ON RATIONALLY CONVEX HULLS

BY

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ABSTRACT. For a compact set $X \subseteq \mathbb{C}^n$, let $h_r(X)$ denote the rationally convex hull of X ; let Δ denote the closed unit disk in $\mathbb{C}$; and, following Wermer, for a compact set S such that $\partial\Delta \subseteq S \subseteq \Delta$ let $X_S = S \times S \cap \partial\Delta^2$. It is shown that

$$h_r(X_S) = \{(z, w) \in S \times S \mid u_S(z) + u_S(w) \leq 1\}$$

where u_S is a function on S which, in the case when S is smoothly bounded, is specified by requiring $u_S|_{\partial\Delta} = 0$, $u_S|_{\partial S \setminus \partial\Delta} = 1$ and $u_S|_{\text{int } S}$ harmonic. In particular this provides a precise description of $h_r(X)$ for certain sets $X \subseteq \mathbb{C}^2$ with the property that $h_r(X) \neq X$, but $h_r(X)$ does not contain analytic structure (as Wermer demonstrated, there are S for which $X = X_S$ has these properties). Furthermore, it follows that whenever $h_r(X_S) \neq X_S$ then there is a Gleason part of $h_r(X_S)$ for the algebra $R(X_S)$ with positive four-dimensional measure. In fact, the Gleason part of any point $(z, w) \in h_r(X_S) \cap \text{int } \Delta^2$ such that $u_S(z) + u_S(w) < 1$ has positive four-dimensional measure.

A similar idea is then used to construct a compact rationally convex set $Y \subseteq \mathbb{C}^2$ such that each point of Y is a peak point for $R(Y)$ even though $R(Y) \neq C(Y)$; namely, $Y = \tilde{X}_T = \{(z, w) \in \mathbb{C}^2 \mid z \in T, |w| = \sqrt{1 - |z|^2}\}$ where T is any compact subset of $\text{int } \Delta$ having the property that $R(T) \neq C(T)$ even though there are no nontrivial Jensen measures for $R(T)$. This example is more concrete than the original example of such a uniform algebra which was discovered by Cole. It is possible to show, for instance, that $R(\tilde{X}_T)$ is not even in general locally dense in $C(\tilde{X}_T)$, a possibility which had been suggested by Stuart Sidney.

Finally, smooth examples (3-spheres in $\mathbb{C}^6$) with the same pathological properties are obtained from X_S and $\tilde{X}_T$.

Introduction. Let X be a compact subset of $\mathbb{C}^n$. $R_0(X)$ is the algebra of all

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Received by the editors November 8, 1972.

AMS (MOS) subject classifications (1970). Primary 32E99, 46J10.

Key words and phrases. Rationally convex hull, analytic structure, peak point.

⁽¹⁾ This paper contains the results of the author's Ph.D. thesis written at Brown University under the direction of John Wermer, to whom the author is deeply grateful. This research was supported in part by the National Science Foundation under Grant GP-28574.

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rational functions P/Q on $\mathbb{C}^n$ with P, Q polynomials and $Q \neq 0$ on X . $R(X)$ is the uniform closure of $R_0(X)$ in $C(X)$. $b_r(X)$, the rationally convex hull of X , is the set $\{z \in \mathbb{C}^n \mid \text{for all polynomials } P \text{ such that } P(z) = 0, P \text{ has a zero on } X\}$.

While every $X \subseteq \mathbb{C}$ is rationally convex, i.e., $b_r(X) = X$, in $\mathbb{C}^2$ even a compact arc may fail to be rationally convex. (See the example of Wermer [1] and Rudin [2] or [3, p. 70]. Wermer's example is in $\mathbb{C}^3$.) The rationally convex hull of a general "circled" compact set in $\mathbb{C}^n$ has been described very thoroughly by de Leeuw [4], [5] and Rossi [6]. A set $X \subseteq \mathbb{C}^n$ is "circled" if, whenever $(z_1, \dots, z_n) \in X$, $(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ and $|\lambda_j| = 1$ for $1 \leq j \leq n$, it follows that $(\lambda_1 z_1, \dots, \lambda_n z_n) \in X$. Suppose that X is a "circled" compact set in $\mathbb{C}^n$, that $\text{int } X$ is a dense connected subset of X , and that $\text{int } X$ meets every coordinate axis $\{z_j = 0\}$ which X meets. In this case their result is that

$$b_r(X) = \text{closure} \{ (z_1, \dots, z_n) \in \mathbb{C}^n \mid (\log |z_1|, \dots, \log |z_n|) \in \text{linear convex hull} \{ (s_1, \dots, s_n) \in \mathbb{R}^n \mid (e^{s_1}, \dots, e^{s_n}) \in X \} \}.$$

The general case follows by taking a decreasing intersection of such sets (see also [3, pp. 71–76]).

In the first section of this paper we describe $b_r(X)$ for certain sets X contained in the boundary of the bicylinder which were first discussed by Wermer [7]. Given a compact set S contained in the closed unit disk Δ and containing the boundary of the disk $\partial\Delta$, let

$$X_S = \{(z, w) \in \mathbb{C}^2 \mid |z| = 1 \text{ and } w \in S, \text{ or } |w| = 1 \text{ and } z \in S\}.$$

A set $Y \subseteq \mathbb{C}^n$ is said to "contain analytic structure" if there is a nonconstant analytic mapping from a disk to $\mathbb{C}^n$ whose image is contained in Y . As was observed by Soltzenberg [8], the open mapping theorem implies that if Y contains analytic structure, then one of the coordinate projections $z_i(Y)$ has nonempty interior in $\mathbb{C}$. Wermer in [7] constructed an S such that $b_r(X_S) \setminus X_S \neq \emptyset$ but $\text{int } S = \emptyset$, whence $b_r(X_S) \subseteq S \times S$ does not contain analytic structure. (The first example of this general type is due to Stolzenberg [8], who proved the existence of a compact set $X \subseteq \mathbb{C}^2$ which is not polynomially convex but whose polynomially convex hull does not contain analytic structure.)

First assume that S is a compact connected subset of Δ containing $\partial\Delta$, and write $S = \bigcap_{n=1}^{\infty} S_n$ where

- (i) each S_n is a compact connected smoothly bounded subset of Δ ;
- (ii) S_n is a neighborhood of S_{n+1} in Δ .

Define $u_n \in C(S_n)$ by requiring that u_n be harmonic on $\text{int } S_n$, $u_n \equiv 0$ on $\partial\Delta$

and $u_n \equiv 1$ on $\partial S_n \setminus \partial \Delta$. For $z \in S$ let

$$u_S(z) = \lim_{n \rightarrow \infty} u_n(z).$$

The main result of §1 may then be stated as

Theorem 1. $b_r(X_S) = \{(z, w) \in S \times S \mid u_S(z) + u_S(w) \leq 1\}$.

The determination of $b_r(X_S)$ yields some quantitative information:

Theorem 2. *If $(z, w) \in b_r(X_S) \setminus X_S$ and $u_S(z) + u_S(w) < 1$, then the Gleason part of (z, w) in $R(X_S)$ has positive 4-dimensional measure.*

This leads almost at once to

Corollary. *If $b_r(X_S) \neq X_S$, then the 4-dimensional measure of $b_r(X_S) \setminus X_S$ is positive.*

One consequence of Theorem 1 is that there are compact connected subsets S of Δ which contain $\partial \Delta$ and for which X_S is rationally convex even though $R(X_S) \neq C(X_S)$. For example, let S be a compact connected subset of Δ such that $S \supseteq \partial \Delta$, $R(S) \neq C(S)$, but the only Jensen measures for $R(S)$ are point masses (such an S can be obtained by a simple modification of an example due to Browder (see [9, p. 192])). It can be shown that $u_S \equiv 1$ on $S \setminus \partial \Delta$, so that $b_r(X_S) = X_S$ by Theorem 1. This suggests that such S might yield interesting rationally convex sets in other ways.

The second section of this paper is essentially independent of the first. In it we exhibit a concrete example of a uniform algebra A defined on a compact metric space (its maximal ideal space) with the properties:

- (I) A does not contain all continuous functions on its maximal ideal space;
- (II) every point of the maximal ideal space of A is a peak point for A ;
- (III) A is finitely generated.

The first example of such an algebra is due to Cole (unpublished); his example is a modification of an algebra satisfying (I) and (II) which is constructed in his thesis ([10]; or see [9, pp. 255–262]).

Let T be any compact subset of $\text{int} \Delta$ such that the only Jensen measures for $R(T)$ are trivial, but $R(T) \neq C(T)$; Cole's construction also begins with such a T (examples of such T , besides the previously mentioned one in [9], are found in [11] and [12]). We use T to obtain a compact subset of the boundary of the unit ball in $\mathbb{C}^2$ by defining

$$\tilde{X}_T = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1, z \in T\}.$$

Theorem 3. *If $A = R(\tilde{X}_T)$, then conditions (I), (II) and (III) are satisfied.*

In the third section we show that under suitable restrictions on T , the properties of $\tilde{X}_T$ are enjoyed locally. This does not follow from the general arguments used in the second section, because there the construction of an annihilating measure for $R(\tilde{X}_T)$ is not local.

Let $T \subseteq \{|z| < 1\}$ be a "Swiss cheese" in a sense to be made precise later. (The previously mentioned example of Browder satisfies our definition.)

Theorem 4. *If K is a compact subset of $\tilde{X}_T$ with nonempty interior in the topology of $\tilde{X}_T$, then the restriction of $R(\tilde{X}_T)$ to K is not dense in $C(K)$.*

In the fourth and concluding section we demonstrate that no smoothness assumptions alone can be sufficient to avoid the pathological behavior exhibited by the sets X_S and $\tilde{X}_T$ discussed in the first two sections. Here we will be interested in the algebra $P(X)$ and its maximal ideal space $b(X)$, defined for a compact set $X \subseteq \mathbb{C}^n$ as follows:

$P(X)$ is the uniform closure in $C(X)$ of the polynomials in the coordinate functions $z_1, \dots, z_n$.

$b(X)$, the polynomially convex hull of X , is the set $\{z \in \mathbb{C}^n \mid \text{for all polynomials } p \text{ in } z_1, \dots, z_n, |p(z)| \leq \max_X |p|\}$.

Starting with the example in [7], or with the alternative use of Wermer's construction on p. 365, we prove the following.

Theorem 5. *There is a $\mathbb{C}^\infty$ 3-sphere Σ_1 contained in $\mathbb{C}^6$ such that $b(\Sigma_1)$ contains points not in Σ_1 but does not contain analytic structure.*

We obtain a similar result by starting instead with the set $\tilde{X}_T$ of the second section.

Theorem 6. *There is a $\mathbb{C}^\infty$ 3-sphere Σ_2 contained in $\mathbb{C}^6$ such that $b(\Sigma_2) = \Sigma_2$, every point of Σ_2 is a peak point for $P(\Sigma_2)$, but $P(\Sigma_2) \neq C(\Sigma_2)$. In other words $P(\Sigma_2)$ satisfies the above conditions (I), (II) and (III).*

1. We will employ the following notation throughout:

Δ will denote the closed unit disk; Δ^2 is then the closed unit bicylinder $\{(z, w) \in \mathbb{C}^2 \mid |z| \leq 1, |w| \leq 1\}$;

∂X will denote the topological boundary of a set $X \subseteq \mathbb{C}, \mathbb{C}^2$ or $\mathbb{C}^n$ depending on the context;

X_T , where $T \subseteq \Delta$, will denote the set $\{(z, w) \in \Delta^2 \mid z \in T \text{ and } |w| = 1 \text{ or } |z| = 1 \text{ and } w \in T\}$;

S (or S_n) will denote a closed connected subset of Δ such that $\partial\Delta \subseteq S$ (or S_n).

The following lemma, which is stated in a slightly different form in [7], will be useful to us:

Lemma 1. Fix $p \in \Delta^2$. If $p \notin b_r(X_S)$, then $p \in b(K)$ for some compact set $K \subseteq X_{\Delta \setminus S}$.

Proof. If $p \notin b_r(X_S)$, then there is some polynomial P such that $P(p) = 0$, but P has no zero on X_S . Let V be the component of $\{x \in \mathbb{C}^2 \mid P(x) = 0\}$ which contains p , and let $K = V \cap \partial\Delta^2$. By assumption, $K \cap X_S = \emptyset$, so that $K \subseteq \{(z, w) \in \partial\Delta^2 \mid z \notin S \text{ or } w \notin S\} = X_{\Delta \setminus S}$. By the maximum principle, if Q is any polynomial on $\mathbb{C}^2$, then $|Q(p)| \leq \sup_{x \in K} |Q(x)|$, so $p \in b(K)$ as claimed.

Definition. Given S as above, it is possible to choose a sequence $\{S_n\}_{n=1}^\infty$ of closed connected subsets of Δ which contain $\partial\Delta$ and satisfy

- (i) S_n is bounded by finitely many disjoint smooth curves;
- (ii) S_n is a neighborhood of S_{n+1} in Δ ;
- (iii) $S = \bigcap_{n=1}^\infty S_n$.

Let u_n be the unique function in $C(S_n)$ which is harmonic on $\text{int } S_n$ and satisfies $u_n|_{\partial\Delta} \equiv 0$, $u_n|_{S_n \setminus \partial\Delta} \equiv 1$. For $z \in S$ we define $u_S(z) = \lim_{n \rightarrow \infty} u_n(z)$. By the maximum principle the functions u_n are monotone increasing, so $u_S(z)$ exists for each $z \in S$. For future reference note that $u_S(z) > 0$ on $S \setminus \partial\Delta$ whenever $S \neq \Delta$.

Lemma 2. u_S is independent of the choice of $\{S_n\}$.

Proof. Let $\{S'_m\}_{m=1}^\infty$ be another sequence of compact sets as above, and let $\{u'_m\}$ be the corresponding functions in $C(S'_m)$.

Given any n , S_n is a neighborhood in Δ of S_{n+1} , hence also of S . Since $\bigcap_{m=1}^\infty S'_m = S$, there is an integer m_0 such that S_n is a neighborhood of S'_{m_0} ; in particular, $\partial S'_{m_0} \subseteq S_n$, so $u'_{m_0} \geq u_n$ on S'_{m_0} . Thus we have for $z \in S$:

$$\lim_{m \rightarrow \infty} u'_m(z) \geq u'_{m_0}(z) \geq u_n(z).$$

Since n was arbitrary, $\lim_{m \rightarrow \infty} u'_m(z) \geq \lim_{n \rightarrow \infty} u_n(z) = u_S(z)$. Similarly, $\lim_{m \rightarrow \infty} u'_m(z) \leq u_S(z)$ for all $z \in S$.

Observe that if S is "nice," then u_S has the following properties.

Lemma 3. If S is bounded by a finite number of disjoint smooth curves, then $u_S \in C(S)$, u_S is harmonic on $\text{int } S$, and $u_S|_{\partial\Delta} \equiv 0$, $u_S|_{S \setminus \partial\Delta} \equiv 1$.

Proof. Pick $\{S_n\}$ satisfying (i), (ii), (iii) in the definition of u_S , and let $\{u_n\}$ be the corresponding functions used to define u_S . Let $u \in C(S)$ be harmonic on

int S and satisfy $u|_{\partial\Delta} \equiv 0$, $u|_{\partial S \setminus \partial\Delta} \equiv 1$. Since S is smoothly bounded, there is a sequence of harmonic functions $\{v_m\}$ each defined in some neighborhood of S , such that v_m converges to u uniformly on S (see for example, [9, p. 184]).

If $z \in S$, then for all n , $u_n(z) \leq u(z)$ by the maximum principle, so $u_S(z) \leq u(z)$. To prove the converse, we may assume that $v_m \leq 0$ on $\partial\Delta$ and $v_m \leq 1 - 1/m$ on S . Given any m , we choose n sufficiently large that $S_n \subseteq \{v_m \leq 1\}$; then $v_m|_{\partial S_n} \leq u_n|_{\partial S_n}$, so that $v_m(z) \leq u_n(z) \leq u_S(z)$. Since m was arbitrary, $u(z) = \lim_{m \rightarrow \infty} v_m(z) \leq u_S(z)$, whence $u_S = u$.

Our main result may now be stated.

Theorem 1. $h_r(X_S) = \{(z, w) \in S \times S \mid u_S(z) + u_S(w) \leq 1\}$.

We first prove two lemmas.

Lemma 4. Let B be the annulus $\{z \in \mathbb{C} \mid 1 \geq |z| \geq r\}$. Then $h_r(X_B) = \{(z, w) \in \Delta^2 \mid |zw| \geq r\}$. (This is a special case of Theorem 1, since by Lemma 3, $u_B = (\log |z|)/(\log r)$. It is also a special case of a similar theorem for "circled" subsets of $\mathbb{C}^n$, referred to in the introduction.)

Proof. Fix $(z_0, w_0) \in \Delta^2$. If $|z_0 w_0| < r$, the polynomial $P(z, w) = zw - z_0 w_0$ vanishes at (z_0, w_0) , but has no zero on X_B , so $(z_0, w_0) \notin h_r(X_B)$.

Conversely, suppose that $|z_0 w_0| \geq r$. The polynomial $P(z, w) = zw$ is larger in modulus at (z_0, w_0) than it is on any compact subset of $X_{\Delta \setminus B}$. By Lemma 1, $(z_0, w_0) \in h_r(X_B)$.

Lemma 5. Let X be a compact subset of $\mathbb{C}^m$ and let $\Phi: X \rightarrow \mathbb{C}^m$. Suppose that $z_i \circ \Phi \in R(X)$ for each coordinate projection z_i , $i = 1, 2, \dots, m$. Then $\Phi(h_r(X)) \subseteq h_r(\Phi(X))$.

Proof. Fix $p_0 \in h_r(X)$, and suppose that P is a polynomial on $\mathbb{C}^m$ such that $P(\Phi(p_0)) = 0$. We must show that P has a zero on $\Phi(X)$.

Any function $F \in R(X)$ which has a zero on $h_r(X)$ must have a zero on X . Indeed, suppose $F(p) = 0$ for some $p \in h_r(X)$. Choose $f_n \in R_0(X)$ such that $\sup_X |F - f_n| \leq 1/n$. Let $g_n = f_n - f_n(p)$, then $g_n(p) = 0$, and $g_n = P_n/Q_n$ for some polynomials P_n, Q_n with $Q_n \neq 0$ on X hence on $h_r(X)$. So $P_n(p) = 0$, and since $p \in h_r(X)$, $P_n(p_n) = 0$ for some $p_n \in X$. Then $g_n(p_n) = 0$, so that

$$\begin{aligned} |F(p_n)| &\leq |F(p_n) - f_n(p_n)| + |f_n(p_n) - f_n(p)| + |f_n(p) - F(p)| \\ &\leq 1/n + |g_n(p_n)| + 1/n = 2/n. \end{aligned}$$

Some subsequence $p_{n_k} \rightarrow p_\infty \in X$, so that $F(p_\infty) = 0$ for some $p_\infty \in X$ as claimed.

In our case, $P(\Phi) \in R(X)$ and $P(\Phi)(p_0) = 0$, so we conclude that $P(\Phi)(p) = 0$ for some $p \in X$. Thus P has a zero at $\Phi(p) \in \Phi(X)$ as desired.

Corollary. *Let S be finitely connected. Suppose that ϕ is a one to one continuous map from S into Δ such that $\phi(\partial\Delta) = \partial\Delta$ and ϕ is holomorphic on $\text{int } S$. Define $\Phi: b_r(X_S) \rightarrow \Delta^2$ by $\Phi(z, w) = (\phi(z), \phi(w))$. Then $\Phi(b_r(X_S)) = b_r(X_{\phi(S)})$.*

Proof. $\phi \in R(S)$ since S is finitely connected (see, for example, [9, p. 231] or [3, p. 51]), and similarly $\phi^{-1} \in R(\phi(S))$. Since evidently $X_S \subseteq S \times S$, $b_r(X_S) \subseteq S \times S$, so that $z \circ \Phi$ and $w \circ \Phi$ are in $R(b_r(X_S))$; similarly $z \circ \Phi^{-1}$ and $w \circ \Phi^{-1}$ are in $R(b_r(X_{\phi(S)}))$. Observe that

$$\begin{aligned} X_{\phi(S)} &= \{(\zeta, \eta) \mid |\zeta| = 1, \eta \in \phi(S) \text{ or } |\eta| = 1, \zeta \in \phi(S)\} \\ &= \{(\phi(z), \phi(w)) \mid |z| = 1, w \in S \text{ or } |w| = 1, z \in S\} = \Phi(X_S). \end{aligned}$$

Lemma 5 applied to Φ yields

$$\Phi(b_r(X_S)) \subseteq b_r(\Phi(X_S)) = b_r(X_{\phi(S)}),$$

and applying Lemma 5 to Φ^{-1} we have conversely

$$\begin{aligned} b_r(X_{\phi(S)}) &= \Phi(\Phi^{-1}[b_r(X_{\phi(S)})]) \\ &= \Phi(\Phi^{-1}[b_r(\Phi(X_S))]) \subseteq \Phi(b_r(\Phi^{-1} \circ \Phi(X_S))) = \Phi(b_r(X_S)). \end{aligned}$$

Proof of Theorem 1. We first prove the theorem for those S of the form $S = \Delta \setminus \bigcup_{i=1}^n \text{int } D_i$, where the D_i are disjoint closed disks in $\text{int } \Delta$; then we shall deduce the general case by a "limit" argument. To avoid triviality we assume $n > 0$. In the simplest case $n = 1$, we can reduce the problem to the situation of Lemma 4 by mapping S onto an annulus via a linear fractional transformation, and applying the corollary to Lemma 5. For $n > 1$ we would like to imitate this procedure by defining $\phi: S \rightarrow \{1 \geq |\zeta| \geq r\}$ such that $\phi(\partial\Delta) = \{|\zeta| = 1\}$ and $\phi(\partial D_i) = \{|\zeta| = r\}$ for $i = 1, 2, \dots, n$, with ϕ analytic on $\text{int } S$. Although in general one can only construct a ϕ with approximately these properties—and, of course, ϕ will not be 1-to-1—it turns out that we can again reduce the problem to the situation of Lemma 4.

Fix $(a, b) \in S \times S$.

Case 1. Suppose that $u_S(a) + u_S(b) < 1$. We will show that $(a, b) \in b_r(X_S)$ by proving that any polynomial which vanishes at (a, b) also has a zero on X_S . So let $P(a, b) = 0$, and let

$$(1) \quad \delta = 1 - u_S(a) - u_S(b) > 0.$$

For convenience we divide the proof of Case 1 into three steps.

Step A: Construction of ϕ . We shall construct a ϕ_δ —call it ϕ —which approximately maps S onto an annulus.

Let u_i be the continuous function on S which is harmonic on $\text{int } S$ and satisfies:

$$u_i|_{\partial D_i} \equiv 1, \quad u_i|_{\partial S \setminus \partial D_i} \equiv 0; \quad i = 1, 2, \dots, n.$$

Note that $u_S = \sum_{i=1}^n u_i$ by Lemma 3. Each u_i extends to be harmonic in a neighborhood of S by the reflection principle. In this way we obtain an open set $\mathcal{O} \supseteq S$ which we may take small enough that

- (i) $\sum_{i=1}^n |u_i(z)| < 2$ for $z \in \mathcal{O}$;
- (ii) $u_S(z) > 1$ for $z \in \mathcal{O} \cap \text{int } D_i$, $i = 1, 2, \dots, n$;
- (iii) $u_i(z) < 0$ for $z \in \mathcal{O} \setminus \Delta$, $i = 1, 2, \dots, n$.

We may also assume that $\partial \mathcal{O}$ consists of circles, one in each component of $\mathbb{C} \setminus S$.

Let $\gamma_0: [0, 1] \rightarrow \mathcal{O}$ be a positively oriented circle contained in $\mathcal{O} \setminus \Delta$.

Similarly, let γ_j be a positively oriented circle contained in $\mathcal{O} \cap \text{int } D_j$, $j = 1, 2, \dots, n$. To avoid confusion we will distinguish between the map γ_j and its image which we will denote by Γ_j , $j = 0, 1, \dots, n$. If γ is any closed curve in $\mathcal{O}$, then γ is homologous in $\mathcal{O}$ to $\sum_{j=1}^n c_j \gamma_j$ for some integers $c_1, \dots, c_n$.

Let v_S be the (multiple-valued) conjugate harmonic function to u_S , and let v_i be the harmonic conjugate of u_i , $i = 1, 2, \dots, n$. Define

$$\alpha_j = \frac{1}{2\pi} \text{var}_{\gamma_j} v_S, \quad \omega_{ij} = \frac{1}{2\pi} \text{var}_{\gamma_j} v_i.$$

Observe that $\det((\omega_{ij})) \neq 0$. If not, there would be real constants c_i , not all zero, such that

$$\sum_{i=1}^n c_i \omega_{ij} = 0 \quad \text{for } j = 1, 2, \dots, n.$$

Then $f = \sum_{i=1}^n c_i (u_i + iv_i)$ would be a nonconstant single-valued analytic function on $\mathcal{O}$, with

$$f(\partial S) \subseteq \{\text{Re } z = 0\} \cup \{\text{Re } z = c_1\} \cup \dots \cup \{\text{Re } z = c_n\}.$$

As this is absurd, we conclude that there are indeed constants η_{ki} , $1 \leq k, i \leq n$, such that $\sum_{i=1}^n \eta_{ki} \omega_{ij} = \delta_k^j$, $1 \leq j, k \leq n$.

Now choose real numbers $\beta_1, \dots, \beta_n$ in such a way that $\alpha_j + \beta_j$ is rational

for each j , making sure that the β_j are sufficiently small that if we let $d_i = \sum_{k=1}^n \beta_k \eta_{ki}$, then

$$(2) \quad K = 2 \max_i |d_i| < \delta/2.$$

Now we define u on $\mathcal{O}$ by

$$(3) \quad u = u_S + \sum_{i=1}^n d_i u_i = \sum_{i=1}^n (1 + d_i) u_i.$$

The harmonic conjugate of u is $v = v_S + \sum_{i=1}^n d_i v_i$, so

$$\begin{aligned} \frac{1}{2\pi} \operatorname{var}_{\gamma_j} v &= \frac{1}{2\pi} \operatorname{var}_{\gamma_j} v_S + \frac{1}{2\pi} \sum_{i=1}^n d_i \operatorname{var}_{\gamma_j} v_i = \alpha_j + \sum_{i=1}^n d_i \omega_{ij} \\ &= \alpha_j + \sum_{i=1}^n \sum_{k=1}^n \beta_k \eta_{ki} \omega_{ij} = \alpha_j + \sum_{k=1}^n \beta_k \delta_k^j = \alpha_j + \beta_j. \end{aligned}$$

Choose a positive integer N large enough that $N(\alpha_j + \beta_j)$ is an integer for $j = 1, 2, \dots, n$, and let $\phi = e^{-N(u+iv)}$. By construction, ϕ is a single-valued analytic function on $\mathcal{O}$.

Step B: Properties of ϕ . Let $B = \{\zeta \in \mathbb{C} \mid 1 \geq |\zeta| \geq e^{-N(1-K)}\}$. We claim that $\phi^{-1}(B) \subseteq S$. Note that for $z \in \phi^{-1}(B)$ we have $0 \leq u(z) \leq 1 - K$, and suppose that $z \in \mathcal{O} \setminus S$. There are two cases to consider.

If $z \in \mathcal{O} \setminus \Delta$, then $u_i(z) < 0$ for $i = 1, 2, \dots, n$ by condition (iii) above. Note that $\delta \leq 1$ by equation (1), so that

$$u(z) = \sum_{i=1}^n (1 + d_i) u_i(z) < \sum_{i=1}^n (1 - \delta/4) u_i(z) < 0$$

by equations (2) and (3). Thus $z \notin \phi^{-1}(B)$.

If $z \in \operatorname{int} D_j \cap \mathcal{O}$, then

$$\begin{aligned} u(z) &= u_S(z) + \sum_{i=1}^n d_i u_i(z) \\ &\geq u_S(z) - \sum_{i=1}^n |d_i| |u_i(z)| > u_S(z) - K/2 \cdot 2 \geq 1 - K; \end{aligned}$$

this follows from equations (2) and (3) and conditions (i) and (ii) above. Again, $z \notin \phi^{-1}(B)$.

In particular, we have established that $\phi^{-1}(\{|\zeta| = 1\})$ is contained in S . But if $z \in S \setminus \partial\Delta$, then $u_i(z) \geq 0$ for $i = 1, 2, \dots, n$, and $u_S(z) > 0$ as noted earlier, so that

$$u(z) = \sum_{i=1}^n (1 + d_i)u_i(z) \geq (1 - \delta/4)u_S(z) > 0.$$

Thus $\phi^{-1}(\partial\Delta) \subseteq \partial\Delta$.

Now let U be the subset of $\mathcal{O}$ bounded by the images Γ_j of the maps γ_j , $j = 0, 1, \dots, n$. Let

$$R = \inf_{z \in \Gamma_0} e^{-Nu(z)}, \quad r = \sup_{z \in \Gamma_j; 1 \leq j \leq n} e^{-Nu(z)}.$$

$\Gamma_0 \subseteq \mathcal{O} \setminus \Delta$, so by the above $R > 1$. Similarly, $\Gamma_j \subseteq \mathcal{O} \cap \text{int } D_j$, so that $r < e^{-N(1-K)} < 1$. Let

$$U' = \{z \in U \mid R > e^{-Nu(z)} > r\},$$

and note that $\phi(U') \supseteq B$, since $\phi(\partial U') = \{|\zeta| = r\} \cup \{|\zeta| = R\}$.

For $\zeta \in \phi(\mathcal{O})$ let $m(\zeta)$ denote the number of times including multiplicities that ϕ takes on the value ζ in U . $m(\zeta)$ is constant on each component of $\phi(\mathcal{O}) \setminus \bigcup_{j=0}^n \phi(\Gamma_j)$. By choice of r and R , the set $\{\zeta \in \mathbb{C} \mid r < |\zeta| < R\}$ is contained in a component of $\phi(\mathcal{O}) \setminus \bigcup_{j=0}^n \phi(\Gamma_j)$. Thus $m(\zeta)$ is constant on $\phi(U')$; we use m to denote this constant value. $m > 0$ since if $z_0 \in \partial\Delta$, then $r < 1 = |\phi(z_0)| < R$.

We conclude that for all $\zeta \in \phi(U')$, ζ has precisely m inverses in U' under ϕ . Call these inverses $\psi_1(\zeta), \dots, \psi_m(\zeta)$ in any order.

Step C: Reduction to Lemma 4. Recall that we are trying to show that a given polynomial P which vanishes at (a, b) also has a zero on X_S . We are now ready to use P to define a function $Q \in R(X_B)$. In fact, let

$$Q(\zeta, \eta) = \prod_{i,j=1}^m P(\psi_i(\zeta), \psi_j(\eta)), \quad (\zeta, \eta) \in \phi(U') \times \phi(U').$$

Note that Q is well defined since it is independent of the numbering of the ψ_i . Fix η . Except for ζ in a discrete subset of $\phi(U')$, ζ will have m distinct inverse images in U' under ϕ . For such ζ one may renumber the ψ_i near ζ to make them analytic near ζ . Thus $Q(\zeta, \eta)$ is analytic in ζ except possibly on a discrete set. For any of the isolated points ζ_0 with less than m distinct inverse images, $\zeta \rightarrow \prod_{i,j=1}^m P(\psi_i(\zeta), \psi_j(\eta))$ is analytic and bounded in a deleted neighborhood of ζ_0 ,

and so extends to be analytic at ζ_0 . Similarly, for fixed ζ the function $\eta \rightarrow \prod_{i,j=1}^m P(\psi_i(\zeta), \psi_j(\eta))$ is analytic on $\phi(U')$, so that Q is analytic on $\phi(U') \times \phi(U')$. Now it was noted previously that $B \subseteq \phi(U')$, so $X_B \subseteq B \times B \subseteq \phi(U') \times \phi(U')$. Since $B \times B$ is rationally convex, $Q \in R(X_B)$.

Observe now that

$$\begin{aligned} |\phi(a)\phi(b)| &= |\exp\{-N(u(a) + iv(a)) - N(u(b) + iv(b))\}| \\ &= \exp\{-N(u(a) + u(b))\} \\ &\geq \exp\left\{-N\left[\sum_{i=1}^n (1 + |d_i|)(u_i(a) + u_i(b))\right]\right\} \quad \text{by (3)} \\ &\geq \exp\{-N[(1 + K/2)(u_S(a) + u_S(b))]\} \quad \text{by (2)} \\ &\geq \exp\{-N[1 - \delta + K/2]\} \quad \text{by (1)} \\ &\geq \exp\{-N[1 - K]\} \quad \text{by (2).} \end{aligned}$$

By Lemma 4, $h_r(X_B) = \{(\zeta, \eta) \in \Delta^2 \mid 1 \geq |\zeta\eta| \geq e^{-N(1-K)}\}$, so we have that $(\phi(a), \phi(b)) \in h_r(X_B)$. By definition of the ψ_i there exist i_0 and j_0 such that $\psi_{i_0}(\phi(a)) = a$ and $\psi_{j_0}(\phi(b)) = b$. (Note that $\psi_i(\phi(a))$ and $\psi_j(\phi(b))$ are well defined for all i and j , since $(\phi(a), \phi(b)) \in h_r(X_B)$ implies $\phi(a), \phi(b) \in B$, and so by the fact that $B \subseteq \phi(U')$ we have $\phi(a), \phi(b) \in \phi(U')$ as desired.) Thus,

$$Q(\phi(a), \phi(b)) = P(a, b) \prod_{i \neq i_0 \text{ or } j \neq j_0} P(\psi_i(\phi(a)), \psi_j(\phi(b))) = 0.$$

It follows that there is some point $(\zeta_0, \eta_0) \in X_B$ such that $Q(\zeta_0, \eta_0) = 0$, as was shown in the proof of Lemma 5. Hence

$$\prod_{i,j=1}^m P(\psi_i(\zeta_0), \psi_j(\eta_0)) = 0,$$

so that for some i_1, j_1 we have $P(\psi_{i_1}(\zeta_0), \psi_{j_1}(\eta_0)) = 0$. As noted earlier, $\phi^{-1}(\partial\Delta) \subseteq \partial\Delta$ and $\phi^{-1}(B) \subseteq S$, so $(z_0, w_0) = (\psi_{i_1}(\zeta_0), \psi_{j_1}(\eta_0)) \in X_S$. Thus P has the zero $(z_0, w_0) \in X_S$ as claimed.

Case 2. Suppose that $u_S(a) + u_S(b) > 1$. Let

$$(1') \quad \delta = u_S(a) + u_S(b) - 1 > 0.$$

Proceed to define ϕ exactly as in Case 1. Now, however, let

$$B' = \{\zeta \in \mathbb{C} \mid 1 \geq |\zeta| \geq e^{-N(1+K)}\},$$

a larger annulus than before. In fact, if $z \in S$ then $\phi(z) \in B'$ because

$$\begin{aligned} |\phi(z)| &= \exp\{-Nu(z)\} = \exp\left\{-N\left(u_S(z) + \sum_{i=1}^n d_i u_i(z)\right)\right\} \\ &\geq \exp\{-N(u_S(z) + K/2)\} \geq \exp\{-N(1 + K/2)\} \end{aligned}$$

by using equations (2) and (3) which are still valid.

Now define $\Phi(z, w) = (\phi(z), \phi(w))$ for $(z, w) \in \mathcal{O} \times \mathcal{O}$. Φ is analytic on $\mathcal{O} \times \mathcal{O}$, so $z \circ \Phi$ and $w \circ \Phi$ are in $R(X_S)$. We have

$$\begin{aligned} |\phi(a)\phi(b)| &= \exp\{-N(u(a) + u(b))\} \\ &\leq \exp\left\{-N\left(u_S(a) + u_S(b) - \sum_{i=1}^n |d_i|(u_i(a) + u_i(b))\right)\right\} \quad \text{by (3)} \\ &\leq \exp\{-N(1 + \delta - K)\} \quad \text{by (1'), (2)} \\ &< \exp\{-N(1 + K)\} \quad \text{by (2).} \end{aligned}$$

Thus by Lemma 4, $\Phi(a, b) \notin h_r(X_{B'})$. The remark at the end of the preceding paragraph shows that $\phi(S) \subseteq B'$; as in Case 1, $\phi(\partial\Delta) = \partial\Delta$, so $\Phi(X_S) = X_{\phi(S)} \subseteq X_{B'}$. Thus $\Phi(a, b) \notin h_r(\Phi(X_S))$. By Lemma 5, $(a, b) \notin h_r(X_S)$.

Case 3. Suppose, finally, that $u_S(a) + u_S(b) = 1$. We may assume that $u_S(a) > 0$. Choose $a_n \in S$ with $u_S(a_n) < u_S(a)$ and $a_n \rightarrow a$. We have $(a_n, b) \rightarrow (a, b)$, and each $(a_n, b) \in h_r(X_S)$, since $u_S(a_n) + u_S(b) < 1$. Since $h_r(X_S)$ is closed, $(a, b) \in h_r(X_S)$.

Given an arbitrary set S , one can find a sequence of regions S_n with properties (i), (ii) and (iii) in the definition of u_S following Lemma 1. For each n , int S_n is conformally equivalent to a domain bounded by circles; the equivalence extends to a continuous 1 to 1 map on S_n . By the corollary to Lemma 5 and the special case above, the theorem holds for each S_n .

Since $S \subseteq S_n$ for each n , $h_r(X_S) \subseteq \bigcap_{n=1}^{\infty} h_r(X_{S_n})$. Conversely, suppose that $p \in h_r(X_{S_n})$ for all n . If P is any polynomial which vanishes at p , then there is some point $p_n \in X_{S_n}$ such that $P(p_n) = 0$. By compactness there is a subsequence p_{n_k} converging to $p_0 \in \bigcap_{n=1}^{\infty} X_{S_n} = X_S$, and clearly $P(p_0) = 0$. Thus $p \in h_r(X_S)$. So we have $h_r(X_S) = \bigcap_{n=1}^{\infty} h_r(X_{S_n})$. Now $u_S(z) = \lim_{n \rightarrow \infty} u_{S_n}(z)$ for $z \in S$, and the u_{S_n} are monotone increasing, so

$$\begin{aligned} b_r(X_S) &= \bigcap_{n=1}^{\infty} b_r(X_{S_n}) = \bigcap_{n=1}^{\infty} \{(z, w) \in S_n \times S_n \mid u_{S_n}(z) + u_{S_n}(w) \leq 1\} \\ &= \{(z, w) \in S \times S \mid u_S(z) + u_S(w) \leq 1\} \end{aligned}$$

and the theorem is proven.

Note 1. The result can, of course, be generalized somewhat. For example, by modifying Lemma 4 and the proof of Theorem 1, one can show

$$\begin{aligned} &\text{given } S_1, S_2, \dots, S_n \text{ then } b_r(S_1 \times S_2 \times \dots \times S_n \cap \partial\Delta^n) \\ &= \{z \in S_1 \times S_2 \times \dots \times S_n \mid u_{S_1}(z_1) + \dots + u_{S_n}(z_n) \leq n-1\}. \end{aligned}$$

Note 2. The restriction to sets S which are connected is not serious, since more general subsets of Δ are of no additional interest. In fact, if $\partial\Delta \subseteq T \subseteq \Delta$ and T is closed, let S be the component of $\partial\Delta$ in T . One readily verifies that

$$(4) \quad b_r(X_T) = b_r(X_S) \cup X_T,$$

so that the other components of T contribute nothing to $b_r(X_T) \setminus X_T$. We outline a proof.

Choose nonempty compact sets $K_n \subseteq \Delta \setminus S$ such that $K_{n+1} \supseteq K_n$ and $\bigcup_{n=1}^{\infty} K_n = \Delta \setminus S$. One can then choose smoothly bounded compact sets C_n, D_n such that $T \subseteq C_n \cup D_n$, $S \subseteq C_n$, $K_n \subseteq D_n$, and $C_n \cap D_n = \emptyset$. Let l_n be a finite union of line segments which meet C_n and D_n precisely at their endpoints, chosen so that the compact set S_n defined by $S_n = C_n \cup D_n \cup l_n$ is connected.

Observe that $u_S \equiv 1$ on $D_n \cup l_n$. As remarked earlier, $u_S > 0$ on $S_n \setminus \partial\Delta$, so, by Theorem 1, $b_r(X_{S_n}) = b_r(X_{C_n}) \cup X_{S_n}$. The result (4) follows by using the facts that $T \subseteq S_n$ and that $\bigcap_{n=1}^{\infty} C_n = S$.

Example. In [7], Wermer shows that $b_r(X_S)$ does not "contain analytic structure" precisely when $\text{int } S = \emptyset$. He proceeds to construct a specific S such that $b_r(X_S) \neq X_S$, although $\text{int } S = \emptyset$. Using Theorem 1 we have an alternative method for constructing such an S .

If $|a| < 1$ and $0 < r < 1 - |a|$, let $D(a, r) = \{z - a \mid |z - a| < r\}$. Then $u_{\Delta \setminus D(a, r)}(z) \rightarrow 0$ as $r \rightarrow 0$ if $z \in \Delta \setminus \{a\}$. (For example, if $a = 0$, then $u_{\Delta \setminus D(a, r)}(z) = \log |z| / \log r$ and this assertion is clear; the general case can be essentially reduced to the case $a = 0$ by appropriate linear fractional transformations.)

Let $\{a_j\}$ be a countable dense subset of $\text{int } \Delta$, and let b be any other point of $\text{int } \Delta$. Take D_1 to be an open disk centered at a_1 , such that $\bar{D}_1 \subseteq \text{int } \Delta$ and $u_{\Delta \setminus D_1}(b) \leq 1/4$; this is possible by the preceding remark. Suppose that

$D_1, D_2, \dots, D_{n-1}$ have been chosen. Let a be the first of the a_j which is not in $\bigcup_{i=1}^{n-1} \bar{D}_i$. Choose an open disk D_n with center a such that $\bar{D}_n \subseteq \text{int } \Delta \setminus \bigcup_{i=1}^{n-1} \bar{D}_i$ and $u_{\Delta \setminus D_n}(b) \leq 1/2^{n+1}$. Let $S = \Delta \setminus \bigcup_{i=1}^{\infty} D_i$. By construction $\text{int } S = \emptyset$, and $u_S(b) \leq \sum_{i=1}^{\infty} u_{\Delta \setminus D_i}(b) \leq 1/2$, so that $(b, b) \in h_r(X_S) \setminus X_S$ by Theorem 1.

Note. Theorem 1 also shows that one must be careful in the choice of S to guarantee $h_r(X_S) \neq X_S$. Suppose, for example, that S is a connected compact subset of Δ containing $\partial\Delta$, such that the only Jensen measures for $R(S)$ are point masses. Then X_S is rationally convex, even when $R(S) \neq C(S)$. To see this, let $\{S_n\}_{n=1}^{\infty}$ be as in the definition of u_S .

Given $z \in S \setminus \partial\Delta$, let μ_n be harmonic measure for z as a point of S_n ; each μ_n is then a Jensen measure for $R(S_n)$. Each μ_n may be regarded as a measure on Δ , and so, by passing to a subsequence, we may assume that $\{\mu_n\}$ converges weak-star on Δ to a measure μ with support contained in S . μ represents z for $R(S)$ and is in fact a Jensen measure as we will now show.

Let $f \in R(S)$. Choose f_m analytic on a neighborhood of S so that f_m converges to f uniformly on S as $m \rightarrow \infty$. If n is large, $f_m \in R(S_n)$ and so

$$\log |f_m(z)| \leq \int \log |f_m| d\mu_n.$$

Thus for any $\delta > 0$ we have

$$\log |f_m(z)| \leq \int \log (|f_m| + \delta) d\mu_n$$

and so

$$\log |f_m(z)| \leq \lim_{\delta \downarrow 0} \lim_{n \rightarrow \infty} \int \log (|f_m| + \delta) d\mu_n = \int \log |f_m| d\mu.$$

Let $\epsilon > 0$; then $\log (|f_m| + \epsilon)$ converges uniformly on S to $\log (|f| + \epsilon)$ as $m \rightarrow \infty$, and so

$$\lim_{m \rightarrow \infty} \int \log (|f_m| + \epsilon) d\mu = \int \log (|f| + \epsilon) d\mu.$$

On the other hand we have

$$\begin{aligned} \lim_{m \rightarrow \infty} \int \log (|f_m| + \epsilon) d\mu &\geq \lim_{m \rightarrow \infty} \int \log |f_m| d\mu \\ &\geq \lim_{m \rightarrow \infty} \log |f_m(z)| = \log |f(z)|. \end{aligned}$$

Combining these results and applying monotone convergence we see that

$$\int \log |f| d\mu = \lim_{\epsilon \downarrow 0} \int \log (|f| + \epsilon) d\mu \geq \log |f(z)|.$$

Thus μ is a Jensen measure for the point z with respect to $R(S)$. By our assumption on S , μ is the unit point mass at z .

Choose $\chi \in C(\Delta)$ such that $0 \leq \chi \leq 1$, $\chi(z) = 1$, and $\text{supp } \chi \subseteq \text{int } \Delta$. Then

$$\begin{aligned} 1 &\geq u_S(z) = \lim_{n \rightarrow \infty} u_{S_n}(z) = \lim_{n \rightarrow \infty} \int_{\partial S_n \setminus \partial \Delta} 1 d\mu_n \\ &\geq \lim_{n \rightarrow \infty} \int_{\partial S_n} \chi d\mu_n = \int_S \chi d\mu = \chi(z) = 1. \end{aligned}$$

Suppose now that $(z, w) \in b_r(X_S)$ and $z \in S \setminus \partial \Delta$. By Theorem 1 we have $u_S(w) \leq 1 - u_S(z) = 0$. As noted earlier, $u_S(w) = 0$ implies $w \in \partial \Delta$. Thus $b_r(X_S) = X_S$.

Note. It follows rather readily from Theorem 1 that if $b_r(X_S) \neq X_S$, then the 4-dimensional measure of $b_r(X_S) \setminus X_S$ is positive (in contrast to that of X_S which, consisting of two copies of $S \times \partial \Delta$, has only positive 3-dimensional measure). To see this, extend u_S to all of Δ by

$$u_S(z) = 1 \quad \text{if } z \in \Delta \setminus S.$$

Then u_S is superharmonic on $\text{int } \Delta$ (see Lemma 6 below), so that the function g defined on $\text{int } \Delta \times \text{int } \Delta$ by $g(z, w) = u_S(z) + u_S(w)$ is superharmonic. If $(z_0, w_0) \in b_r(X_S) \setminus X_S$, then $(z_0, w_0) \in \text{int } \Delta \times \text{int } \Delta$ and, by Theorem 1, $u_S(z_0) + u_S(w_0) \leq 1$. Thus if B_ϵ is any small ball around (z_0, w_0) and if $\bar{m}$ denotes 4-dimensional Lebesgue measure on $\mathbb{C}^2$ we have by the superharmonicity of g :

$$\frac{1}{\text{volume } B_\epsilon} \int_{B_\epsilon} g d\bar{m} \leq 1.$$

This implies that $\{(z, w) \in B_\epsilon \mid g(z, w) = u_S(z) + u_S(w) \leq 1\}$, which by Theorem 1 is contained in $b_r(X_S)$, has positive 4-dimensional measure.

A more careful analysis leads to a significant sharpening of this result; in fact, we are able to show that some of the "Gleason parts" for $R(X_S)$ have positive 4-dimensional measure. The concept of "Gleason part" has been useful in the study of analytic structure in the maximal ideal space of a uniform algebra (see, e.g., [3, Chapter VI]). If T is a compact set in $\mathbb{C}^n$, the Gleason part (for the algebra $R(T)$ "=" $R(b_r(T))$) of a point $p \in b_r(T)$ is the equivalence class of p under the equivalence relation:

$$p \sim q \text{ if } \sup \left\{ |f(p) - f(q)| \mid f \in R(T) \text{ and } \sup_T |f| \leq 1 \right\} < 2.$$

(Thus, for instance, if $h_r(T)$ contains a connected analytic variety V , a normal family argument shows that the points of V belong to a single part of $h_r(T)$.)

Theorem 2. *If $h_r(X_S) \setminus X_S \neq \emptyset$, then there is a point $(z_0, w_0) \in h_r(X_S) \setminus X_S$ with $u_S(z_0) + u_S(w_0) < 1$. Any such point belongs to a Gleason part of $h_r(X_S)$ with positive 4-dimensional measure.*

The proof of Theorem 2 depends on several lemmas, for which we establish the following notation:

m = 2-dimensional Lebesgue measure on the plane $\mathbb{C}$;

$S^r = \{z \in S \mid u_S(z) \leq r\}$, $0 < r \leq 1$ (note that S^r is closed since by Lemma 6 below, u_S is superharmonic).

Lemma 6. *u_S is superharmonic on $\text{int } \Delta$ (as above, we consider u_S extended to Δ by $u_S \equiv 1$ on $\Delta \setminus S$).*

Proof. Take $\{S_n\}_{n=1}^\infty$ as in the definition of u_S . One verifies that each u_{S_n} is superharmonic on $\text{int } \Delta$ by considering three cases:

- (i) if $z \in \Delta \setminus S_n$, then $u_{S_n} \equiv 1$ near z ;
- (ii) if $z \in \text{int } S_n$, then u_{S_n} is harmonic near z by definition;
- (iii) if $z \in \partial S_n \cap \text{int } \Delta$, and $r < 1 - |z|$, then

$$u_{S_n}(z) = 1 = \frac{1}{2\pi} \int_0^{2\pi} d\theta \geq \frac{1}{2\pi} \int_0^{2\pi} u_{S_n}(z + re^{i\theta}) d\theta.$$

Thus u_S is the pointwise limit on $\text{int } \Delta$ of the monotone increasing sequence of superharmonic functions $\{u_{S_n}\}$. So u_S is superharmonic on $\text{int } \Delta$.

Corollary. *If $0 < u_S(z) < 1$, there is a point z' near z where $u_S(z') < u_S(z)$.*

Proof. If $z \in \text{int } S$, the statement is trivial since u_S is harmonic near z . So suppose that $z \in \partial S \setminus \partial \Delta$. Then we can find r arbitrarily small so that the set $E = \{\theta \mid z + re^{i\theta} \in \Delta \setminus S\}$ has positive 1-dimensional measure. If we also had $u(z + re^{i\theta}) \geq u(z)$ for all θ , we would have

$$\frac{1}{2\pi} \int_{[0, 2\pi]} u(z + re^{i\theta}) d\theta \geq \frac{1}{2\pi} \int_E 1 d\theta + \frac{1}{2\pi} \int_{[0, 2\pi] \setminus E} u(z) d\theta > u(z).$$

This contradiction establishes that $u(z + re^{i\theta}) < u(z)$ for some θ .

This corollary verifies the easy half of Theorem 2, since $(z, w) \in h_r(X_S) \setminus X_S$ and $u_S(z) + u_S(w) = 1$ imply $0 < u_S(z) < 1$. Applying the corollary to z , we have z' near z with $u_S(z') < u_S(z)$. Then (z', w) has the property that $u_S(z') + u_S(w) < 1$, and, of course, by Theorem 1 $(z', w) \in h_r(X_S) \setminus X_S$.

Lemma 7. Fix r , $0 < r \leq 1$. If $0 < u_S(z_0) < r$, then z_0 is not a peak point for $R(S^r)$.

Proof. Again take $\{S_n\}$ as in the definition of u_S . Since $u_{S_n}(z_0) \leq u_S(z_0)$ for all n , $z_0 \in (S_n)^r$ for all n . Let μ_n be a probability measure supported on ∂S_n^r such that for all functions f harmonic in a neighborhood of S_n^r we have

$$\int_{\partial S_n^r} f d\mu_n = f(z_0).$$

By passing to a subsequence we may assume that μ_n converges weak-star to a probability measure μ on ∂S^r which then has the property that, for all f harmonic in a neighborhood of S^r , $\int_{\partial S^r} f d\mu = f(z_0)$. It follows that μ represents z_0 for $R(S^r)$.

Choose $\chi \in C(\Delta)$ such that $0 \leq \chi \leq 1$, $\chi(z_0) = 1$, and $\text{supp } \chi \subseteq \text{int } \Delta$. Then

$$\begin{aligned} \int_{\partial S^r} \chi d\mu &= \lim_{n \rightarrow \infty} \int_{\partial S_n^r} \chi d\mu_n \leq \lim_{n \rightarrow \infty} \int_{\partial S_n^r} \frac{u_{S_n}}{r} d\mu_n \\ &= \lim_{n \rightarrow \infty} \frac{u_{S_n}(z_0)}{r} = \frac{u_S(z_0)}{r} < 1, \end{aligned}$$

where we have made use of the fact that $u_{S_n} = r$ on $\partial S_n^r \setminus \partial \Delta$ (if $u_S(z) < r$, then $u_{S_n} < r$ near z). So μ is not a point mass and therefore z_0 is not a peak point for $R(S^r)$.

Lemma 8. Let $(z_0, w_0) \in b_r(X_S) \setminus (X_S)$, and suppose that $z_0 \sim z_1$ in $S^{1-u_S(w_0)}$. Then $(z_1, w_0) \in b_r(X_S) \setminus X_S$, and $(z_0, w_0) \sim (z_1, w_0)$ in $b_r(X_S)$. Similarly, if $w_1 \sim w_0$ in $S^{1-u_S(z_1)}$, then $(z_1, w_1) \in b_r(X_S) \setminus X_S$ and $(z_1, w_1) \sim (z_1, w_0)$ in $b_r(X_S)$.

Proof. Let

$$\begin{aligned} K &= b_r(X_S) \cap \{(z, w) \mid w = w_0\} \\ &= \{(z, w_0) \in \Delta^2 \mid u_S(z) \leq 1 - u_S(w_0)\} = S^{1-u_S(w_0)} \times \{w_0\}. \end{aligned}$$

K is a compact subset of the plane $\{w = w_0\}$, and is nonempty since $(z_0, w_0) \in K$.

Since we have assumed that $z_1 \in S^{1-u_S(w_0)}$, we have $u_S(z_1) \leq 1 - u_S(w_0)$, so $(z_1, w_0) \in b_r(X_S)$, whence $(z_1, w_0) \in K$.

Let $F_n \in R(b_r(X_S))$, $|F_n| \leq 1$, and $F_n(z_0, w_0) \rightarrow +1$. Then $F_n(z, w_0) \in R(K)$, $|F_n(z, w_0)| \leq 1$, so by the assumption that $z_0 \sim z_1$ in $S^{1-u_S(w_0)}$, $F_n(z_1, w_0) \rightarrow +1$.

Thus $(z_0, w_0) \sim (z_1, w_0)$ in $h_r(X_S)$. The second half of the lemma follows by applying the same argument with the roles of the coordinates reversed.

Proof of Theorem 2. Choose δ so that $0 < \delta < 1 - u_S(z_0) - u_S(w_0)$. Let

$$T_1 = \{z \in S^{1-u_S(w_0)} \mid u_S(z) \leq u_S(z_0) + \delta, z \sim z_0 \text{ in } S^{1-u_S(w_0)}\},$$

$$T_2 = \{w \in S^{1-u_S(z_0)-\delta} \mid w \sim w_0 \text{ in } S^{1-u_S(z_0)-\delta}\}.$$

If $z \in T_1$, $(z, w_0) \sim (z_0, w_0)$ in $h_r(X_S)$ by Lemma 8. We also have $u_S(z) \leq u_S(z_0) + \delta$, so that $S^{1-u_S(z)} \supseteq S^{1-u_S(z_0)-\delta}$. Hence if $w \in T_2$, then $w \sim w_0$ in $S^{1-u_S(z)}$. By Lemma 8, $(z, w) \sim (z, w_0)$ in $h_r(X_S)$. Thus $(z, w) \sim (z_0, w_0)$ in $h_r(X_S)$, and so the part of $R(h_r(X_S))$ to which (z_0, w_0) belongs contains $T_1 \times T_2$. Since $R(X_S)$ can be identified with $R(h_r(X_S))$, the theorem will be proved if we show that $m(T_1) > 0$ and $m(T_2) > 0$.

For $0 < R < 1 - |z_0|$, let

$$E_R = \{z \in S \mid u_S(z) > u_S(z_0) + \delta\} \cap \{|z - z_0| < R\}.$$

By the superharmonicity of u_S , we have

$$\begin{aligned} u_S(z_0) &\geq \frac{1}{\pi R^2} \int_{|z-z_0| < R} u_S(z) dm \geq \frac{1}{\pi R^2} \int_{E_R} (u_S(z_0) + \delta) dm \\ &= [u_S(z_0) + \delta] m(E_R) / \pi R^2. \end{aligned}$$

Thus $m(E_R) \leq \pi R^2 (u_S(z_0) / (u_S(z_0) + \delta))$. Note also that $u_S(z_0) < 1 - u_S(w_0)$, so by Lemma 7, z_0 is not a peak point of $R(S^{1-u_S(w_0)})$.

Let K be a compact subset of $\mathbb{C}$, and suppose that $x \in K$ is not a peak point for $R(K)$. Given $\epsilon > 0$, let

$$P_\epsilon = \left\{ y \in K \mid \sup \left\{ |f(x) - f(y)| \mid f \in R(K), \sup_K |f| \leq 1 \right\} < \epsilon \right\}.$$

A theorem due to Browder says that

$$\lim_{n \rightarrow \infty} \frac{m(P_\epsilon \cap \{y \in \mathbb{C} \mid |x - y| < 1/n\})}{m(\{y \in \mathbb{C} \mid |x - y| < 1/n\})} = 1$$

(see [13], or [9, pp. 175–177]). In particular, if $\epsilon = 1$ this says that the part of $R(K)$ to which x belongs has density one at x . (This consequence of Browder's theorem was also proved by Melnikov [14] and by Wilken.) Applying this to

$z_0 \in S^{1-u_S(w_0)}$, we obtain

$$m(\{z \in S^{1-u_S(w_0)} \mid |z - z_0| < R, z \sim z_0 \text{ in } S^{1-u_S(w_0)}\}) > \pi R^2(u_S(z_0)/(u_S(z_0) + \delta))$$

for R sufficiently small.

Combining this result with the above we see that for R small and positive:

$$\begin{aligned} m(T_1 \cap \{|z - z_0| < R\}) &= m(\{|z| \mid |z - z_0| < R, u_S(z) \leq u_S(z_0) + \delta, z \sim z_0 \text{ in } S^{1-u_S(w_0)}\}) \\ &= m(\{|z - z_0| < R\}) - m(E_R) \\ &\quad - m(\{|z| \mid |z - z_0| < R, u_S(z) \leq u_S(z_0) + \delta, \text{ but not } z \sim z_0 \text{ in } S^{1-u_S(w_0)}\}) \\ &> \pi R^2 - \pi R^2 \left(\frac{u_S(z_0)}{u_S(z_0) + \delta} \right) - \left[\pi R^2 - \pi R^2 \left(\frac{u_S(z_0)}{u_S(z_0) + \delta} \right) \right] = 0. \end{aligned}$$

In particular, $m(T_1) > 0$.

Finally, $u_S(w_0) < 1 - u_S(z_0) - \delta$ by definition of δ , so by Lemma 7 w_0 is not a peak point for $R(S^{1-u_S(z_0)-\delta})$. Thus $m(T_2) > 0$ since nontrivial parts have positive measure. This and the last corollary complete the proof of Theorem 2.

2. Let B denote the closed unit ball in $\mathbb{C}^2 = \{(z, w) \mid |z|^2 + |w|^2 \leq 1\}$. If T is an arbitrary compact subset of the open unit disk, we associate with T a subset of the 3-sphere $\{|z|^2 + |w|^2 = 1\}$ by defining

$$\tilde{X}_T = \{(z, w) \in \partial B \mid z \in T\}.$$

Two observations about $R(\tilde{X}_T)$ can be made immediately. First, since it is well known that if X is a compact subset of $\mathbb{C}^n$ then $R(X)$ is generated by $n + 1$ functions (for a proof see Lemma 11), the algebra $R(\tilde{X}_T)$ is finitely generated. Secondly, each point of $\tilde{X}_T$ is a peak point for $R(\tilde{X}_T)$; this is in itself a trivial observation, since if Y is any compact subset of ∂B then each point $(z_0, w_0) \in Y$ is a peak point for $P(Y)$ —hence for $R(Y)$ —because $P(z, w) = (\bar{z}_0 z + \bar{w}_0 w + 1)/2$ peaks at (z_0, w_0) . (This explicit peaking function was pointed out to me by several people.) This latter fact, however, becomes particularly significant in conjunction with the following result:

Theorem 3. *Suppose that the only Jensen measures for $R(T)$ are trivial, and $R(T) \neq C(T)$. Then*

- (i) *the maximal ideal space of $R(\tilde{X}_T)$ is $\tilde{X}_T$, i.e., $\tilde{X}_T$ is rationally convex;*
- (ii) *$R(\tilde{X}_T) \neq C(\tilde{X}_T)$.*

Thus for such a T properties I, II, III mentioned in the introduction are satisfied by the algebra $A = R(\tilde{X}_T)$. The proof of Theorem 3 can be divided into two lemmas.

Lemma 9. *Suppose that T is a compact subset of the open unit disk, and the only Jensen measures for $R(T)$ are trivial. Then $\tilde{X}_T$ is rationally convex.*

Proof. Suppose $(\alpha, \beta) \in b_r(\tilde{X}_T)$. Let μ be a Jensen measure on $\tilde{X}_T$ for the homomorphism corresponding to evaluation at (α, β) . We assert that

$$\text{supp } \mu \subseteq \{(z, w) \in \tilde{X}_T \mid z = \alpha\}.$$

The latter set is a circle which we denote by Γ_α .

Observe that $\alpha \in T$, since otherwise $z - \alpha$ vanishes at (α, β) but has no zero on $\tilde{X}_T$. Define a measure $\bar{\mu}$ on T by

$$\int_T g d\bar{\mu} = \int_{\tilde{X}_T} g(z) d\mu(z, w), \quad g \in C(T).$$

Let $f \in R(T)$; then since μ is a Jensen measure for (α, β)

$$\log |f(\alpha)| \leq \int_{\tilde{X}_T} \log |f(z)| d\mu(z, w) = \int_T \log |f(z)| d\bar{\mu}(z).$$

So $\bar{\mu}$ is a Jensen measure for α with respect to $R(T)$. By assumption, $\bar{\mu}$ is the unit point mass at α . Hence

$$\mu(\Gamma_\alpha) = \int_{\Gamma_\alpha} d\mu = \int_{\{\alpha\}} d\bar{\mu} = 1.$$

Since μ is a probability measure, $\text{supp } \mu \subseteq \Gamma_\alpha$.

Now $w \neq 0$ on $\tilde{X}_T$, so $1/w \in R(\tilde{X}_T)$. Thus

$$\left| \frac{1}{\beta} \right| = \left| \int_{\tilde{X}_T} \frac{1}{w} d\mu \right| \leq \int_{\Gamma_\alpha} \frac{1}{|w|} d\mu = \frac{1}{(1 - |\alpha|^2)^{1/2}}$$

or $|\beta| \geq (1 - |\alpha|^2)^{1/2}$. Since $(\alpha, \beta) \in B$ we must have $|\beta| = (1 - |\alpha|^2)^{1/2}$, whence $(\alpha, \beta) \in \tilde{X}_T$.

Lemma 10. *Let T be a compact subset of the open disk such that $R(T) \neq C(T)$. Then $R(\tilde{X}_T) \neq C(\tilde{X}_T)$.*

Proof. Let $\mu \in C(T)^*$, $\mu \perp R(T)$, $\mu \neq 0$. If $z_0 \in \text{int } \Delta$ let m_{z_0} be normalized Lebesgue measure on the circle $\Gamma_{z_0} = \{(z_0, w) \mid |w| = (1 - |z_0|^2)^{1/2}\}$. Define a measure $\bar{\mu}$ on $\tilde{X}_T$ by

$$\int_{\tilde{X}_T} g d\bar{\mu} = \int_T \left[\int_{\Gamma_z} g(z, w) dm_z(w) \right] d\mu(z), \quad g \in C(\tilde{X}_T).$$

Note that $\bar{\mu} \neq 0$. We wish to show that $\bar{\mu} \perp R(\tilde{X}_T)$, so let $f(z, w)$ be a rational function holomorphic in a neighborhood of $\tilde{X}_T$. For z near T in $\mathbb{C}$ we may define a function

$$a(z) = \int_{\Gamma_z} f(z, w) dm_z(w);$$

we assert that $a(z)$ is analytic in a neighborhood of T .

Fix $z_0 \in T$. For some $\epsilon > 0$ and $R_1 > (1 - |z_0|^2)^{1/2} > R_2 > 0$, $f(z, w)$ is analytic on

$$\Omega = \{(z, w) \in \mathbb{C}^2 \mid |z - z_0| < \epsilon, R_1 > |w| > R_2\}.$$

On Ω , $f(z, w)$ may be expanded in a Laurent series in w :

$$f(z, w) = \sum_{n=-\infty}^{\infty} a_n(z) w^n,$$

where

$$a_n(z) = \frac{1}{2\pi i} \int_{|w|=(1-|z_0|^2)^{1/2}} \frac{f(z, w) dw}{w^{n+1}}$$

is analytic in z . But then

$$a(z) = \int_{\Gamma_z} f(z, w) dm_z(w) = \int_{\Gamma_z} \sum_{n=-\infty}^{\infty} a_n(z) w^n dm_z(w) = a_0(z)$$

is analytic for z near z_0 . So the assertion is proven.

Consequently we have

$$\int_{\tilde{X}_T} f d\bar{\mu} = \int_T \left[\int_{\Gamma_z} f(z, w) dm_z(w) \right] d\mu(z) = \int_T a(z) d\mu(z) = 0.$$

Thus $R(\tilde{X}_T) \neq C(\tilde{X}_T)$. This completes the proof of Theorem 3.

3. We wish to consider sets of the type $\tilde{X}_T$ described in §2, where we again require that T be a compact subset of $\text{int } \Delta$. We impose the additional requirement that T be a "Swiss cheese", so that $R(T) \neq C(T)$. For our purposes a Swiss

cheese T is a compact subset of $\mathbb{C}$ with empty interior, formed by deleting from a piecewise smoothly bounded connected compact set S a sequence of nonempty piecewise smoothly bounded simply connected open sets O_m subject to the conditions:

- (i) $\overline{O}_m \cap \overline{O}_n = \emptyset$ if $m \neq n$;
- (ii) $\overline{O}_m \subseteq \text{int } S$ for all m ;
- (iii) $\sum_{m=1}^{\infty} \text{length}(\partial O_m) < \infty$.

With this definition the example in Browder's book [9, p. 192] is a Swiss cheese.

Theorem 4. *Let $T \subseteq \text{int } \Delta$ be a Swiss cheese. Let K be a compact subset of $\tilde{X}_T$ with nonempty interior in the topology of $\tilde{X}_T$. Then $R(K) \neq C(K)$. Consequently, the restriction of $R(\tilde{X}_T)$ to K is not dense in $C(K)$. (Note that, in particular, T may be chosen so that $\tilde{X}_T$ satisfies Theorem 3.)*

Proof. We will construct a measure μ on K such that $\mu \perp R(K)$ and $\mu \neq 0$. For simplicity we may suppose that $(0, 1)$ is in the interior of K relative to $\tilde{X}_T$. It follows that, for some positive $\epsilon < 1$ and some positive $\theta_0 < \pi$, K contains the set $F \cap \tilde{X}_T$ where

$$F = \{(z, w) \in \partial B \mid |z| \leq \epsilon, |\arg w| \leq \theta_0\}.$$

Since T is a Swiss cheese, we may write $T = S \setminus \bigcup_{m=1}^{\infty} O_m$ with S and O_m as in the above definition. Let

$$K_n = \left\{ (z, w) \in F \mid z \in S \setminus \bigcup_{m=1}^n O_m \right\}, \text{ if } n > 0;$$

so that $K_{n+1} \subseteq K_n$ for all $n > 0$, and $\bigcap_{n=1}^{\infty} K_n = K \cap F$. If U is any small open neighborhood of

$$\{(z, \theta) \in \mathbb{C} \times \mathbb{R} \mid |z| \leq \epsilon, |\theta| \leq \theta_0\},$$

then the map $\psi: U \rightarrow \partial B$ defined by

$$\psi(z, \theta) = (z, e^{i\theta}(1 - |z|^2)^{1/2}), \quad (z, \theta) \in U,$$

establishes a real analytic diffeomorphism of U with an open neighborhood of K_1 as a subset of ∂B . For $n = 1, 2, \dots$ let $\partial_0 K_n$ denote the boundary of K_n relative to ∂B .

We may visualize $K_1, K_2, \dots$ and $\partial_0 K_1, \partial_0 K_2, \dots$ by means of the map ψ^{-1} :

$$\psi^{-1}(K_n) = \left\{ z \in S \setminus \bigcup_{m=1}^n O_m \mid |z| \leq \epsilon \right\} \times \{|\theta| \leq \theta_0\}, \quad n > 0;$$

$$\psi^{-1}(\partial_0 K_n) = \left\{ z \in \partial S \cup \left(\bigcup_{m=1}^n \partial O_m \right) \mid |z| \leq \epsilon \right\} \times \{|\theta| \leq \theta_0\}$$

$$\cup \left\{ z \in S \setminus \bigcup_{m=1}^n O_m \mid |z| = \epsilon \right\} \times \{|\theta| \leq \theta_0\}$$

$$\cup \left\{ z \in S \setminus \bigcup_{m=1}^n O_m \mid |z| \leq \epsilon \right\} \times \{-\theta_0, \theta_0\}, \quad n > 0.$$

From this it is evident that, for each n , $n = 1, 2, \dots$, the set K_n is the closure of an open subset V_n of ∂B and

(i) $\partial_0 K_n$ is "piecewise smooth";

(ii) V_n "lies on one side of $\partial_0 K_n$ ".

We may therefore apply Stokes' theorem to V_n and $\partial_0 K_n$. (See [15, pp. 271–275] for precise definitions and a statement of an appropriate form of Stokes' theorem.)

Let f be analytic in a neighborhood of K_n . The preceding remarks show that

$$\begin{aligned} (1) \quad & \int_{\partial_0 K_n} f dz \wedge dw \\ &= \int_{V_n} \left(\frac{\partial f}{\partial \bar{z}} d\bar{z} + \frac{\partial f}{\partial \bar{w}} d\bar{w} \right) \wedge dz \wedge dw = 0. \end{aligned}$$

Let μ_n denote the measure on $\partial_0 K_n$ corresponding to the form $dz \wedge dw$. We may regard $\{\mu_n\}$ as a sequence of measures on K_1 , because $K_n \subseteq K_1$ for all $n > 0$. We assert that the condition

$$(2) \quad \sum_{m=1}^{\infty} \text{length}(\partial O_m) < \infty$$

implies that the μ_n converge in norm. In fact, suppose that f is C^∞ in a neighborhood of K_1 and $\sup_{K_1} |f| \leq 1$. If $N > M > 0$, then

$$\begin{aligned}
\left| \int_{K_1} f d\mu_N - \int_{K_1} f d\mu_M \right| &= \left| \int_{\partial_0 K_N} f(z, w) dz \wedge dw - \int_{\partial_0 K_M} f(z, w) dz \wedge dw \right| \\
&= \left| \int_{\psi^{-1}(\partial_0 K_N)} f(\psi(z, \theta)) d(z \circ \psi) \wedge d(w \circ \psi) - \int_{\psi^{-1}(\partial_0 K_M)} f(\psi(z, \theta)) d(z \circ \psi) \wedge d(w \circ \psi) \right| \\
&= \left| \sum_{m=M+1}^N \left(\int_{\{(z, \theta) \mid |z| \leq \epsilon, z \in \partial O_m, |\theta| \leq \theta_0\}} f(z, e^{i\theta}(1-|z|^2)^{1/2}) dz \wedge \left(\frac{\partial w}{\partial \bar{z}} d\bar{z} + \frac{\partial w}{\partial \theta} d\theta \right) \right. \right. \\
&\quad \left. \left. - \int_{\{(z, \theta) \mid |z| = \epsilon, z \in O_m, |\theta| \leq \theta_0\}} f(z, e^{i\theta}(1-|z|^2)^{1/2}) dz \wedge \left(\frac{\partial w}{\partial \bar{z}} d\bar{z} + \frac{\partial w}{\partial \theta} d\theta \right) \right. \right. \\
&\quad \left. \left. - \int_{\{(z, \theta) \mid |z| \leq \epsilon, z \in O_m, \theta = \pm \theta_0\}} f(z, e^{i\theta}(1-|z|^2)^{1/2}) dz \wedge \left(\frac{\partial w}{\partial \bar{z}} d\bar{z} + \frac{\partial w}{\partial \theta} d\theta \right) \right) \right| \\
&\leq \sum_{m=M+1}^N \left(\left| \int_{\{|z| \leq \epsilon, z \in \partial O_m, |\theta| \leq \theta_0\}} f(z, e^{i\theta}(1-|z|^2)^{1/2}) dz \wedge (ie^{i\theta}(1-|z|^2)^{1/2} d\theta) \right| \right. \\
&\quad \left. + \left| \int_{\{|z| = \epsilon, z \in O_m, |\theta| \leq \theta_0\}} f(z, e^{i\theta}(1-|z|^2)^{1/2}) dz \wedge (ie^{i\theta}(1-|z|^2)^{1/2} d\theta) \right| \right. \\
&\quad \left. + \left| \int_{\{|z| \leq \epsilon, z \in O_m, \theta = \pm \theta_0\}} f(z, e^{i\theta}(1-|z|^2)^{1/2}) dz \wedge \left(\frac{-e^{i\theta} z}{2(1-|z|^2)^{1/2}} d\bar{z} \right) \right| \right) \\
&\leq \sum_{m=M+1}^N \left(\sup_{K_1} |f| \cdot \text{length}(\partial O_m) \cdot 2\theta_0 + \sup_{K_1} |f| \cdot \text{length}(\partial O_m) \cdot 2\theta_0 \right. \\
&\quad \left. + 2 \sup_{K_1} |f| \cdot 2 \text{area}(O_m) \cdot \frac{\epsilon}{2(1-\epsilon^2)^{1/2}} \right) \\
&\leq \sum_{m=M+1}^N \left(4\theta_0 \text{length}(\partial O_m) + \frac{2\epsilon}{(1-\epsilon^2)^{1/2}} \text{area}(O_m) \right)
\end{aligned}$$

which by (2) converges to zero independent of f as $M, N \rightarrow \infty$.

Let μ be the uniform limit of the μ_n . Since $\text{supp } \mu_n \subseteq K_n$ and $K_{n+1} \subseteq K_n$, we have $\text{supp } \mu \subseteq \bigcap_{n=1}^{\infty} K_n = K \cap F$. If f is analytic in a neighborhood of K , then f is analytic in a neighborhood of some K_M , and for $N \geq M$ we have $\int_{\partial_0 K_N} f d\mu_N = 0$, by (1). So

$$\int_K f d\mu = \lim_{N \rightarrow \infty} \int_{\partial_0 K_N} f d\mu_N = 0.$$

Thus $\mu \perp R(K)$, and it remains only to show that $\mu \neq 0$.

Choose $M > 0$ so that $O_M \cap \{|z| < \epsilon\} \neq \emptyset$. Let $g(z)$ be a bounded Borel function on $\mathbb{C}$ such that $g(z) = 0$ unless $z \in \{|z| \leq \epsilon\} \cap \partial O_M$, and such that

$$\int_{\{|z| \leq \epsilon, z \in \partial O_M\}} g(z) dz = 1.$$

Let $f(z, w) = g(z)/iw$; then for $N \geq M$, μ_N agrees with μ_M on the support of f , so

$$\begin{aligned} \int_{K_1} f d\mu_N &= \int_{K_1} f d\mu_M = \int_{\partial_0 K_M} \frac{g(z)}{iw} dz \wedge dw \\ &= \int_{\{|z| \leq \epsilon, z \in \partial O_M, |\theta| \leq \theta_0\}} \frac{g(z)}{ie^{i\theta}(1-|z|^2)^{1/2}} dz \wedge (ie^{i\theta}(1-|z|^2)^{1/2} d\theta) \\ &= 2\theta_0 \int_{\{|z| \leq \epsilon, z \in \partial O_M\}} g(z) dz = 2\theta_0 \neq 0. \end{aligned}$$

Thus $\int_K f d\mu \neq 0$, and so $\mu \neq 0$.

4. Our final objective is the construction of smooth sets with the pathological behavior discussed in the first two sections. The key result we need is that for any compact subset X of $\mathbb{C}^2$ we can find a smooth function f such that z, w, f generate $R(X)$. More precisely, we have the following result:

Lemma 11. *Let X be a compact subset of $\mathbb{C}^n$. There is an $f \in C_0^\infty(\mathbb{C}^n)$ such that the restrictions to X of $z_1, \dots, z_n$ and f generate $R(X)$. Furthermore, f may be chosen so that $f|_{b_r(X)} \in R(b_r(X))$, i.e., $f|_{b_r(X)}$ coincides with the natural extension of $f|_X$ to the maximal ideal space of $R(X)$, $b_r(X)$.*

As the proof is simply a modification of the proof that $R(X)$ has $n+1$ generators (see, e.g., [9, p. 16]), we leave it to be included as an appendix.

One addition to the notation previously introduced is needed.

ρ_X , for X a compact subset of ∂B , will denote an arbitrarily chosen C^∞ real-valued function on ∂B which is zero precisely on X . It is easy to construct such a function. A similar function was used by Freeman in Example 5.3 in [16].

Lemma 12. *Let X be a compact subset of ∂B . Let $\mathfrak{U}$ be a uniform algebra on X which contains the polynomials in z and w , and suppose that there is a function $f \in C^\infty(\partial B)$ such that z, w, f generate $\mathfrak{U}$. Define a uniform algebra $\mathfrak{U}'$ on ∂B by*

$$\mathfrak{U}' = \{g \in C(\partial B) \mid g|_X \in \mathfrak{U}\}.$$

Let $\Phi: \partial B \rightarrow \mathbb{C}^6$ by

$$\Phi(z, w) = (z, w, f(z, w), \rho_X(z, w), \bar{z}\rho_X(z, w), \bar{w}\rho_X(z, w)).$$

Then Φ establishes an isometric isomorphism between the algebras $\mathfrak{U}'$ and $P[\Phi(\partial B)]$ by $F \mapsto F \circ \Phi^{-1}$ for $F \in \mathfrak{U}'$.

Proof. First observe that $z, w, f, \rho_X, \bar{z}\rho_X$ and $\bar{w}\rho_X$ are in $\mathfrak{U}'$. In fact, we claim that they generate $\mathfrak{U}'$. Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{U}'$ generated by these

functions, and let μ be any measure on ∂B orthogonal to $\mathfrak{B}$. We will show that $\mathfrak{U}' = \mathfrak{B}$ by showing that $\mu \perp \mathfrak{U}'$.

Now if $g \in C(\partial B)$ and $g|_X \equiv 0$, then g is uniformly approximable on ∂B by polynomials in $z, w, \rho_X, \bar{z}\rho_X, \bar{w}\rho_X$ —this follows from the Stone-Weierstrass theorem for locally compact spaces, since the real-valued functions $\rho_X(z + \bar{z})$, $\rho_X(z - \bar{z})$, $\rho_X(w + \bar{w})$ and $\rho_X(w - \bar{w})$ separate the points of $\partial B \setminus X$. Thus $\text{supp } \mu \subseteq X$. But $\mathfrak{B}$ contains all polynomials in z, w and f , so by the hypothesis that z, w and f generate $\mathfrak{U}$, we have $\mu \perp \mathfrak{U}'$.

Since the coordinates of the map Φ generate $\mathfrak{U}'$, it follows that Φ establishes the required isomorphism.

Theorem 5. *There is a C^∞ 3-sphere $\Sigma_1 \subseteq \mathbb{C}^6$ such that $b(\Sigma_1) \setminus \Sigma_1$ is nonempty but does not contain analytic structure.*

Proof. Let $Y = X_S$ with S chosen as in Wermer [7] or as in the example on p.365 with $b = 0$. Y has the following relevant properties:

- (i) $(0, 0) \in b_r(Y)$;
- (ii) $b_r(Y)$ does not contain analytic structure;
- (iii) $Y \subseteq \partial\Delta^2$, the topological boundary of the unit bicylinder.

By Lemma 11 there is an $f \in C^\infty(\mathbb{C}^2)$ such that z, w and f generate $R(Y)$, and $f \in R(b_r(Y))$.

Let $X = b_r(Y) \cap \partial B$, $\mathfrak{U}$ = the uniform algebra on X generated by z, w, f .

We observe as follows that the maximal ideal space of $\mathfrak{U}$, which we shall denote by M , may be identified with $b_r(Y) \cap B$.

If $(z_0, w_0) \in b_r(Y) \cap B$, and G is a polynomial in z, w, f , then

$$\begin{aligned} |G(z_0, w_0)| &\leq \max\{|G(z, w)| \mid (z, w) \in [(\text{Silov boundary } R(Y)) \cap (B \cap b_r(Y))] \\ &\quad \cup (\partial B \cap b_r(Y))\} \\ &= \max_X |G| \end{aligned}$$

by the Rossi local maximum modulus principle. Thus evaluation at (z_0, w_0) induces a homomorphism on $\mathfrak{U}$.

If, on the other hand, m is a homomorphism on $\mathfrak{U}$, then since z, w, f generate $R(Y)$, m induces a homomorphism of $R(Y)$ and so corresponds to evaluation of z, w, f at some point $(z_0, w_0) \in b_r(Y)$. Since B is polynomially convex, we must have $(z_0, w_0) \in b_r(Y) \cap B$.

We have therefore established the following facts about the maximal ideal space of $\mathfrak{U}$:

(i) $(0, 0) \in M$;

(ii) M does not contain analytic structure.

Let $\mathcal{U}' = \{F \in C(\partial B) \mid F|_X \in \mathcal{U}\}$. The maximal ideal space of $\mathcal{U}'$ may evidently be identified with $\partial B \cup M$, and the Silov boundary of $\mathcal{U}'$ is ∂B . Clearly the maximal ideal space of $\mathcal{U}'$ does not contain analytic structure.

Let Φ be the map of $(z, w, f, \rho_X, \bar{z}\rho_X, \bar{w}\rho_X)$ as described in Lemma 12. Let $\Sigma_1 = \Phi(\partial B)$. By Lemma 12, $\mathcal{U}'$ and $P(\Sigma_1)$ are equivalent uniform algebras, so $b(\Sigma_1) \setminus \Sigma_1$ is nonempty but Σ_1 does not contain analytic structure.

Note. The same technique may be applied to Stolzenberg's example [8] to obtain such a Σ_1 in $\mathbb{C}^5$ rather than $\mathbb{C}^6$.

Theorem 6. *There is a polynomially convex C^∞ 3-sphere $\Sigma_2 \subseteq \mathbb{C}^6$ such that every point of Σ_2 is a peak point for $P(\Sigma_2)$, but $P(\Sigma_2) \neq C(\Sigma_2)$.*

Proof. Let X be the set $\tilde{X}_T$ of Theorem 3. Since $X \subseteq \partial B$, by Lemma 11 there is an $f \in C^\infty(\partial B)$ such that z, w and f generate $R(X)$. Let

$$\mathcal{U} = R(X), \quad \mathcal{U}' = \{F \in C(\partial B) \mid F|_X \in \mathcal{U}\}.$$

Clearly the maximal ideal space of $\mathcal{U}'$ is ∂B , every point of ∂B is a peak point for $\mathcal{U}'$, but $\mathcal{U}' \neq C(\partial B)$. Let Φ be the corresponding map as described in Lemma 12, and let $\Sigma_2 = \Phi(\partial B)$. Then $\mathcal{U}'$ and $P(\Sigma_2)$ are equivalent uniform algebras, so Σ_2 has the required properties.

APPENDIX (PROOF OF LEMMA 11)

Notation. If $\alpha = (\alpha_1, \dots, \alpha_{2n})$ with each α_i a nonnegative integer, then D^α is the differential operator on $C_0^\infty(\mathbb{C}^n)$ defined by

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial^{a_1} z_1 \dots \partial^{a_n} z_n \partial^{a_{n+1}} \bar{z}_1 \dots \partial^{a_{2n}} \bar{z}_n}, \quad f \in C_0^\infty(\mathbb{C}^n),$$

where $|\alpha| = \sum_{i=1}^{2n} \alpha_i$.

Lemma 11. *Let X be a compact subset of $\mathbb{C}^n$. There is an $f \in C_0^\infty(\mathbb{C}^n)$ such that the restrictions to X of $z_1, \dots, z_n$ and f generate $R(X)$. Furthermore, f may be chosen so that $f|_{b_r(X)} \in R(b_r(X))$, i.e., $f|_{b_r(X)}$ coincides with the natural extension of $f|_X$ to $b_r(X)$.*

Proof of lemma. Choose a sequence of polynomials $g_1, g_2, \dots$ so that each g_m has no zero on X , and so that $\{p/g_m \mid p \text{ is a polynomial}, 1 \leq m < \infty\}$ is dense in $R(X)$. For each m let h_m be a C_0^∞ -extension of $1/g_m$ from $b_r(X)$ to $\mathbb{C}^n$ (note

that g_m does not vanish on $b_r(X)$, and arrange so that the supports of the b_m 's are all contained in some fixed compact subset of $\mathbb{C}^n$.

Now choose positive constants $c_1, c_2, \dots$ so that

- (i) $c_m \sup_{x \in \mathbb{C}^n} \sup_{0 \leq |\alpha| \leq m} |D^\alpha b_m(x)| < 2^{-m}$,
 (ii) for $1 \leq k < m$, $c_m \sup_{x \in X} |g_k(x)/g_m(x)| < 2^{-m} c_k$, for $m = 1, 2, \dots$. Let $f = \sum_{k=1}^{\infty} c_k b_k$. By (i), $f \in C_0^\infty(\mathbb{C}^n)$, and $f|_{b_r(X)} \in R(b_r(X))$.

The proof may now be concluded exactly as in [9]; for this purpose we henceforth regard all functions as functions on X , so that for example $b_k = 1/g_k$. We will show that the algebra A generated by $z_1, z_2, \dots, z_n, f$ is $R(X)$. By choice of $\{g_m\}$, it is sufficient to show that $1/g_k \in A$ for all k .

Let $f_k = \sum_{m=k}^{\infty} c_m/g_m$, $k = 1, 2, \dots$. Then $f_1 = f \in A$, and if $1/g_1, 1/g_2, \dots, 1/g_{k-1} \in A$, then $f_k \in A$. Proceed by induction: if $f_k \in A$, then $f_k g_k \in A$, and $f_k g_k = c_k + \sum_{m=k+1}^{\infty} c_m g_k/g_m$. By (ii), $c_k > \sup_X |\sum_{m=k+1}^{\infty} c_m g_k/g_m|$, so $(f_k g_k)^{-1} \in A$, so $1/g_k \in A$. Thus the induction is complete and so is the proof of the lemma.

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