LATTICE POINTS AND LIE GROUPS. I

BY

ROBERT S. CAHN⁽¹⁾

ABSTRACT. Assume that G is a compact semisimple Lie group and $\mathfrak{g}$ its associated Lie algebra. It is shown that the number of irreducible representations of G of dimension less than or equal to n is asymptotic to $kn^{a/b}$, where a = the rank of $\mathfrak{g}$ and b = the number of positive roots of $\mathfrak{g}$.

Let G be a simple, compact or complex, simply connected Lie group and $\mathfrak{g}$ its associated Lie algebra. If G is compact a representation is a real analytic group homomorphism $f: G \rightarrow GL(V)$ where V is a complex vector space. If G is complex a representation is a complex analytic group homomorphism $f: G \rightarrow GL(V)$. In either case f will be called irreducible if V has no nontrivial invariant subspaces under the action of $f(G)$. A homomorphism of Lie groups induces a homomorphism of the associated Lie algebras,

$$f^*: \mathfrak{g} \rightarrow \mathfrak{gl}(V),$$

a Lie algebra representation, and f^* will be called irreducible if V has no nontrivial invariant subspaces under the action of $f^*(\mathfrak{g})$. It is seen from this definition that f is irreducible $\leftrightarrow f^*$ is irreducible. If G is simply connected a Lie algebra representation of $\mathfrak{g}$ induces a group representation of G and we thus have a bijection between irreducible representations of G and $\mathfrak{g}$. By the dimension of a representation we mean the dimension of V . Identifying conjugate representations we ask, "How many irreducible representations of G (or equivalently $\mathfrak{g}$) are of dimension $\leq T$?". The question is simpler when asked of Lie algebras since the structure of the representations is less complex.

$\mathfrak{g}$ is a complex simple Lie algebra if G is a complex simple Lie group or a compact real form of a complex simple Lie algebra when G is a compact simple Lie group. In the latter case there is a bijection between the complex representations of $\mathfrak{g}$ defined over $\mathbb{R}$ and the complex representations of its complexification, $\mathfrak{g} \otimes \mathbb{C}$, a complex simple Lie algebra so that we need only consider the case of $\mathfrak{g}$ complex and simple.

The root space decomposition of a simple complex Lie algebra is well known

---

Received by the editors March 18, 1971 and, in revised form, October 13, 1972.

AMS (MOS) subject classifications (1970). Primary 22E45; Secondary 10E10.

Key words and phrases. Semisimple Lie group, irreducible representation, lattice points, Weyl's character formula.

(1) The results in this paper constitute part of the author's thesis.

and is found in [1] and [2]. We let $\mathfrak{H}$ be a Cartan subalgebra, $\mathfrak{H}^*$ its dual and $\mathfrak{G} = \mathfrak{H} \oplus_{\alpha} \mathfrak{G}_{\alpha}$ be the canonical root space decomposition of $\mathfrak{G}$,

$$\mathfrak{G}_{\alpha} = \{X \in \mathfrak{G} \mid [H, X] = \alpha(H)X, H \in \mathfrak{H}\}.$$

$R = \{\alpha \in \mathfrak{H}^* \mid \mathfrak{G}_{\alpha} \neq 0\}$ is called the set of roots. A subset of $R, \{\alpha_1, \dots, \alpha_{a_{\mathfrak{G}}}\}$, will be called simple if they are linearly independent, span $\mathfrak{H}^*$ and form an integer basis for R . The dimension of $\mathfrak{H} = a_{\mathfrak{G}}$ is the rank of $\mathfrak{G}$.

The Killing form is defined by $(X, Y) = \text{Tr}(\text{Ad } X \circ \text{Ad } Y)$. Restricted to $\mathfrak{H}$ it is symmetric and nondegenerate. $(,)$ induces a dual form on $\mathfrak{H}^*$ so we may speak of (α, β) when α and β are roots. Further, there are unique vectors $H_{\alpha}, H_{\beta} \in \mathfrak{H}$ such that $(\alpha, \beta) = \alpha(H_{\beta}) = \beta(H_{\alpha}) = (H_{\alpha}, H_{\beta})$.

If $f^*: \mathfrak{G} \rightarrow \mathfrak{gl}(V)$ is a representation it has a weight space decomposition, $V = \bigoplus_{\lambda} V_{\lambda}$, where

$$V_{\lambda} = \{v \neq 0 \mid f^*(H)v = \lambda(H)v, \text{ any } H \in \mathfrak{H}\}.$$

If f^* is finite dimensional it is necessary that

$$\lambda(H_i) = \lambda(2H_{\alpha_i}/(\alpha_i, \alpha_i)) = 2(\lambda, \alpha_i)/(\alpha_i, \alpha_i) \in \mathbb{Z}$$

for any $\alpha_i, i = 1, \dots, a$. If f^* is irreducible there exists a weight λ , called the dominant weight, such that $\lambda \geq \lambda'$ for any other λ' in f^* and $\lambda(H_i) \in \mathbb{Z}^+, i = 1, \dots, a$. Furthermore, if $f^{*'}$ is another irreducible representation with λ as dominant weight then f^* is conjugate to $f^{*'}$. Thus we may identify f^* with its dominant weight and we will write π_{λ} for f^* . The lattice of dominant weights is $\mathbb{Z}^+ \lambda_1 \oplus \dots \oplus \mathbb{Z}^+ \lambda_a$ where $\lambda_i(H_j) = \delta_{ij}$. The interest of this is that the dimension of π_{λ} is a polynomial in λ . By the Weyl character formula

$$f'_{\mathfrak{G}}(\lambda) = \dim \pi_{\lambda} = \prod_{\alpha > 0} (\lambda + \delta, \alpha) / \prod_{\alpha > 0} (\delta, \alpha)$$

where $\delta = \frac{1}{2} \sum_{\alpha > 0} \alpha \cdot \delta = \sum \lambda_i$ [1, p. 257], so if λ belongs to the lattice of dominant weights then $\lambda + \delta$ belongs to the lattice of dominant weights. If we change coordinates to $\Lambda = \lambda + \delta = \sum \Lambda_i \lambda_i$ where $\Lambda_i \in \mathbb{R}$, then

$$\dim \pi_{\lambda} = f_{\mathfrak{G}}(\Lambda) = \prod_{\alpha > 0} (\Lambda, \alpha) / \prod_{\alpha > 0} (\delta, \alpha).$$

The number of irreducible representations of $\mathfrak{G}$ of dimension $\leq n$ is then equal to the number of lattice points, Λ , such that $\Lambda_i > 0$ and $f_{\mathfrak{G}}(\Lambda) \leq n$. We now state

Theorem. *Let G be a simply connected, simple, complex or compact Lie group. The number of irreducible representations of G of dimension $\leq n$ is asymptotic to $kn^{a_{\mathfrak{G}}/b_{\mathfrak{G}}}$; $b_{\mathfrak{G}}$ = the number of positive roots of $\mathfrak{G}$.*

Proof. We first note that (Λ, α) is a linear homogeneous polynomial in the coefficients of Λ since

$$\left(\sum_{i=1}^a X_i \lambda_i, \sum_{i=1}^a m_i \alpha_i \right) = \sum_{i=1}^a m_i (\lambda_i, \alpha_i) X_i.$$

If $e_i, \dots, e_a$ is an orthonormal basis of $\mathfrak{H}^*$ and if $M: \lambda_i \rightarrow e_i$, then if M^t is the transpose of M with respect to $(\cdot, \cdot)$

$$(\Lambda, \alpha) = (M^{-1}M\Lambda, \alpha) = (M\Lambda, (M^{-1})^t\alpha)$$

and $M\Lambda$ lies in the regular integer lattice in $\mathbf{R}^a$. Thus if $L = \sum_{i=1}^a X_i e_i$, $X_i > 0$, and

$$f_{\mathfrak{H}}^0(L) = \prod_{\alpha > 0} (L, (M^{-1})^t\alpha) / \prod_{\alpha > 0} (M\delta, (M^{-1})^t\alpha)$$

then $f_{\mathfrak{H}}^0(\sum_{i=1}^a X_i e_i) = f_{\mathfrak{H}}(\sum_{i=1}^a X_i \lambda_i)$ so we may regard $f_{\mathfrak{H}}$ as having asymptotes $e_i = 0$ and the lattice of weights as the ordinary integer lattice. We now prove a lemma on homogeneous functions.

Lemma 1. *Let f be a homogeneous function on $\mathbf{R}^a$ of degree b which is the product of linear forms $\sum m_i x_i$, $m_i \geq 0$. If $f = 0$ on the planes $x_i = 0$, $i = 1, \dots, a$, and if*

$$S(1) = \{x \in \mathbf{R}^a \mid f(x) \leq 1, x_i \geq 0\}$$

has finite volume then the number of lattice points in

$$S(r) = \{x \in \mathbf{R}^a \mid f(x) \leq r, x_i \geq 0\}$$

is asymptotic to $\text{Vol}(S(1))r^{a/b}$.

Proof. It is clear that the volume of $S(r) = \text{Vol}(S(1))r^{a/b}$. If $x \in S(r)$ then

$$f(x/(r^{1/b})) = (r^{-1/b})^b f(x) = r^{-1} f(x) \leq 1.$$

Since we are in $\mathbf{R}^a$ the Jacobian of the coordinate change $x \rightarrow \alpha x$ is α^a so $\text{Vol}(S(r)) = r^{a/b} \text{Vol}(S(1))$. We will be done if the number of lattice points in $S(r) \sim \text{Vol}(S(r))$. To see this, draw a unit a -cube at every lattice point of $S(r)$, w , with vertices at $w, w + e_i$ any i . Call the union of these cubes $\bar{L}(r)$; a set which will contain $S(r) \cap \{x_i \geq 1 \text{ all } i\}$ since f will be increasing in each coordinate. Now at each lattice point, w , draw a unit cube with vertices $w, w - e_i$ any i . Call the union of these cubes $\underline{L}(r)$. $\underline{L}(r) \subset S(r)$ and $\text{Vol } \underline{L}(r) = \text{Vol } \bar{L}(r)$. Call $E(r) = S(r) \cap \{x_i \leq 1 \text{ some } i\}$. Then

$$\underline{L}(r) \subset S(r) \subset \bar{L}(r) \cup E(r)$$

which implies $|\text{Vol } S(r) - \text{the number of lattice points}| \leq \text{Vol } E(r)$. However

$$\text{Vol } E(r) = r^{a/b} \text{Vol} \{x \in S(1) \mid x_i \leq r^{-1/b} \text{ some } i\}$$

and since $\text{Vol } S(1) < \infty$ the volume of this latter set $\rightarrow 0$ by dominated convergence. Thus $\text{Vol } E(r)$ is $o(\text{Vol } S(r))$ and the number of lattice points in $S(r)$ is asymptotic to $\text{Vol } S(r)$. $\square$

We now have a criterion we would like to apply to the polynomials $f_{\mathfrak{U}}$. A canonical example is the algebra A_2 . The positive roots of A_2 are $\alpha_1, \alpha_2, \alpha_1 + \alpha_2$ and the polynomial $f_{A_2}^0(x, y) = kxy(x + y)$. We wish to show

$$\text{Vol}\{x, y \mid x > 0, y > 0, kxy(x + y) \leq 1\} < \infty$$

or equivalently $\text{Vol } A < \infty$ where

$$A = \{x, y \mid x > 0, y > 0, xy(x + y) \leq 1\}.$$

We divide A into two subsets, $A_x = A \cap \{x \geq y\}$, $A_y = A \cap \{x \leq y\}$. If $(x, y) \in A_x$, $xy(x + y) \leq 1$ which implies $x^2y \leq 1$.

$$A_x \subset \{(x, y) \mid x > y > 0, x^2y \leq 1\}.$$

$\text{Vol } A_x \cap \{x \in [0, 1]\} \leq 1/2$ so $\text{Vol } A_x$ is finite if

$$\text{Vol}\{(x, y) \mid x > y, x > 1, x^2y \leq 1\} < \infty.$$

The volume of this set is $\int_1^\infty x^{-2} dx = 1$ so $\text{Vol } A_x \leq 3/2$. Similarly, $\text{Vol } A_y \leq 3/2$ so $\text{Vol } A < 3$ and the theorem is true for the algebra A_2 . We now extend this method to higher dimensions.

Lemma 2. In $\mathbb{R}^a$ let $f(x)$ be a sum of monomials of degree b . If for every permutation i of $\{1, \dots, a\}$ there exists in $f(x)$ a monomial $X_{i(1)}^{s_1} \cdots X_{i(a)}^{s_a}$ where $s_1 > \dots > s_a > 0$, then the volume of the set $S(1) = \{x \mid f(x) \leq 1, x_i \geq 0\}$ is finite.

Remark. From Lemma 1 this implies $\text{Vol } S(r) = \text{Vol } S(1)r^{a/b}$.

Proof of Lemma 2. We proceed by induction. If $a = 2$ we have monomials $X_1^{s_1}X_2^{s_2}$ and $X_1^{s'_2}X_2^{s'_1}$, $s_1 > s_2$, $s'_1 > s'_2$. Again partitioning $S(1)$ into A_x and A_y we see

$$\begin{aligned} \text{Vol } A_x &\leq 1/2 + \int_1^\infty x^{-s_1/s_2} \\ &= 1/2 + (s_1/s_2 - 1)^{-1} < \infty \quad \text{since } s_1 > s_2. \end{aligned}$$

Similarly $\text{Vol } A_y \leq 1/2 + (s'_1/s'_2 - 1)^{-1}$.

Now assume the lemma true for $a - 1$. Partition $S(1)$ into the sets

$$A_{i_1, \dots, i_a} = S(1) \cap \{x_{i_1} \geq \dots \geq x_{i_a}\}.$$

We wish to show $\text{Vol } A_{i_1, \dots, i_a} < \infty$ for any i . As before

$$A_{i_1, \dots, i_a} \subset \{x \mid x_{i_1} \geq \dots \geq x_{i_a}, x_{i_1}^{s_1} \dots x_{i_a}^{s_a} \leq 1\}.$$

If $x_{i_1} \geq 1$ a cross-section of this set at x_{i_1} is the set

$$\{(x_{i_2}, \dots, x_{i_a}) \mid x_{i_2} \geq \dots \geq x_{i_a} \geq 0, x_{i_2}^{s_2} \dots x_{i_a}^{s_a} \leq 1/x_{i_1}^{s_1}\}.$$

By induction and the previous remark the volume of the cross-section $= kx_{i_1}^{-\gamma}$ where $\gamma = s_1(a-1)/(\sum_{i=2}^a s_i)$. The volume of

$$A_{i_1, \dots, i_a} \leq \text{Vol}(A_{i_1, \dots, i_a} \cap \{x_{i_1} \in [0, 1]\}) + \int_1^\infty y^{-\gamma} dy.$$

The first set is contained in the unit cube so it has volume ≤ 1 and the integral is finite as long as $\gamma \geq 1$. But $s_1 > s_i \forall i > 1$ so $(a-1)s_1 > \sum_{j=2}^a s_j \Rightarrow \gamma > 1$. $\square$

The proof of Theorem 1 will be complete if we show the criterion of Lemma 2 applies to the polynomials $f_{\mathfrak{g}}$ for all simple complex Lie algebras.

If $\Lambda = \sum_{i=1}^a X_i \lambda_i$ then for each $\alpha = \sum_{i=1}^a m_i \alpha_i$

$$(\Lambda, \alpha) = \sum_{i=1}^a m_i (\lambda_i, \alpha_i) X_i.$$

Thus to determine f we must list all the positive roots of $\mathfrak{g}$ in terms of the simple roots. We begin with the A_n algebras.

Lemma 3. *The monomial $X_1^{s(1)} \dots X_n^{s(n)}$ is found in the expansion of f_{A_n} for every permutation s of $(1, \dots, n)$.*

Proof. By referring to Serre [2] the positive roots of A_n are $\alpha_1, \dots, \alpha_n$; $\alpha_1 + \alpha_2, \dots, \alpha_{n-1} + \alpha_n$; $\dots$; $\alpha_1 + \dots + \alpha_n$. Since $(\lambda_i, \alpha_i) = c$, $f_{A_n} = kX_1 \dots X_n (X_1 + X_2) \dots (X_{n-1} + X_n) \dots (X_1 + \dots + X_n)$. We now apply induction. If $n = 2$, $f_{A_2} = X_1^2 X_2 + X_1 X_2^2$. Now assume the lemma for $n-1$. We write $f_{A_n} = X_n (X_n + X_{n-1}) \dots (X_1 + \dots + X_n) f_{A_{n-1}}$. Pick an arbitrary permutation s . Then $s(n) = j$. By induction $X_1^{s(1)'} \dots X_{n-1}^{s(n-1)'}$ occurs in $f_{A_{n-1}}$ where

$$s(i)' = \begin{cases} s(i) & \text{if } s(i) < j, \\ s(i) - 1 & \text{if } s(i) > j. \end{cases}$$

Multiply this monomial by X_n in the first j factors $X_n, \dots, (X_n + \dots + X_{n+j-1})$. Now pick the least i such that $s(i)' < s(i)$. Multiply the monomial by X_i in $(X_1 + \dots + X_n)$. Then pick the next i' such that $s(i') < s(i)'$ and multiply by $X_{i'}$ in $(X_2 + \dots + X_n)$. Since $i' > i \Rightarrow i' \geq 2$, $X_{i'}$ is found in $(X_2 + \dots + X_n)$. We may thus continue until we have $X_1^{s(1)} \dots X_n^{s(n)}$. $\square$

Remark. The degree of f_{A_n} is minimal such that we may find monomials

$X_{i(1)}^{s_1} \cdots X_{i(n)}^{s_n}$ where $s_1 > \cdots > s_n > 0$ since $s_n \geq 1$, $s_{n-1} \geq 2$, $\cdots$, $s_1 \geq n$ so that the degree of $f = \sum_{i=1}^n s_i \geq \sum_{i=1}^n i =$ the degree of f_{A_n}

Lemma 4. *The monomial $X_1^{2s(1)-1} \cdots X_n^{2s(n)-1}$ is found in the polynomials f_{B_n} and f_{C_n} for any permutation s .*

Proof. The positive roots of B_n are $\alpha_1, \cdots, \alpha_n; \alpha_1 + \alpha_2, \cdots, \alpha_{n-1} + \alpha_n; \cdots; \alpha_1 + \cdots + \alpha_n$ and $\alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_n$ where $i < j \leq n$ [2]. $f_{B_n} = k f_{A_n} \prod_{i=1}^{n-1} \prod_{j=i+1}^n (X_i + \cdots + X_{j-1} + 2X_j + \cdots + 2X_n)$. From Lemma 3 we know the monomial $X_1^{s(1)} \cdots X_n^{s(n)}$ is in f_{A_n} . We wish then to show that $X_1^{s(1)-1} \cdots \hat{X}_j \cdots X_n^{s(n)-1}$ where $s(j) = 1$ lies in

$$\prod_{i=1}^{n-1} \prod_{j=i+1}^n (X_i + \cdots + X_{j-1} + 2X_j + \cdots + 2X_n).$$

We proceed as follows. There are $n-1$ factors containing X_1 . $s(1) - 1 \leq n-1$ so we may choose X_1 in $s(1) - 1$ of these factors. There are $(n-1) + (n-2)$ factors containing X_2 and

$$(s(1) - 1) + (s(2) - 1) \leq (n-1) + (n-2)$$

so choose X_2 in the next $s(2) - 1$ factors. Thus we may proceed at each stage being able to choose $s(i) - 1$ X_i 's. Multiplying we have the monomial $X_1^{2s(1)-1} X_2^{2s(2)-1} \cdots X_n^{2s(n)-1}$.

For C_n the positive roots are $\alpha_1, \cdots, \alpha_n; \alpha_1 + \alpha_2, \cdots, \alpha_{n-1} + \alpha_n; \cdots; \alpha_1 + \cdots + \alpha_n; \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{n-1} + \alpha_n$, $i < n$, $i \leq j \leq n-1$. The roots are different from B_n but contain the same α_i so the argument is the same. $\square$

Lemma 5. *f_{D_n} contains monomials of descending degrees for $n \geq 6$.*

Proof. Referring to Serre the positive roots of D_n are $\alpha_1, \cdots, \alpha_{n-1}; \alpha_1 + \alpha_2, \cdots, \alpha_{n-2} + \alpha_{n-1}; \cdots; \alpha_1 + \cdots + \alpha_{n-1}; \alpha_{n-2} + \alpha_n, \cdots, \alpha_1 + \cdots + \alpha_{n-2} + \alpha_n; \alpha_{n-2} + \alpha_{n-1} + \alpha_n, \cdots, \alpha_1 + \cdots + \alpha_{n-1} + \alpha_n; \alpha_i + \cdots + 2\alpha_j + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$. We may write

$$\begin{aligned} f_{D_n} = k f_{A_{n-1}} (X_{n-2} + X_n) (X_{n-3} + X_{n-2} + X_n) \cdots (X_1 + \cdots + X_{n-2} + X_n) \\ \cdot \underbrace{X_n (X_{n-2} + X_{n-1} + X_n) \cdots (X_1 + \cdots + X_n)}_{\text{---}} \\ \cdot \prod_{i=1}^{n-3} \prod_{j=i+1}^{n-2} (X_i + \cdots + 2X_j + \cdots + 2X_{n-2} + X_{n-1} + X_n). \end{aligned}$$

The bracketed expression is what is needed along with $f_{A_{n-1}}$ to create f_{A_n} except for the missing factor $(X_{n-1} + X_n)$. We compensate by adding the term $(X_{n-3} + 2X_{n-2} + X_{n-1} + X_n)$ to create a function containing every monomial of f_{A_n} . The remaining terms we write as

$$g_{D_n} = \prod_{i=1}^{n-2} (X_i + \cdots + X_{n-2} + X_n)$$

$$\prod_{i=1}^{n-4} \prod_{j=i+1}^{n-2} (X_i + \cdots + 2X_j + \cdots + 2X_{n-2} + X_{n-1} + X_n).$$

We know from Lemma 3 that $X_{s(1)}^1 \cdots X_{s(n)}^n$ is found in f_{A_n} for any permutation s . We wish to produce a monomial with descending degrees in the $X_{s(i)}$ in g_{D_n} for any permutation s . There are two cases. First assume that $s(1) \neq n-1$. Then we will be done if the monomial

$$X_{s(n)}^{n-2} \cdots X_{s(6)}^4 X_{s(5)}^2 X_{s(4)} X_{s(3)} X_{s(2)}$$

is in g_{D_n} . First choose $n-2$ different X_i from

$$\prod_{i=1}^{n-2} (X_i + \cdots + X_{n-2} + X_n), \quad i \neq n-1, s(1).$$

We then proceed to the second factor. There are $n-3$ terms containing X_1 so if $s(j) = 1$ we may pick X_1 in $j-3$ terms. Mimicking Lemma 4 we may continue by picking $j'-3$ X_2 's; where $s(j') = 2$ and so on to X_{n-2} . The sole difference in the procedure will be that if $j \in (1, 2, 3, 4)$ we choose no $X_{s(j)}$'s. After X_{n-2} every term contains X_{n-1} and X_n so we may arbitrarily choose $k-2$ X_{n-1} 's and $k'-3$ X_n 's; $s(k) = n-1$, $s(k') = n$. We have thus produced the desired monomial belonging to g_{D_n} and multiplying by $X_{s(1)}^1 \cdots X_{s(n)}^n$ we have a monomial with strictly decreasing degrees.

If $n-1 = s(1)$ we will be done if

$$X_{s(n)}^{n-3} X_{s(n-1)}^{n-3} \cdots X_{s(6)}^4 X_{s(5)}^2 X_{s(4)} X_{s(3)} X_{s(2)} X_{s(1)}$$

is in g_{D_n} . First pick $\{X_{s(n-1)}, \dots, X_{s(2)}\}$ in $\prod_{i=1}^{n-2} (X_i + \cdots + X_{n-2} + X_n)$.

Then proceed as before choosing $j-3$ X_1 's, $j'-3$ X_2 's and so on again skipping $X_{s(1)}, \dots, X_{s(4)}$. Proceed to X_{n-2} and then to X_n . There will be one remaining term which *a priori* contains X_{n-1} . Multiplying by X_{n-1} from this factor we produce our monomial.

We have proved Theorem 1 for A_n, B_n, C_n and D_n for $n \geq 6$. These are all the complex simple Lie algebras except for the algebras $G_2, F_4, D_4, D_5, E_6, E_7$ and E_8 . In these cases the conditions of Lemma 2 may be verified directly.

We now summarize the results:

Algebra	$a_{\mathfrak{G}}$	$b_{\mathfrak{G}}$	$c_{\mathfrak{G}} = a_{\mathfrak{G}}/b_{\mathfrak{G}}$
$A_n, n \geq 1$	n	$n(n+1)/2$	$2/n+1$
$B_n, C_n, n \geq 2$	n	n^2	$1/n$
$D_n, n \geq 4$	n	$n(n-1)$	$1/n-1$
G_2	2	6	$1/3$
F_4	4	24	$1/6$
E_6	6	36	$1/6$
E_7	7	63	$1/9$
E_8	8	120	$1/15$

We now extend our results to semisimple Lie algebras.

Corollary. Let $\mathfrak{G}$ be a semisimple Lie algebra, $\mathfrak{G} = \bigoplus_{i=1}^n \mathfrak{G}_i$, with $\mathfrak{G}_i$ the simple components. If $c_{\mathfrak{G}_1} = \dots = c_{\mathfrak{G}_s} > c_{\mathfrak{G}_{s+1}} \geq \dots \geq c_{\mathfrak{G}_n}$, then the number of irreducible representations of $\mathfrak{G}$ of dimension less than or equal to T is asymptotic to $kT^{c_{\mathfrak{G}_1}} \log^{s-1} T$.

Proof. We first assume that $\mathfrak{G}$ has two simple factors, $\mathfrak{G} = \mathfrak{G}_1 \oplus \mathfrak{G}_2$. The irreducible representations of $\mathfrak{G}$ are tensor products of irreducible representations of the simple factors and the dimension of the tensor representation is a product of the dimensions of the factor representations. The number of irreducible representations of $\mathfrak{G}$ of dimension $\leq r$ is $b(r) = \sum_{m,n \in \mathbb{Z}^+; mn \leq r} M_1(m)M_2(n)$, where $M_i(x)$ is the number of irreducible representations of $\mathfrak{G}_i$ of dimension x .

We partition $S = \{x, y \mid xy \leq r, x, y \geq 0\}$ into $S_x = S \cap \{x \in [0, r^{1/2}]\}$, $S_y = S \cap \{y \in [0, r^{1/2}]\}$. $S = S_x \cup S_y$ so if we estimate both $b_x(r) = \sum_{(m,n) \in S_x} M_1(m)M_2(n)$ and $b_y(r) = \sum_{(m,n) \in S_y} M_1(m)M_2(n)$ asymptotically, then $b(r) \sim \text{Max}(b_x(r), b_y(r))$. Assume $c_{\mathfrak{G}_1} > c_{\mathfrak{G}_2}$ (c_1 and c_2 for brevity); we will deal with $c_1 = c_2$ later.

Theorem 1 states $\sum_{j=1}^n M_i(j) \sim \mu_i n^{c_i}$. Thus

$$b_x(r) = \sum_{i=1}^{[r^{1/2}]} M_1(i) \sum_{j=1}^{[r/i]} M_2(j).$$

For any ϵ there exists r_2 such that

$$\left| \left(\sum_{j=1}^L M_2(j) - \mu_2 L^{c_2} \right) / \sum_{j=1}^L M_2(j) \right| < \epsilon \quad \text{any } L \geq r_2.$$

Then

$$b_x(r) = \mu_2 r^{c_2} \sum_{i=1}^{[r^{1/2}]} M_1(i)/i^{c_2} + \epsilon' b_x(r)$$

where $|\epsilon'| < \epsilon$ if $r > r_2^2$. Thus

$$b_x(r) \sim \mu_2 r^{c_2} \sum_{i=1}^{[r^{1/2}]} M_1(i)/i^{c_2}.$$

By the Abel summation formula

$$\sum_{i=1}^{[r^{1/2}]} M_1(i)/i^{c_2} = \sum_{i=1}^{[r^{1/2}]-1} \left(\sum_{j=1}^i M_1(j) \right) (1/i^{c_2} - 1/(i+1)^{c_2}) + \sum_{i=1}^{[r^{1/2}]} M_1(i) \cdot r^{-c_2/2}.$$

Now $c_2/i^{c_2+1} > 1/i^{c_2} - 1/(i+1)^{c_2} > c_2/(i+1)^{c_2+1}$, so

$$\begin{aligned} & \sum_{i=1}^{[r^{1/2}]-1} \left(\sum_{j=1}^i M_1(j) \right) \cdot c_2/i^{c_2+1} \\ & > \sum_{i=1}^{[r^{1/2}]-1} \left(\sum_{j=1}^i M_1(j) \right) (1/i^{c_2} - 1/(i+1)^{c_2}) > \sum_{i=1}^{[r^{1/2}]-1} \left(\sum_{j=1}^i M_1(j) \right) \cdot c_2/(i+1)^{c_2+1}. \end{aligned}$$

For any $\epsilon > 0$ there exists r_1 such that

$$\left| \left(\sum_{j=1}^L M_1(j) - \mu_1 L^{c_1} \right) / \sum_{j=1}^L M_1(j) \right| < \epsilon \quad \text{any } L \geq r_1.$$

If $r \gg r_1^2, r_2^2$

$$\sum_{i=1}^{[r^{1/2}]-1} \left(\sum_{j=1}^i M_1(j) \right) \cdot c_2/i^{c_2+1} = \sum_{i=1}^{[r^{1/2}]-1} \mu_1 c_2 i^{c_1-c_2-1} + E + A$$

where

$$|E| < \epsilon \sum_{i=r_1}^{[r^{1/2}]-1} \left(\sum_{j=1}^i M_1(j) \right) \cdot c_2/i^{c_2+1}$$

and

$$A = \sum_{i=1}^{r_1} \left(\sum_{j=1}^i M_1(j) - \mu_1 i^{c_1} \right) \cdot c_2/i^{c_2+1}.$$

$$\begin{aligned} \sum_{i=1}^{[r^{1/2}]-1} \mu_1 c_2 i^{c_1-c_2-1} & \sim \mu_1 c_2 \int_1^{[r^{1/2}]-1} x^{c_1-c_2-1} dx \\ & = \mu_1 c_2 / (c_1 - c_2) x^{c_1-c_2} \Big|_1^{[r^{1/2}]-1} = k r^{(c_1-c_2)/2} + k'. \end{aligned}$$

Also

$$\sum_{i=1}^{[r^{1/2}]} M_1(i) \cdot r^{-c_2/2} = k_0 r^{(c_1 - c_2)/2} + E',$$

where $|E'| < \epsilon \sum_{i=1}^{[r^{1/2}]} M_1(i) \cdot r^{-c_2/2}$. Thus

$$\sum_{i=1}^{[r^{1/2}]} M_1(i)/i^{c_2} = (k + k_0) r^{c_1 - c_2/2} + (k' + A) + (E + E').$$

From this

$$(1 + 2\epsilon) b_x(r) > (k + k') r^{c_1 + c_2/2} + (k' + A) r^{c_2} > (1 - 2\epsilon) b_x(r).$$

Thus $b_x(r) \sim c r^{c_2} + c' r^{c_1 + c_2}$. Similarly $b_y(r) \sim \mu_1 r^{c_1} \sum_{i=1}^{[r^{1/2}]} M_2(i)/i^{c_1}$. But in this case $\sum_{i=1}^{[r^{1/2}]} M_2(i)/i^{c_1}$ is asymptotic to a constant. To see this

$$\sum_{i=1}^{[r^{1/2}]} M_2(i)/i^{c_1} = \sum_{i=1}^{[r^{1/2}]-1} \left(\sum_{j=1}^i M_2(j) \right) (1/i^{c_1} - 1/(i+1)^{c_1}) + \sum_{j=1}^{[r^{1/2}]} M_2(j) r^{-c_1/2}.$$

$\sum_{j=1}^x M_2(j)$ is $O(x^{c_2})$ and $(1/i^{c_1} - 1/(i+1)^{c_1}) < c_1/i^{c_1+1}$, so

$$\begin{aligned} \sum_{i=1}^{[r^{1/2}]} M_2(i)/i^{c_2} &\leq k \int_1^{r^{1/2}} x^{c_2 - c_1 - 1} dx + k_0 r^{c_2 - c_1/2} \\ &= k/(c_1 - c_2) (1 - r^{c_2 - c_1/2}) + k_0 r^{c_2 - c_1/2}. \end{aligned}$$

But $c_2 - c_1 < 0$ so the above sum is $\leq 2k/(c_1 - c_2)$ if r is sufficiently large and $\lim_{r \rightarrow \infty} \sum_{i=1}^r M_2(i)/i^{c_1}$ exists and is equal to k' . Thus $b_y(r) \sim k' r^{c_1}$ and $b(r) \sim b_y(r)$.

This settles the case of $\mathcal{G} = \bigoplus_{i=1}^n \mathcal{G}_i$ where $c_1 > c_i$, $i > 1$. By the above argument $\mathcal{G}_1 \oplus \mathcal{G}_2$ has asymptotically $k' n^{c_1}$ irreducible representations of dimension $\leq n$. By iteration $(\mathcal{G}_1 \oplus \mathcal{G}_2) \oplus \mathcal{G}_3$ still has $\sim k' n^{c_1}$ irreducible representations and so on. This leaves the case of $c_1 = \dots = c_s$. Let $\mathcal{G} = \mathcal{G}_1 \oplus \mathcal{G}_2$. Tracing the argument for $c_1 \neq c_2$ nothing is changed until we arrive at $\int_1^{[r^{1/2}]} x^{c_1 - c_2 - 1} dx$. This integral now equals $\int_1^{[r^{1/2}]} x^{-1} dx = \frac{1}{2} \log r$ so that $b_x(r) \sim k r^{c_1} \log r$ and $b_y(r) \sim k' r^{c_1} \log r$. Now

$$b(r) = b_x(r) + b_y(r) - \sum_{i,j \in S_x \cap S_y} M_1(i) M_2(j)$$

and the latter sum equals

$$\sum_{i=1}^{[r^{1/2}]} M_1(i) M_2(j) = \sum_{i=1}^{[r^{1/2}]} M_1(i) \cdot \sum_{j=1}^{[r^{1/2}]} M_2(j)$$

which is $O(r^{c_1})$ so that $b(r) \sim kr^{c_1} \log r$. Taking $\mathcal{G} = (\mathcal{G}_1 \oplus \mathcal{G}_2) \oplus \mathcal{G}_3$ we arrive at the integral

$$\int_1^{[r^{1/2}]} (\log x)/x \, dx = \frac{1}{8} \log^2 r.$$

So $b_x(r) \sim kr^{c_1} \log^2 r$, $b_y(r) \sim k' r^{c_1} \log^2 r$, $\sum_{i,j \in S_x \cap S_y} M_1(i)M_2(j)$ is $O(r^{c_1} \log r)$ and $b(r) \sim k_0 r^{c_1} \log^2 r$. Continuing to the case $\mathcal{G} = \bigoplus_{i=1}^s \mathcal{G}_i$ we have $b(r) \sim kr^{c_1} \log^{s-1} r$ and our corollary is proven. $\square$

BIBLIOGRAPHY

1. N. Jacobson, *Lie algebras*, Interscience Tracts in Pure and Appl. Math., no. 10, Interscience, New York, 1962. MR 31 #2354.
2. J.-P. Serre, *Algèbres de Lie semi-simples complexes*, Benjamin, New York, 1966. MR 35 #6721.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MIAMI, CORAL GABLES, FLORIDA 33124