the multiplication described above is, in fact, a binary operation on U . Define $\psi: U \rightarrow E(S)$ by $(a, v, u, c)\psi = \rho_a \rho_b^{-1} \rho_c$ where $b = ua = cv$; since $E(S)$ is, by Theorem 3.2, an inverse semigroup of strong quotients of $S\rho$, ψ is onto. Further, easy calculation shows that $\Delta \rho_a \rho_b^{-1} \rho_c = \{(xu, ay)\tau^*: x, y \in S\}$, $\nabla \rho_a \rho_b^{-1} \rho_c = \{(xc, vy)\tau^*: x, y \in S\}$ and thus, because τ^* obeys (3), that

$$\Delta \rho_a \rho_b^{-1} \rho_c \rho_d \rho_e^{-1} \rho_f = \{(x(p *_l c)u, a(v *_r d)y)\tau^*: x, y \in S\},$$

$$\nabla \rho_a \rho_b^{-1} \rho_c \rho_d \rho_e^{-1} \rho_f = \{(x(c *_l p)f, q(d *_r v)y)\tau^*: x, y \in S\}.$$

Thus, because of the action of $\rho_a \rho_b^{-1} \rho_c \rho_d \rho_e^{-1} \rho_f$ we find

$$\begin{aligned} \rho_a \rho_b^{-1} \rho_c \rho_d \rho_e^{-1} \rho_f &= \rho_{(p *_l c)u} \rho_{(p *_l c)ua(v *_r d)(c *_l p)f} \\ &= [(a, v, u, c)(d, q, p, f)]\psi. \end{aligned}$$

Hence ψ is a homomorphism.

Finally, the proof of Theorem 4.4 shows that $\rho_a \rho_b^{-1} \rho_c = \rho_d \rho_e^{-1} \rho_f$ if and only if $(a, v, u, c) \sim (d, q, p, f)$. Hence $\sim$ is the congruence of ψ and so $U/\sim$ is isomorphic to $E(S)$.

Theorem 4.8. *Let $S = S^1$ be a naturally quasisemilatticed semigroup and let V be the set of all triples (a, b, c) of elements of S with $b \in Sa \cap cS$. Define a binary operation on V by*

$$(a, b, c)(d, e, f) = (a(b *_r cd), (e *_l cd)cd(cd *_r b), (cd *_l e)f)$$

and a relation $\sim$ on V by

$(a, b, c) \sim (d, e, f) \Leftrightarrow b = ua = cv, e = pd = fq$ and there exist $x, y, z, w \in S$ such that $(u, a) \tau^(xp, dy), (c, v) \tau^*(xf, qy), (p, d) \tau^*(zu, aw), (f, q) \tau^*(zc, vw)$. Then $\sim$ is a congruence on V and $V/\sim$ is isomorphic to $E(S)$.*

Proof. First

$$(e *_l cd)cd(cd *_r b) = (e *_l cd)b(b *_r cd) = (e *_l cd)ua(b *_r cd) \in Sa(b *_r cd)$$

while

$$(e *_l cd)cd(cd *_r b) = (cd *_l e)e(cd *_r b) = (cd *_l e)fq(cd *_r b) \in (cd *_l e)fS$$

so that the multiplication is a binary operation on V .

Define $\psi: E(S)$ by $(a, b, c)\psi = \rho_a \rho_b^{-1} \rho_c$. Then, by Theorem 3.2, ψ is onto and further, from the proof of that theorem, ψ is a homomorphism. Finally, as in the proof of Theorem 4.7, $\sim$ is the congruence of ψ so that $E(S) \approx V/\sim$.

The coordinatisation given in Theorem 4.8 reduces to that given by Eberhart and Selden when S is a subsemigroup of the positive reals ≤ 1 [5]. It has, however, the drawback that, when restricted to a Brandt $\mathcal{J}$ -class of $E(S)$ it does not give the usual Brandt multiplication. The latter can be recovered if we give $E(S)$ the coordinates described in the next theorem.

Theorem 4.9. *Let $S = S^1$ be a naturally quasisemilatticed semigroup and let W be the set of all triples (a, b, c) of elements of S with $b \in Sa \cap Sc$. Define a binary operation on W by*

$$(a, b, c)(d, e, f) = (a(c *_r d), b(c *_r d) \wedge_l e(d *_r c), f(d *_r c))$$

and a relation $\sim$ by

$(a, b, c) \sim (d, e, f) \Leftrightarrow b = ua = vc, e = pd = qf$ and there exist $x, y, z, w \in S$ such that $(u, a) \tau^(xp, dy), (v, c) \tau^*(xq, fy), (p, d) \tau^*(zu, aw), (qf) \tau^*(zv, cw)$. Then $\sim$ is a congruence on W and $E(S) \approx W/\sim$.*

Proof. Since

$$\begin{aligned} b(c *_r d) \wedge_l e(d *_r c) &= \{b(c *_r d) *_l e(d *_r c)\}qf(d *_r c) \\ &= \{e(d *_r c) *_l b(c *_r d)\}ua(c *_r d) \end{aligned}$$

where $b = ua = vc$, $e = pd = qf$, the multiplication described is, in fact, a binary operation on W .

Define $(a, b, c)\psi = ap(bp)^{-1}vp$ if $b = vc$. Then, firstly, ψ is well defined.

For, if $b = vc = wc$, then

$$\begin{aligned} ap(bp)^{-1}vp &= ap(vpcp)^{-1}vp = apcp^{-1}vp^{-1}vp \\ &= apcp^{-1}vp^{-1}vpcpcp^{-1} \quad \text{since idempotents commute} \\ &= ap(vc)p^{-1}(vc)pcp^{-1} = ap(wc)p^{-1}(wc)pcp^{-1} = ap(bp)^{-1}wp. \end{aligned}$$

Next we show that ψ is a homomorphism of W onto $E(S)$; the onto-ness is obvious.

Since $(a, b, c)\psi(d, e, f)\psi = (ap(bp)^{-1}vp)(dp(ep)^{-1}qp)$, it follows from the multiplication in $\mathcal{G}((S \times S)/\tau^*)$ that

$$(a, b, c)\psi(d, e, f)\psi = \{a(c *_r d)\}p\{p \wedge_l v\}(d \wedge_r c)\}p^{-1}\{v *_l p\}q\}p.$$

On the other hand, from the multiplication in W ,

$$\begin{aligned} \{(a, b, c)(d, e, f)\}\psi \\ = \{a(c *_r d)\}p\{b(c *_r d) \wedge_l e(d *_r c)\}p^{-1}\{b(c *_r d) *_l e(d *_r c)\}q\}p. \end{aligned}$$

Since S is naturally quasisemilatticed,

$$(p \wedge_l v)(d \wedge_r c) \mathcal{L} \{p(d \wedge_r c) \wedge_l v(d \wedge_r c)\} = e(d *_r c) \wedge_l b(c *_r d)$$

so there exist $x, z \in S$ such that

$$\begin{aligned} (p \wedge_l v)(d \wedge_r c) &= x\{b(c *_r d) \wedge_l e(d *_r c)\}, \\ z\{(p \wedge_l v)(d \wedge_r c)\} &= b(c *_r d) \wedge_l e(d *_r c). \end{aligned}$$

Hence, working with x alone,

$$((p \wedge_l v)(d \wedge_r c), 1) \tau^* (x\{b(c *_r d) \wedge_l e(d *_r c)\}, 1)$$

so that, since τ^* is a shift and

$$(p \wedge_l v)(d \wedge_r c) = (p *_l v)ua(c *_r d) = (v *_l p)qf(d *_r c),$$

$$\begin{aligned} b(c *_r d) \wedge_l e(d *_r c) &= \{e(d *_r c) *_l b(c *_r d)\}ua(c *_r d) \\ &= \{b(c *_r d) *_l e(d *_r c)\}qf(d *_r c), \end{aligned}$$

we get

$$((p *_l v)u, a(c *_r d)) \tau^* (x\{e(d *_r c) *_l b(c *_r d)\}u, a(c *_r d)),$$

$$((v *_l p)q, f(d *_r c)) \tau^* (x\{b(c *_r d) *_l e(d *_r c)\}q, f(d *_r c)).$$

Hence, by Lemma 4.3,

$$(a, b, c) \psi (d, e, f) \psi \leq [(a, b, c)(d, e, f)] \psi.$$

Operating with z gives the reverse inequality so that ψ is a homomorphism.

Finally, if $b = ua = vc$, $e = pd = qf$, Lemma 4.3 and the definition of ρ shows that

$$(a, b, c)\psi = (d, e, f)\psi \Leftrightarrow (a, b, c) \sim (d, e, f).$$

Hence $E(S) \approx W/\sim$.

The congruences in Theorems 4.7, 4.8, 4.9, and thus the coordinatisations for $E(S)$, undergo considerable simplification in two cases: (i) S is cancellative; the results for this case are stated in Theorem 6.2. (ii) $\mathcal{D}$ is trivial on S ; in this case $\tau = \tau^* = \tau_0$ is a semilattice congruence on $S \times S$ and the congruences reduce to

$$(a, v, u, c) \sim (d, q, p, f) \text{ in } U \Leftrightarrow (u, a) \tau_0 (p, d), (c, v) \tau_0 (f, q),$$

$$(a, b, c) \sim (d, e, f) \text{ in } V \Leftrightarrow (u, a) \tau_0 (p, d), (c, v) \tau_0 (f, q)$$

$$\text{where } b = ua = cv, e = pd = fq,$$

$$(a, b, c) \sim (d, e, f) \text{ in } W \Leftrightarrow (u, a) \tau_0 (p, d), (v, c) \tau_0 (q, f)$$

$$\text{where } b = ua = vc, c = pd = qf.$$

To end this section, we give an example to show how the coordinatisation in Theorem 4.9 gives rise to the Brandt multiplication in Brandt $\mathcal{J}$ -classes of $E(S)$. Suppose that $S \times S^1$ is a naturally quasisemilatticed cancellative semigroup on which $\mathcal{J}$ is trivial. Then it follows from Theorem 5.2 that, in $E(S) = W/\sim$,

$$J_b = \{(a, b, c) : b \in Sa \cap Sc\}$$

is a $\mathcal{J}$ -class for each $b \in S$: in this case $\sim$ is, in fact, the identity congruence. By Theorem 4.9,

$$(a, b, c)(d, b, f) = (a(c *_r d), b(c *_r d) \wedge_l b(d *_r c), f(d *_r c)).$$

This belongs to J_b if and only if $b = b(c *_r d) \wedge_l b(d *_r c)$. But the latter implies $b \in Sb(c *_r d)S \subseteq SbS$ and $b \in Sb(d *_r c)S \subseteq SbS$ whence, since $\mathcal{J}$ is trivial and S is cancellative, $(c *_r d) = 1 = (d *_r c)$; thus $c = d$. Hence, modulo the ideal generated by J_b ,

$$(a, b, c)(d, b, f) = \begin{cases} (a, b, f) & \text{if } c = d, \\ 0 & \text{otherwise.} \end{cases}$$

This is just the multiplication in the Brandt semigroup

$$\mathcal{M}^0(\{1\}; X, X, \Delta) \quad \text{where } X = \{x \in S : b \in Sx\}.$$

5. Green's relations and congruences on $E(S)$. In this section $S = S^1$ denotes a naturally quasisemilatticed semigroup and $E(S)$ denotes the quotient of $I(S)$, modulo the relations

$$aa^{-1}bb^{-1} = (a \wedge_r b)(a \wedge_r b)^{-1}, \quad a^{-1}ab^{-1}b = (a \wedge_l b)^{-1}(a \wedge_l b)$$

for all $a, b \in S$, regarded as a subsemigroup of $\mathcal{I}((S \times S)/\tau^*)$. The results are easily translated into the coordinatised forms of $E(S)$.

Lemma 5.1. *Let $\rho_a \rho_b^{-1} \rho_c \in E(S)$ where $b = ua = cv$. Then*

- (i) $(\rho_a \rho_b^{-1} \rho_c)^{-1} (\rho_a \rho_b^{-1} \rho_c) = \rho_c^{-1} \rho_c \rho_v \rho_v^{-1}$,
- (ii) $(\rho_a \rho_b^{-1} \rho_c) (\rho_a \rho_b^{-1} \rho_c)^{-1} = \rho_u^{-1} \rho_u \rho_a \rho_a^{-1}$.

Theorem 5.2. *Let $\rho_a \rho_b^{-1} \rho_c, \rho_d \rho_e^{-1} \rho_f \in E(S)$ where $b = ua = cv, e = pd = fq$.*

- (i) $\rho_a \rho_b^{-1} \rho_c \mathcal{L} \rho_d \rho_e^{-1} \rho_f \Leftrightarrow (c, v) \tau (f, q)$.
- (ii) $\rho_a \rho_b^{-1} \rho_c \mathcal{R} \rho_d \rho_e^{-1} \rho_f \Leftrightarrow (u, a) \tau (p, d)$.
- (iii) $\rho_a \rho_b^{-1} \rho_c \mathcal{H} \rho_d \rho_e^{-1} \rho_f \Leftrightarrow (u, a) \tau (p, d), (c, v) \tau (f, q)$.
- (iv) $\rho_a \rho_b^{-1} \rho_c \mathcal{D} \rho_d \rho_e^{-1} \rho_f \Leftrightarrow b \mathcal{D} e$.
- (v) $\rho_a \rho_b^{-1} \rho_c \leq \rho_d \rho_e^{-1} \rho_f \Leftrightarrow b \leq e$.

Proof. (i)

$$\begin{aligned} \rho_a \rho_b^{-1} \rho_c \mathcal{L} \rho_d \rho_e^{-1} \rho_f &\Leftrightarrow (\rho_a \rho_b^{-1} \rho_c)^{-1} (\rho_a \rho_b^{-1} \rho_c) = (\rho_d \rho_e^{-1} \rho_f)^{-1} (\rho_d \rho_e^{-1} \rho_f) \\ &\Leftrightarrow \rho_c^{-1} \rho_c \rho_v \rho_v^{-1} = \rho_f^{-1} \rho_f \rho_q \rho_q^{-1} \Leftrightarrow (c, v) \tau (f, q) \end{aligned}$$

since, by Theorem 3.8 and Lemma 3.4, $(S \times S)/\tau$ is the semilattice of idempotents of $E(S)$.

(ii) is dual to (i) while (iii) is immediate from (i) and (ii).

(iv) If $\rho_a \rho_b^{-1} \rho_c \mathcal{D} \rho_d \rho_e^{-1} \rho_f$ then $\rho_a \rho_b^{-1} \rho_c \mathcal{L} \rho_x \rho_y^{-1} \rho_z \mathcal{R} \rho_d \rho_e^{-1} \rho_f$ for some $x, y, z \in S$ with $y = rx = zs$. By (i) and (ii), these imply $(c, v) \tau (z, s), (r, x) \tau (p, d)$. Hence, from the definition of τ , $b = cv \mathcal{D} zs = rx \mathcal{D} pd = e$.

Conversely, if $b \mathcal{D} e$ then, for some $t \in S$, $b \mathcal{L} t \mathcal{R} e$. Hence there exist $\alpha, \beta, \gamma, \delta \in S$ such that

$$b = \alpha t, \quad t = \beta b = e \gamma, \quad e = t \delta;$$

thus $e = \beta b \delta$. Let $g = \beta u$, $x = a \delta$ and set $y = gx$, $z = f$; so $y = e = zq$. Then $ua = b \mathcal{L} t = \beta ua = ga$, $t \mathcal{R} e = t \delta = \beta ua \delta = gx$. That is, $ua \mathcal{L} ga \mathcal{R} gx$ which implies $(u, a) \tau (g, x)$. Hence, by (i), (ii),

$$\rho_a \rho_b^{-1} \rho_c \mathcal{R} \rho_x \rho_y^{-1} \rho_f \mathcal{L} \rho_f^{-1} \rho_f \rho_q \rho_q^{-1} \mathcal{L} \rho_d \rho_e^{-1} \rho_f.$$

Thus $\rho_a \rho_b^{-1} \rho_c \mathcal{D} \rho_d \rho_e^{-1} \rho_f$.

If $\rho_a \rho_b^{-1} \rho_c \in E(S) \rho_d \rho_e^{-1} \rho_f \notin E(S)$ then $\rho_a \rho_b^{-1} \rho_c \mathcal{R} \rho_x \rho_y^{-1} \rho_z$ and $\rho_x \rho_y^{-1} \rho_z \in E(S) \rho_d \rho_e^{-1} \rho_f$ for some $x, y, z \in S$ with $y = rx = zs$. Since $(S \times S)/\tau$ is the semilattice of idempotents $E(S)$, these relations imply $(u, a) \tau (r, x)$ and $(z, s) \tau (z \wedge_1 f, s \wedge_1 q)$. Hence $b = ua \mathcal{D} rx = y$ and $y = zs \mathcal{D} (z \wedge_1 f)(s \wedge_1 q) = (z *_1 f)(q *_1 s)$ which implies $b \in Sfqs = SeS$.

Conversely, if $b \in SeS$, $\rho_b \in E(S)\rho_e E(S)$ and so, since $\rho_a \rho_b^{-1} \rho_c \mathcal{D} \rho_b$ and $\rho_a \rho_e^{-1} \rho_f \mathcal{D} \rho_e$, $\rho_a \rho_b^{-1} \rho_c \leq_g \rho_a \rho_e^{-1} \rho_f$.

Corollary 5.3. *Let I be an ideal of S and set $I^* = \{\rho_a \rho_b^{-1} \rho_c \in E(S) : b \in I\}$. Then I^* is an ideal of $E(S)$ and each ideal of $E(S)$ has this form.*

Corollary 5.4. *If S has a kernel, so has $E(S)$; the kernel of $E(S)$ is bisimple if the kernel of S is a $\mathcal{D}$ -class of S (even if $\text{Ker } S$ is not bisimple).*

An equivalence relation β on the set E of idempotents of an inverse semigroup T is called a normal partition if there is a congruence ρ on T such that $\beta = \rho \cap (E \times E)$. Reilly and Scheiblich [14] have shown that an equivalence β on E is a normal partition if and only if

- (i) $(a, b) \in \beta$, $(c, d) \in \beta$ implies $(a \wedge c, b \wedge d) \in \beta$,
- (ii) $(a, b) \in \beta$ implies $(x^{-1}ax, x^{-1}bx) \in \beta$ for all $a, b, c, d \in E$, $x \in S$.

It is shown in [14] that the mapping $\Theta: \sigma \rightarrow \sigma \cap (E \times E)$ is a complete lattice homomorphism of the complete lattice Λ of congruences on T onto the complete lattice of normal partitions on E . Thus each Θ -class is a complete sublattice of Λ ; in particular, it has a greatest and a least element; if β is a normal partition on E we shall denote the greatest and least elements of $\beta\Theta^{-1}$ by β^V and $\beta^\wedge$ respectively.

Theorem 5.5. *The lattice of Θ -classes of congruences of $E(S)$ is isomorphic to the lattice of semilattice congruences on $S \times S$ which obey (1).*

If β is the normal partition corresponding to the semilattice congruence σ on $S \times S$ then $E(S)/\beta^V$ is isomorphic to the inverse hull of Sp in $\mathcal{H}((S \times S)/\sigma)$, where ρ is the shift representation of S associated with σ .

Proof. Since every homomorphic image of $E(S)$ is separated over S , it is immediate from Theorem 3.6 and Lemma 3.4 that the normal partitions on $E(S)$ are precisely the shift semilattice congruences on $S \times S$. Further, from its definition, $E(S)/\beta^V$ is, up to isomorphism, the only fundamental homomorphic image of $E(S)$ with normal partition β . Hence the rest of the theorem follows from Theorem 3.6.

As a consequence of Theorem 5.5, we can regard the normal partitions β of $E(S)$, and the corresponding semigroups $E(S)/\beta^V$, as known. Although Theorem 3.8 gives a method for constructing all shift semilattice congruences on $S \times S$ from equivalences on S , it does not give a unique method of construction. Hence the situation is not entirely satisfactory. However, in the case when S is the positive cone of an archimedean ordered group, it is easy to see that congruences on S which obey the conditions of Theorem 3.8 are the Rees factor congruences on S . This, together with the fact that a semigroup, with a left and right zero, has a zero, gives Theorem 4.4 of [5].

6. **The cancellative case.** If the semigroup $S \times S^1$ is cancellative, the theory in the previous two sections undergoes considerable simplification.

Lemma 6.1. *Let $S = S^1$ be a cancellative naturally quasisemilatticed semigroup. Then $(a, b) \tau (c, d) \Leftrightarrow a = gc, b = db$ for some units $g, h \in S$ while τ^* is the identity on $S \times S$.*

Hence the results in Theorems 4.7, 4.8, 4.9 reduce to the results in Theorem 6.2.

Theorem 6.2. *Let $S = S^1$ be a cancellative naturally quasisemilatticed semigroup.*

(i) *Let $U = \{(a, v, u, c) \in S \times S \times S \times S : ua = cv\}$; define*

$$(a, v, u, c)(d, q, p, f) = (a(v *_r d), q(d *_r v), (p *_l c)u, (c *_l p)f)$$

and

$$(a, v, u, c) \sim (d, q, p, f) \Leftrightarrow u = gp, c = gf, a = db, v = qb$$

for some units $g, h \in S$.

Then $\sim$ is a congruence on U and $E(S) \approx U/\sim$.

(ii) *Let $V = \{(a, b, c) \in S \times S \times S : b \in Sa \cap cS\}$; define*

$$(a, b, c)(d, e, f) = (a(b *_r cd), (e *_l cd)cd(cd *_r b), (cd *_l e)f)$$

and

$$(a, b, c) \sim (d, e, f) \Leftrightarrow a = db, b = geh, c = gf \quad \text{for some units } g, h \in S.$$

Then $\sim$ is a congruence on V and $E(S) \approx V/\sim$.

(iii) *Let $W = \{(a, b, c) \in S \times S \times S : b \in Sa \cap Sc\}$; define*

$$(a, b, c)(d, e, f) = (a(c *_r d), b(c *_r d) \wedge_l e(d *_r c), f(d *_r c))$$

and

$$(a, b, c) \sim (d, e, f) \Leftrightarrow a = db, b = geh, c = fh \quad \text{for some units } g, h \in S.$$

Then $\sim$ is a congruence on W and $E(S) \approx W/\sim$.

Definition. An inverse semigroup T is an *inverse semigroup of quotients* of a subsemigroup $S = S^1$ if each element of T is of the form $ab^{-1}c$ with $a, b, c \in S$.

If $S = S^1$ is a cancellative semigroup in which the sets of principal left and right ideals form chains under inclusion then it follows from Theorem 4.5 that $I(S)$ is a semigroup of quotients of S . In fact the converse is also true. To prove this, we consider a type of representation which generalises the shift representation considered earlier.

A subset H of a semigroup $S = S^1$ is called *right consistent* if $ab \in H$

implies $a \in H$. Suppose that H is a right consistent subset of a cancellative semigroup $S = S^1$ and for each $a \in S$, define

$$(6.1) \quad x\rho_a = xa \quad \text{for each } x \in H \text{ such that } xa \in H.$$

Then the proof of the following lemma is straightforward.

Lemma 6.3. *Let $S = S^1$ be a cancellative semigroup and let H be a right consistent subset of S . Then the mapping $\rho: a \rightarrow \rho_a$ is a representation of S by one-to-one partial transformations of H .*

Lemma 6.4. *Let $S = S^1$ be a cancellative semigroup and let ω be the shift representation S defined by $(x, ay)\omega_a = (xa, y)$ for all $x, y \in S$. Then $\Delta\omega_a^{-1}\omega_a\omega_b\omega_b^{-1} = Sa \times bS$.*

Theorem 6.5. *Let $S = S^1$ be a cancellative semigroup. Then the following statements are equivalent.*

- (i) $I(S)$ is an inverse semigroup of strong quotients of S .
- (ii) $I(S)$ is an inverse semigroup of quotients of S .
- (iii) The sets of principal left and right ideals of S form chains under inclusion.
- (iv) S is naturally quasisemilatticed and $I(S)$ is naturally isomorphic to $E(S)$.
- (v) S is naturally quasisemilatticed and $I(S)$ is separated over S .
- (vi) for each $a, b \in S$ there exist $x, y \in S$ such that

$$aa^{-1}bb^{-1} = xx^{-1}, \quad a^{-1}ab^{-1}b = y^{-1}y$$

in $I(S)$.

Proof. Clearly (i) implies (ii) and (iii) implies (iv) implies (v) implies (vi) so we need only show that (ii) implies (iii) and (vi) implies (i).

(ii) $\Rightarrow$ (iii). Let $a, b \in S$ and set $H = \{x \in S: a^2 \in xS \text{ or } ab \in xS\}$. Then H is easily seen to be right consistent; let ρ be the corresponding representation of S . Then $a \in \Delta\rho_a\rho_a^{-1} \cap \Delta\rho_b\rho_b^{-1}$ so that $\rho_a\rho_a^{-1}\rho_b\rho_b^{-1}$ is nonzero. By hypothesis, $\rho_a\rho_a^{-1}\rho_b\rho_b^{-1} = \rho_x\rho_x^{-1}\rho_y$ for some $x, y, z \in S$. Thus $a \in \rho_a\rho_a^{-1}\rho_b\rho_b^{-1}$ implies $ax = uy$ for some $u \in H$ and so $a\rho_x\rho_x^{-1}\rho_y = uz$. Since $\rho_a\rho_a^{-1}\rho_b\rho_b^{-1}$ is idempotent, $a = uz$ and so $uy = ax = uzx$ whence, because S is cancellative, $y = zx$.

Now let ω be the representation of S by one-to-one partial transformations of $S \times S$ given in Lemma 6.4. Since, in $I(S)$, $aa^{-1}bb^{-1} = xx^{-1}z^{-1}z$, we have

$$S \times (aS \cap bS) = \Delta\omega_a\omega_a^{-1}\omega_b\omega_b^{-1} = \Delta\omega_z^{-1}\omega_z\omega_x\omega_x^{-1} = Sz \times xS.$$

Thus z is a unit in S and so, in $I(S)$, $z^{-1}z = 1$. It follows that $\rho_a\rho_a^{-1}\rho_b\rho_b^{-1} = \rho_x\rho_x^{-1}$ and so $a \in \Delta\rho_x$; this implies $a^2 \in axS$ or $ab \in axS$. Hence $a \in xS = aS \cap bS$ or $b \in xS = aS \cap bS$; that is $aS \subseteq bS$ or $bS \subseteq aS$. This shows that the

set of principal right ideals of S is a chain under inclusion. Dual arguments show that the same is true for principal left ideals so (iii) is proven.

(vi) $\Rightarrow$ (i). Suppose $aa^{-1}bb^{-1} = cc^{-1}$ in $I(S)$; then $\omega_a\omega_a^{-1}\omega_b\omega_b^{-1} = \omega_c\omega_c^{-1}$ and so, by Lemma 6.4, $aS \cap bS = cS$. Hence the set of principal right ideals of S is a semilattice under inclusion and, in $I(S)$, $aa^{-1}bb^{-1} = (a \wedge b)(a \wedge b)^{-1}$. The dual clearly holds, so we may invoke Theorem 3.2 to conclude that $I(S)$ is an inverse semigroup of strong quotients of S .

Theorem 6.5 can be applied to characterise the positive cones of right ordered groups among semigroups.

Theorem 6.6. *Let $S = S^1$ be a semigroup. Then the following are equivalent.*

- (i) S is positive cone of a right ordered group.
- (ii) each element of $I(S)$ has the form $xy^{-1}z$ for a unique triple $x, y, z \in S$ with $y \in Sx \cap zS$.

Proof. (i) $\Rightarrow$ (ii). Since S is cancellative and the sets of principal left and right ideals of S are chains under inclusion, it follows from Theorem 6.5 that each element of $I(S)$ has the form $xy^{-1}z$ where $y \in Sx \cap zS$. Further, by Theorem 6.2, $xy^{-1}z = ab^{-1}c$ if and only if $x = a$, $y = b$, $z = c$ because S has trivial group of units. Hence (ii) holds.

(ii) $\Rightarrow$ (i). Suppose that $ux = uy$ in S and define σ on $S \times S$ by

$$(a, b) \sigma (c, d) \Leftrightarrow b^{-1}(ab) = d^{-1}(cd) \text{ in } I(S);$$

by Proposition 2.2, σ obeys (1). Then, by (1), $(u, x) \sigma (u, y)$ so that $x^{-1}(ux) = y^{-1}(uy)$ in $I(S)$; whence $(ux)^{-1}x = (uy)^{-1}y$. By the uniqueness hypothesis in (ii), this gives $x = y$.

The dual also holds, hence S is cancellative and so, by Theorem 6.5 and Theorem 6.2, the sets of principal left and right ideals form chains under inclusion and further S has trivial group of units. Hence S is the positive cone of a right ordered group.

6. Some examples. 1. Let S be the semigroup of all 2×2 real matrices of the form $\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}$, $a > 0$, $b \geq 0$. Then the sets of principal left and right ideals of S form chains under inclusion. S has group of units

$$H_1 = \left\{ \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} : a > 0, b = 0 \right\}$$

and kernel

$$K = \left\{ \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} : a > 0, b > 0 \right\}.$$

The kernel is not bisimple but is a $\mathcal{D}$ -class of S .

Since S consists of a group of units and a kernel, it follows from Theorem 6.6 and Proposition 5.2 that the same is true of $I(S)$. In fact, since the kernel of S is a $\mathcal{D}$ -class of S , Proposition 5.2 shows that the kernel of $I(S)$ is a $\mathcal{D}$ -class of $I(S)$ and thus, by [2, Example 2.3.6], is a bisimple inverse semigroup.

2. Let S be the semigroup of all 2×2 real matrices of the form $\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}$, $a, b > 0$ or $b = 0$, $a \geq 1$. Then the sets of principal left and right ideals of S form chains under inclusion. S consists of the disjoint union

$$P = \left\{ \begin{pmatrix} a & 0 \\ 1 & 1 \end{pmatrix} : a \geq 1 \right\},$$

which is isomorphic to the semigroup of reals ≥ 1 which was considered in [5], and a kernel K

$$K = \left\{ \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} : a, b > 0 \right\}.$$

Since S has a kernel, so has $I(S)$; in fact $I(S)$ is the disjoint union of $I(P)$ and its kernel which is a simple, but not bisimple, inverse semigroup. It follows, from Theorem 5.2, that each $\mathcal{D}$ -class of $\text{Ker } I(S)$ contains a unique element of S . Thus the $\mathcal{D}$ -classes of $\text{Ker } I(S)$ have S as a transversal but no $\mathcal{D}$ -class of $\text{Ker } I(S)$ is a subsemigroup. Thus $\text{Ker } I(S)$ is a different type of simple inverse semigroup from those considered by Munn [11].

The semigroup S in this example is the positive cone of a right order on the group of all 2×2 real matrices of the form $\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}$, $a > 0$. Similar examples can be obtained by considering $\mathcal{J}$ -classes in the positive cones of right ordered groups which are not ordered.

3. Let S be the positive cone of the l -group. Then, in S , $\mathcal{H} = \mathcal{J}$ and so, by Proposition 5.2, $\mathcal{D} = \mathcal{J}$ in $E(S)$. Regard $E(S)$ as $V/\sim$ where V is as in Theorem 6.2; then $\sim$ is the identity so $E(S) = V$. The idempotents in the $\mathcal{J} = \mathcal{D}$ -class containing (b, b, b) are the triples $\{(a, b, u) : b = ua\}$. Further, from Lemma 5.1,

$$(a, b, u) \leq (c, b, v) \Leftrightarrow u \in Sv, a \in cS.$$

Hence if this inequality holds, $ua = vc = b$, $u = xv$, $a = cy$ for some $x, y \in S$. This implies, $vc = ua = xvcy$ and, since $Sp = pS$ for each $p \in S$, $vcy = y'vc$ for some $y' \in S$, so $vc = xy'vc$. Since S is cancellative with trivial unit group this gives $x = y' = y = 1$. Hence the idempotents in each $\mathcal{J}$ -class are trivially ordered. Thus each $\mathcal{J}$ -class is Brandt and so $E(S)$ is completely semisimple.

4. Let $S = S^1$ be the cyclic monoid of index r and period m [2, p. 20]; thus

$$S = \{a, a^2, \dots, a^{r-1}, a^r, \dots, a^{r+m-1}\}^1.$$

Then the sets of principal left and right ideals of S are chains under inclusion so that Theorem 4.5 may be applied to describe $I(S)$.

It is easy to calculate, using Theorem 3.7 that, on $S \times S$,

$$(a^u, a^v) \tau (a^p, a^q) \Leftrightarrow u = p, v = q \quad \text{on} \quad u + v, p + q \geq r$$

and thus that

$$(a^u, a^v) \tau^* (a^p, a^q) \Leftrightarrow u = p, v = q \quad \text{or} \quad u + v, p + q \geq r \quad \text{and}$$

$$ea^u = ea^p, ea^v = ea^q \quad \text{where} \quad e^2 = e \neq 1.$$

It follows from this that $I(S)$ can be identified with the set of triples $\{(i, k, j) : i, j \leq k \leq r-1\}$ together with the kernel $\{a^r, \dots, a^{r+m-1}\}$ of S . Hence $I(S)$ has order $m + \sum_1^r k^2 = m + r(r+1)(2r+1)/6$. It is easy to see that any non-trivial congruence on $I(S)$ induces a nontrivial congruence on S . Hence, up to isomorphism, $I(S)$ is the only inverse semigroup generated by S .

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