SLICING AND INTERSECTION THEORY FOR CHAINS MODULO ν ASSOCIATED WITH REAL ANALYTIC VARIETIES

BY

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ABSTRACT. In a real analytic manifold a k dimensional (real) analytic chain is a locally finite sum of integral multiples of chains given by integration over certain k dimensional analytic submanifolds (or strata) of some k dimensional real analytic variety. In this paper, for any integer $\nu \geq 2$, the concepts and results of [6] on the continuity of slicing and the intersection theory for analytic chains are fully generalized to the modulo ν congruence classes of such chains.

1. Introduction. For a separable real analytic manifold M , a real analytic mapping $f: M \rightarrow \mathbb{R}^n$, and a k dimensional analytic chain T in M [4, 4.2.28] with $k \geq n$, it was shown in [6, 4.3] that the slice function $\langle T, f, \cdot \rangle$ is continuous on the set of points y in $\mathbb{R}^n$ for which

$$\dim(f^{-1}\{y\} \cap \text{spt } T) \leq k - n \quad \text{and} \quad \dim(f^{-1}\{y\} \cap \text{spt } \partial T) \leq k - n - 1.$$

Geometrically, for almost all y in $\mathbb{R}^n$, the slice $\langle T, f, y \rangle$ is the $k - n$ dimensional analytic chain in M given by oriented integration, counting multiplicities, along the fiber $f^{-1}\{y\} \cap \text{spt } T$.

For any integer $\nu \geq 2$, two k dimensional analytic chains T_1 and T_2 are congruent modulo ν if there exists a third analytic chain Q such that $T_1 - T_2 = \nu Q$. The resulting congruence classes are called k dimensional analytic chains modulo ν in M . With such a congruence class S we associate the set $\text{spt}^\nu S = \bigcap_{T \in S} \text{spt } T$. Generalizing [6, 4.3], we prove in 4.1 our main result:

Slicing Modulo ν Theorem. *Suppose S is a k dimensional analytic chain modulo ν in M , f is an analytic map of M into $\mathbb{R}^n$, $k \geq n$, and*

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$\mathbb{R}^n \cap \{y: \dim(f^{-1}\{y\} \cap \text{spt}^\nu S) \leq k - n \text{ and } \dim(f^{-1}\{y\} \cap \text{spt}_-^\nu \partial S) \leq k - n - 1\}$.

Then there exists a continuous mapping from Y into the $k - n$ dimensional analytic chains modulo ν in M such that for every $y \in Y$ the value $\langle S, f, y \rangle^\nu$, which we call the **slice modulo ν** of S in $f^{-1}\{y\}$, satisfies the condition:

If T is a k dimensional analytic chain in M belonging to S with $\dim(f^{-1}\{y\} \cap \text{spt } T) \leq k - n$ and $\dim(f^{-1}\{y\} \cap \text{spt } \partial T) \leq k - n - 1$, then $\langle T, f, y \rangle$ belongs to $\langle S, f, y \rangle^\nu$.

As an elementary example consider the four linear maps $g_1, g_2, g_3: \mathbb{R} \rightarrow \mathbb{R}^2$, $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $g_1(y) = (y, 0)$, $g_2(y) = (-y, 3^{1/2}y)$, $g_3(y) = (-y, -3^{1/2}y)$ for $y \in \mathbb{R}$ and $f(y, z) = y$ for $(y, z) \in \mathbb{R}^2$, the oriented half-line $H = E^1 \mathbb{L} \{y: y > 0\}$ in $\mathbb{R}$, and the one dimensional analytic chain

$$T = g_{1\#}H + g_{2\#}H + g_{3\#}H$$

in $\mathbb{R}^2$ (see Figure 1). We compute $\partial T = -3\delta_{(0,0)}$ and

$$\langle T, f, a \rangle = -\delta_{(a, -3^{1/2}a)} - \delta_{(a, 3^{1/2}a)} \quad \text{for } a < 0,$$

$$\langle T, f, b \rangle = \delta_{(b, 0)} \quad \text{for } b > 0 \quad (\text{see Figure 2}).$$

Hence $\langle T, f, \cdot \rangle$ is continuous on $\mathbb{R} \sim \{0\} = \mathbb{R} \sim f(\text{spt } \partial T)$, in accord with [6, 4.3]; however,

$$\lim_{y \rightarrow 0-} \langle T, f, y \rangle = -2\delta_{(0,0)} \neq \delta_{(0,0)} = \lim_{y \rightarrow 0+} \langle T, f, y \rangle.$$

Figure 1

Figure 2

Nevertheless, letting S be the congruence class modulo 3 of T , we infer that $\text{spt}^3 \partial S = \emptyset$, that $\langle S, f, 0 \rangle^3$ is the congruence class of the two congruent chains $-2\delta_{(0,0)}, \delta_{(0,0)}$ and that the slice modulo 3, $\langle S, f, \cdot \rangle^3$, is continuous on all of $\mathbb{R}$, in accord with the slicing modulo ν theorem.

It follows from this theorem and the remarks in 4.4 that all of the results of [6, 4.3–5.11] carry over to the modulo ν case, including the resulting intersection theory.

The notation employed in the present paper is consistent with [6] and [4] (see the glossaries of [4, pp. 669–671]). As in [6] we let, for any two maps $f: A \rightarrow B$ and $g: A \rightarrow C$, the mapping $f \sqcup g: A \rightarrow B \times C$ be defined by $(f \sqcup g)(a) = (f(a), g(a))$ for $a \in A$.

Chains modulo ν arose classically in algebraic topology as members of various chain complexes having coefficient group $\mathbb{Z}_\nu = \mathbb{Z}/\nu\mathbb{Z}$ [8, III, 4(c)]. Rectifiable and flat chains modulo ν (especially $\nu = 2$) have been used in geometric measure theory in the study of multidimensional elliptic variation problems ([9], [5], [1], [2], [4, 5.3.21]). In [7] analytic chains and analytic chains modulo ν are used to define homology theories with coefficient groups $\mathbb{Z}$ and $\mathbb{Z}_\nu$ for the category of pairs of semianalytic sets and continuous maps between such pairs; questions concerning the extent of local orientability of real analytic varieties are treated there in these terms.

2. Locally flat chains modulo ν . The basic reference for flat chains modulo ν in $\mathbb{R}^n$ is [4, 4.2.26]. In this section we assume M is a separable Riemannian manifold and state appropriate generalizations of some of the results of [6, §3] and [4, 4.2.26] for chains in M whose supports are not necessarily compact.

Recalling [6, §3] and [4, p. 423] we endow the group $\mathcal{F}_k^{\text{loc}}(M)$ of k dimensional locally integral flat chains in M with the modulo ν topology by associating with each pair (U, δ) such that

$$U \text{ is open, } \text{Clos } U \text{ is compact, } \delta > 0,$$

a basic neighborhood of 0, $\mathcal{N}^\nu(U, \delta)$, consisting of those chains $T \in \mathcal{F}_k^{\text{loc}}(M)$ for which there exist $Q \in \mathcal{R}_k(M)$, $R \in \mathcal{R}_k(M)$, and $S \in \mathcal{R}_{k+1}(M)$ with $\text{spt}(T - R - \partial S - \nu Q) \subset M \sim U$, $\mathbf{M}(R) + \mathbf{M}(S) < \delta$.

The statements and proofs of [6, 3, 3.1, 3.2 (1)(3)(4)(5), 3.3, 3.4] readily generalize to the modulo ν case. Assuming $\{U_1, U_2, \dots\}$ is a cover of M consisting of open sets with compact closures, we also see that the modulo ν topology is induced by the complete pseudometric dist^ν defined by

$$\text{dist}^\nu(T_1, T_2) = \sum_{j=1}^{\infty} [\delta_j / 2^j (1 + \delta_j)]$$

where

$$\delta_j = \inf \{ \delta : T_1 - T_2 \in N^\nu(U_j, \delta) \}$$

for $T_1, T_2 \in \mathcal{F}_k^{\text{loc}}(M)$; moreover, we say that T_1 and T_2 are congruent modulo ν , and write $T_1 \equiv T_2 \pmod{\nu}$, whenever $\text{dist}^\nu(T_1, T_2) = 0$. The resulting congruence classes, which we call k dimensional locally flat chains modulo ν in M , are the cosets of the factor group

$$\mathcal{F}_k^{\text{loc}, \nu}(M) = \mathcal{F}_k^{\text{loc}}(M) / \mathcal{F}_k^{\text{loc}}(M) \cap \{T : \text{dist}^\nu(T, 0) = 0\};$$

this group is complete in the induced metric. The operations on $\mathcal{F}_k^{\text{loc}}(M)$ of restriction [6, 3.3], boundary (for $k \geq 1$), and $f_\#$ (see [6, 3.0(3)]) induce similar continuous operations on $\mathcal{F}_k^{\text{loc}, \nu}(M)$. For $T \in \mathcal{F}_k^{\text{loc}}(M)$ we let $(T)^\nu \in \mathcal{F}_k^{\text{loc}, \nu}(M)$ be the coset of T . We also define the modulo ν supports:

$$\text{spt}^\nu S = \bigcap_{R \in S} \text{spt } R \quad \text{for } S \in \mathcal{F}_k^{\text{loc}, \nu}(M),$$

$$\text{spt}^\nu T = \text{spt}^\nu(T)^\nu \quad \text{for } T \in \mathcal{F}_k^{\text{loc}}(M).$$

From the discussion of [4, p. 430] we infer that two locally rectifiable currents $T_1, T_2 \in \mathcal{R}_k^{\text{loc}}(M)$ are congruent modulo ν if and only if there exists a third chain $Q \in \mathcal{R}_k^{\text{loc}}(M)$ such that $T_1 - T_2 = \nu Q$; if, moreover, T_1 and T_2 are analytic chains in M [6, 4], then Q is also an analytic chain because

$$\text{spt } Q \subset \text{spt } T_1 \cup \text{spt } T_2, \quad \text{spt } \partial Q \subset \text{spt } \partial T_1 \cup \text{spt } \partial T_2.$$

3. A constancy lemma and some consequences.

3.1. Lemma (compare [4, 4.1.31 (2)]). *Suppose B is a connected proper k dimensional class 1 submanifold of M . If $Q, R \in \mathcal{F}_k^{\text{loc}}(M)$, $(\text{spt}^\nu Q) \sim B$ and $(\text{spt}^\nu R) \sim B$ are closed, and $(\text{spt}^\nu \partial Q) \cup (\text{spt}^\nu \partial R) \subset M \sim B$, then there exist integers i and j in $\{0, 1, \dots, \nu - 1\}$ such that*

$$\Theta^k(\|Q\|, x) \equiv i \pmod{\nu} \quad \text{and} \quad \Theta^k(\|R\|, x) \equiv j \pmod{\nu} \quad \text{for all } x \in B,$$

$$\text{spt}^\nu(jQ - iR) \subset M \sim B.$$

Proof. For any $x \in B$, there exist neighborhoods $U \subset \text{Clos } U \subset V$ of x and a class 1 map p from V into $\mathbf{R}^k$ such that $V \cap \text{spt}^\nu Q \subset B$, $V \cap \text{spt}^\nu R \subset B$, $p|_{B \cap V}$ is a proper class 1 diffeomorphism and $C = p(U \cap B)$ is an open cube in $\mathbf{R}^k$. Observing that $p_\#(Q|_V) \ll C$ and $p_\#(R|_V) \ll C$ extend [6, 3.3] to flat chains Q' and R' in $\mathbf{R}^k$ with

$$\text{spt}^\nu \partial Q' \subset [\text{spt}^\nu \partial p_\#(Q|_V)] \cup \text{spt}^\nu (\partial[p_\#(Q|_V) \ll C] - [\partial p_\#(Q|_V) \ll C])$$

$$\subset p[V \cap \text{spt}^\nu \partial Q] \cup \text{Bdry } C = \text{Bdry } C,$$

$$\text{spt}^\nu \partial R' \subset \text{Bdry } C,$$

we infer from the modulo ν version of the second proposition of [4, 4.2.3] that

$$Q' \equiv i E^k \mathbb{L} C \pmod{\nu} \quad \text{and} \quad R' \equiv j E^k \mathbb{L} C \pmod{\nu}$$

for some i and j in $\{0, 1, \dots, \nu - 1\}$; hence,

$$\Theta^k(\|Q\|, x) \equiv i \pmod{\nu} \quad \text{and} \quad \Theta^k(\|R\|, x) \equiv j \pmod{\nu} \quad \text{for all } x \in U \cap B,$$

$$U \cap \text{spt}^\nu(jQ - iR) \subset p^{-1}[C \cap \text{spt}^\nu(jQ' - iR')] = \emptyset$$

by the modulo ν analogue of the second corollary of [4, 4.1.15].

It follows that the sets

$$B_{i,j} = [B \sim \text{spt}^\nu(jQ - iR)]$$

$$\cap \{x: \Theta^k(\|Q\|, x) \equiv i \pmod{\nu} \text{ and } \Theta^k(\|R\|, x) \equiv j \pmod{\nu}\}$$

for i and j in $\{0, 1, \dots, \nu - 1\}$ form a relatively open disjointed cover of B ;

thus $B_{i,j} = B$ for some i, j .

3.2. Proposition (compare [3, 3.17], [6, 3.6]). *Suppose f is a locally Lipschitzian map of M into an open subset N of $\mathbb{R}^n$, $T \in \mathcal{R}_k^{1\infty}(M)$, and μ is a positive integer such that $f|_{\text{spt}^\nu T}$ is proper, $(\text{spt}^\nu T) \sim \text{spt}^\nu \partial T$ is locally connected, and $N \cap \{y: \text{card}(f^{-1}\{y\} \cap \text{spt}^\nu T) \leq \mu\}$ is dense in $N \sim f(\text{spt}^\nu \partial T)$. Let*

$$Y = [N \sim f(\text{spt}^\nu \partial T)] \cap \{y: \text{card}(f^{-1}\{y\} \cap \text{spt}^\nu T) < \infty\},$$

G be the class of all nonempty connected open subsets of $N \sim f(\text{spt}^\nu \partial T)$ and, for $W \in G$, $\Gamma(W)$ be the set of all components of $f^{-1}(W) \cap \text{spt}^\nu T$; also let

$$H = \bigcup \{\Gamma(W): W \in G\}, \quad H^* = \{V \cap f^{-1}(Y): V \in H\}.$$

Then the following eight conclusions hold:

(1) For each $V \in \Gamma(W)$ there exists an integer $\Delta(V) \in \{0, 1, \dots, \nu - 1\}$ such that $f_\#(T \mathbb{L} V) \equiv \Delta(V) E^n \mathbb{L} W \pmod{\nu}$.

(2) If $V \in \Gamma(W)$ and $\Delta(V) \neq 0$, then $f(V) = W$.

(3) $\text{Card}[\Gamma(W) \cap \{V: \Delta(V) \neq 0\}] \leq \mu$.

(4) If $W \subset W'$ belong to G and $V' \in \Gamma(W')$, then $\Delta(V') \equiv \sum_{V' \supset V \in \Gamma(W)} \Delta(V) \pmod{\nu}$.

(5) The family H^* is a base for the relative topology of $f^{-1}(Y) \cap \text{spt}^\nu T$.

(6) If $x \in f^{-1}(Y) \cap \text{spt}^\nu T$, then $\Delta(V)$ has the same value, hereafter denoted $\Delta(x)$, for all sufficiently small neighborhoods V of x belonging to H .

(7) If $y \in Y$, then

$$f_\# [T \mathbb{L} f^{-1}(W)] \equiv \left[\sum_{x \in f^{-1}\{y\} \cap \text{spt}^\nu T} \Delta(x) \right] E^n \mathbb{L} W \pmod{\nu}$$

for all sufficiently small neighborhoods W of y belonging to G .

(8) The function mapping $y \in Y$ onto $\sum_{x \in f^{-1}\{y\} \cap \text{spt}^\nu T} \Delta(x) \delta_x$ is mass bounded by $\mu\nu$ and continuous in the modulo ν topology.

Proof. To prove (1) we observe that

$$\begin{aligned} W \cap \text{spt}^\nu \partial f_\#(T \llcorner V) &\subset f[f^{-1}(W) \cap \text{spt}^\nu \partial(T \llcorner V)] \\ &\subset f([f^{-1}(W) \cap \text{spt}^\nu \partial T] \cup [f^{-1}(W) \cap \text{spt}^\nu T \cap \text{Bdry } V]) = \emptyset, \end{aligned}$$

and apply 3.1 with $B = W$, $Q = E^n \llcorner W$, hence $i = 1$, and $R = f_\#(T \llcorner V)$ to choose $\Delta(V) \in \{0, 1, \dots, \nu - 1\}$ such that

$$\begin{aligned} \text{spt}^\nu [f_\#(T \llcorner V) - \Delta(V)E^n \llcorner W] &\subset \mathbb{R}^n \sim W, \\ f_\#(T \llcorner V) &= f_\#(T \llcorner V) \llcorner W \equiv \Delta(V)E^n \llcorner W \pmod{\nu}. \end{aligned}$$

We verify (2) by noting that if $\Delta(V) \neq 0$, then

$$\begin{aligned} W \subset W \cap \text{spt}^\nu f_\#(T \llcorner V) &\subset f[f^{-1}(W) \cap \text{spt}^\nu (T \llcorner V)] \\ &\subset f[f^{-1}(W) \cap \text{spt}^\nu T \cap \text{Clos } V] = f(V). \end{aligned}$$

To prove (3) we choose a point $y \in W$ with $\text{card}(f^{-1}\{y\} \cap \text{spt}^\nu T) \leq \mu$ and apply (2).

For the proof of (4) we have only to modify the proof of [3, 3.7 (4)] by changing the first and last equalities to congruences modulo ν .

(5) and (6) follow just as in the proofs of [3, 3.17 (5) (6)].

To prove (7) we abbreviate $F = f^{-1}\{y\} \cap \text{spt}^\nu T$, choose open sets W_x and disjoint sets V_x for $x \in F$ so that $y \in W_x \in G$, $x \in V_x \in \Gamma(W_x)$, and $f_\#(T \llcorner V_x) \equiv \Delta(x)E^n \llcorner W_x \pmod{\nu}$, and observe that F is finite,

$$\delta = \text{dist} \left[y, \left(\bigcup_{x \in F} \text{Bdry } W_x \right) \cup f \left(\text{spt}^\nu T \sim \bigcup_{x \in F} V_x \right) \right]$$

is positive, and any $W \in G$ contained in $U(y, \delta)$ satisfies

$$\begin{aligned} f^{-1}(W) \cap \text{spt}^\nu T &\subset \bigcup_{x \in F} V_x \quad \text{and} \\ f_\# [T \llcorner f^{-1}(W)] &\equiv \sum_{x \in F} f_\# (T \llcorner [V_x \cap f^{-1}(W)]) \\ &\equiv \sum_{x \in F} \Delta(x)E^n \llcorner (W_x \cap W) = \left(\sum_{x \in F} \Delta(x) \right) E^n \llcorner W \pmod{\nu}. \end{aligned}$$

The mass bound in (8) follows from (6) and (3). A proof of the modulo ν continuity results from modifying the wording of the proof of [6, 3.6 (9)] by changing $\text{spt } T$ to $\text{spt}^\nu T$ and the second equality in the computation of ∂S to a congruence modulo ν .

4. **Analytic chains modulo ν .** Assuming now that M is a separable m dimensional real analytic Riemannian manifold, we first recall the notion of real analytic dimension (which is defined in [6, 2.2]):

If A is an analytic subset of M , then $\dim A = \sup\{-1, k: A \text{ contains a nonempty } k \text{ dimensional analytic submanifold of } M\}$.

If E is an arbitrary subset of M , then $\dim E = \inf\{k: E \subset \bigcup_{U \in \mathcal{U}} A_U \text{ for some locally-finite open cover } \mathcal{U} \text{ of } M \text{ and } k \text{ dimensional analytic subsets } A_U \text{ of } U \text{ for } U \in \mathcal{U}\}$.

We will call S a k dimensional analytic chain modulo ν in M if and only if (compare [6, §4]) $S \in \mathcal{F}_k^{\text{loc}, \nu}(M)$, $\dim(\text{spt}^\nu S) \leq k$, $\dim(\text{spt}^\nu \partial S) \leq k - 1$. It then follows that *there exists a k dimensional analytic chain T belonging to S with $\text{spt } T = \text{spt}^\nu S$.*

In fact, for every $x \in M$ there exist, by [4, 3.4.8 (11)], an open ball U about x and a finite family $\mathcal{B}$ of disjoint k dimensional orientable analytic blocks B [6, §2] in $U \cap (\text{spt}^\nu S \sim \text{spt}^\nu \partial S)$ with orienting k vectorfields β_B such that the real analytic dimension of $Z = U \cap (\text{spt}^\nu S \sim \text{spt}^\nu \partial S \sim \bigcup \mathcal{B})$ is less than k . Applying, for each $B \in \mathcal{B}$, 3.1 to $Q = (\mathcal{H}^k \llcorner B) \wedge \beta_B$ and any locally flat chain R belonging to S (hence, $i = 1$, $\text{spt}^\nu R = \text{spt}^\nu S$, and $\text{spt}^\nu \partial R = \text{spt}^\nu \partial S$), we choose an integer $j_B \in \{1, 2, \dots, \nu - 1\}$ so that

$$\text{spt}^\nu [S - (j_B [\mathcal{H}^k \llcorner B] \wedge \beta_B)^\nu] = \text{spt}^\nu [R - j_B (\mathcal{H}^k \llcorner B) \wedge \beta_B] \subset M \sim B.$$

Then

$$U \cap \text{spt}^\nu S - \left[\left(\sum_{B \in \mathcal{B}} j_B [\mathcal{H}^k \llcorner B] \wedge \beta_B \right)^\nu \right]$$

is contained in $Z \cup \text{spt}^\nu \partial S$, has $\mathcal{H}^k$ measure zero by [6, 2.2 (5) (4)], and is empty by [4, 4.2.26 (4.2.14) $^\nu$]. Moreover

$$U \cap \text{spt}^\nu S = U \cap \text{Clos } \bigcup \mathcal{B} = U \cap \text{spt} \sum_{B \in \mathcal{B}} j_B (\mathcal{H}^k \llcorner B) \wedge \beta_B.$$

To construct T globally we choose open balls $U_1, U_2, \dots$ along with analytic chains $T_1, T_2, \dots$ in M such that $\{U_1, U_2, \dots\}$ is a locally finite open cover of M and

$$U_i \cap \text{spt}^\nu [S - (T_i)^\nu] = \emptyset, \quad U_i \cap \text{spt}^\nu S = U_i \cap \text{spt } T_i.$$

Then we use [6, 2.2 (7)] to select, for $i \in \{1, 2, \dots\}$, open balls V_i with closure in U_i such that $M \subset \bigcup_{i=1}^\infty V_i$ and

$$\mathcal{H}^k([(spt^\nu S) \cup (spt T_i)] \cap \text{Bdry } V_i) = 0$$

for i in $\{0, 1, \dots\}$ and j in $\{0, 1, \dots, \nu - 1\}$, define the analytic chain

$$T = T_1 \sqcup V_1 + \sum_{i=2}^{\infty} T_i \sqcup \left(V_i \sim \bigcup_{j=1}^{i-1} V_j \right),$$

and conclude that $\text{spt } T = \text{spt}^{\nu} T$ and that $\mathcal{H}^k(\text{spt}^{\nu}[S - (T)^{\nu}]) = 0$; hence $S = (T)^{\nu}$ by [4, 4.2.26 (4.2.14) $^{\nu}$].

4.1. Proof of the Slicing Modulo ν Theorem. We fix an analytic chain T belonging to S with $\text{spt } T = \text{spt}^{\nu} S$; let

$$W = \mathbb{R}^n \cap \{w: \dim(f^{-1}\{w\} \cap \text{spt } T) \leq k - n \text{ and } \dim(f^{-1}\{w\} \cap \text{spt } \partial T) \leq k - n - 1\},$$

hence $W \subset Y$, and observe that it suffices to prove the following statement:

There exists a continuous map $\mathcal{S}$ from Y into the $k - n$ dimensional analytic chains modulo ν in M such that $\mathcal{S}(w) = (\langle T, f, w \rangle)^{\nu}$ for every $w \in W$.

In fact suppose this statement is true and T' is another analytic chain belonging to S with

$$\begin{aligned} W' &= \mathbb{R}^n \cap \{w: \dim(f^{-1}\{w\} \cap \text{spt } T') \leq k - n \text{ and} \\ &\quad \dim(f^{-1}\{w\} \cap \text{spt } \partial T') \leq k - n - 1\}. \end{aligned}$$

Then $T' - T = \nu Q$ for some k dimensional analytic chain Q , and for any $w \in W' \cap W$,

$$\dim(f^{-1}\{w\} \cap \text{spt } Q) \leq k - n, \quad \dim(f^{-1}\{w\} \cap \text{spt } \partial Q) \leq k - n - 1;$$

hence, by [6, 4.3], $\langle Q, f, w \rangle$ is an analytic chain in M , and

$$\langle T', f, w \rangle - \langle T, f, w \rangle = \nu \langle Q, f, w \rangle \equiv 0 \pmod{\nu}.$$

The two continuous maps $\mathcal{S}|_{W'}$ and $(\langle T', f, \cdot \rangle)^{\nu}$ of W' agree on the dense [6, 2.2 (7)] subset $W' \cap W$, hence are equal.

To prove the statement we consider three cases.

Case 1. M is an open subset of $\mathbb{R}^m$, $\text{spt } T$ is compact, and $k = n$. Here we use 3.1 and [4, 3.4.8 (11)], reasoning as in [4, 4.2.28] to see that $\text{spt}^{\nu} T \sim \text{spt}^{\nu} \partial T$ is locally connected, note that $f|_{\text{spt } T}$ is proper because $\text{spt } T$ is compact, choose, according to [6, 2.11 (1)] a positive integer μ so that the set

$$\mathbb{R}^n \cap \{y: \text{card}(f^{-1}\{y\} \cap \text{spt } T) > \mu\}$$

has Lebesgue measure zero, and then apply 3.2 (7) by setting

$$\mathcal{S}(y) = \left(\sum_{x \in f^{-1}\{y\} \cap \text{spt}^{\nu} T} \Delta(x) \delta_x \right)^{\nu} \quad \text{for } y \in Y.$$

The statement follows by comparing 3.2 (8) (6) (1) and [6, 3.6 (8) (6) (1)].

Case 2. M is an open subset of $\mathbb{R}^m$, $\text{spt } T$ is compact, and $k > n$. Here we assume, for contradiction, that the statement is false. Since W is dense in $\mathbb{R}^n$ by [6, 2.2 (7)], there then must exist a point $y \in Y$ such that $\langle (T, f, w) \rangle^\nu$ fails to converge as w approaches y in W . Assured by [4, 4.2.17] and [6, 4.2] that the family $\{(T, f, w): w \in W\}$ is relatively compact in $\mathbf{I}_k(M)$ we choose two sequences $w_{1,1}, w_{1,2}, w_{1,3}, \dots$ and $w_{2,1}, w_{2,2}, w_{2,3}, \dots$ in W converging to y and integral currents L_1 and L_2 in $\mathbf{I}_k(M)$ such that $\text{spt}^\nu(L_1 - L_2)$ is nonempty and $\langle T, f, w_{i,j} \rangle$ approaches L_i , for $i \in \{1, 2\}$, as j approaches ∞ . Since

$$\text{spt}^\nu L_i \subset f^{-1}\{y\} \cap \text{spt}^\nu S, \quad \text{spt}^\nu \partial L_i \subset f^{-1}\{y\} \cap \text{spt}^\nu \partial S,$$

$(L_i)^\nu$ is an analytic chain modulo ν in M for $i \in \{1, 2\}$. Thus we may select an open set U in $M \sim \text{spt}^\nu \partial(L_1 - L_2)$ and an orthogonal projection $p: M \rightarrow \mathbb{R}^{k-n}$ such that $B = U \cap \text{spt}^\nu(L_1 - L_2)$ is a nonempty connected $k - n$ dimensional analytic submanifold of M and $p|_B$ is an analytic isomorphism. Fixing, by [6, 2.2 (7)], $z \in p(B)$ so that

$$\dim(f^{-1}\{y\} \cap p^{-1}\{z\} \cap \text{spt}^\nu S) = 0, \quad f^{-1}\{y\} \cap p^{-1}\{z\} \cap \text{spt}^\nu \partial S = \emptyset$$

we infer from 3.1, 3.2 (6), and Case 1 that

$$\langle (L_1 - L_2)^\nu, p, z \rangle^\nu|_U \neq (0)^\nu.$$

Next using [6, 3.2] we pass to subsequences, without changing notations, so that there exist for $i \in \{1, 2\}$ and $j \in \{1, 2, \dots\}$ rectifiable chains $R_{i,j} \in \mathcal{R}_{k-n}(M)$, $S_{i,j} \in \mathcal{R}_{k-n+1}(M)$ with

$$\text{spt}(L_i - \langle T, f, w_{i,j} \rangle - R_{i,j} - \partial S_{i,j}) \subset M \sim U,$$

$$\mathbf{M}(R_{i,j}) + \mathbf{M}(S_{i,j}) \leq j^{-1} \mathcal{Q}^{k-n} U(z, j^{-1}),$$

and then apply [6, 2.2 (7)] and [4, 4.3.6, 4.3.2 (2)] to choose points $z_{i,j} \in \mathbb{R}^{k-n} \cap U(z, j^{-1})$ such that

$$\dim(p^{-1}\{z_{i,j}\} \cap [(f^{-1}\{w_{i,j}\} \cap \text{spt } T) \cup \text{spt } L_i]) \leq 0,$$

$$p^{-1}\{z_{i,j}\} \cap [(f^{-1}\{w_{i,j}\} \cap \text{spt } \partial T) \cup \text{spt } \partial L_i] = \emptyset,$$

$$\langle R_{i,j}, p, z_{i,j} \rangle \in \mathcal{R}_0(M), \quad \langle S_{i,j}, p, z_{i,j} \rangle \in \mathcal{R}_1(M),$$

$$\mathbf{M}\langle R_{i,j}, p, z_{i,j} \rangle + \mathbf{M}\langle S_{i,j}, p, z_{i,j} \rangle \leq j^{-1},$$

and deduce from [6, 3.5 (2)] that the slices

$$\langle L_i - \langle T, f, w_{i,j} \rangle, p, z_{i,j} \rangle|_U = [\langle R_{i,j}, p, z_{i,j} \rangle + (-1)^{k-n} \partial \langle S_{i,j}, p, z_{i,j} \rangle]|_U$$

approach 0 as j approaches ∞ for $i \in \{1, 2\}$. Finally applying Case 1 twice and [6, 4.5], we obtain the desired contradiction by computing

$$\begin{aligned}
 (0)^\nu &\neq \langle (L_1 - L_2)^\nu, p, z \rangle^\nu | U = \lim_{j \rightarrow \infty} (\langle L_1, p, z_{1,j} \rangle - \langle L_2, p, z_{2,j} \rangle)^\nu | U \\
 &= \lim_{j \rightarrow \infty} (\langle \langle T, f, w_{1,j} \rangle, p, z_{1,j} \rangle - \langle \langle T, f, w_{2,j} \rangle, p, z_{2,j} \rangle)^\nu | U \\
 &= \lim_{j \rightarrow \infty} (\langle T, f \boxplus p, (w_{1,j}, z_{1,j}) \rangle - \langle T, f \boxplus p, (w_{2,j}, z_{2,j}) \rangle)^\nu | U \\
 &= \langle \langle (T)^\nu, f \boxplus p, (y, z) \rangle^\nu - \langle (T)^\nu, f \boxplus p, (y, z) \rangle^\nu \rangle | U.
 \end{aligned}$$

Case 3. General case. Assuming the theorem false, we choose, by the modulo ν analogue of [6, 3.2 (1)] an analytic isomorphism ϕ of some open subset V of M onto $\mathbb{R}^m \cap U(0, 2)$, $U = \phi^{-1}[U(0, 1)]$, a countable subset C of W , and a point $y \in Y \cap \text{Clos } C$ such that $\langle T, f, w \rangle^\nu | U = \langle T | U, f | U, w \rangle^\nu$ fails to converge as w approaches y in C . Choosing, by [6, 2.2 (7)], r between 1 and 2 such that

$$\begin{aligned}
 \dim(|\phi|^{-1}\{r\} \cap \text{spt } T) &\leq k - 1, \\
 \dim(|\phi|^{-1}\{r\} \cap f^{-1}\{y\} \cap \text{spt } T) &\leq k - n - 1, \\
 \dim(|\phi|^{-1}\{r\} \cap f^{-1}\{w\} \cap \text{spt } T) &\leq k - n - 1,
 \end{aligned}$$

we infer that

$$\langle \phi_\#(T|V) \llcorner U(0, r), f \circ \phi^{-1}, w \rangle^\nu | U(0, 1) = \langle \phi|U \rangle_\# \langle T|U, f|U, w \rangle^\nu$$

fails to converge as w approaches y in C , which contradicts either Case 1 or Case 2 with S, T, f replaced by $[\phi_\#(T|V) \llcorner U(0, r)]^\nu$, $\phi_\#(T|V) \llcorner U(0, r)$, $f \circ \phi^{-1}$.

4.2. Remark. With T, f, Y, W as in 4.1, we see that the equation $\langle \langle T, f, y \rangle \rangle^\nu = \langle \langle T \rangle^\nu, f, y \rangle^\nu$ holds whenever $y \in W$. For $y \in Y \sim W$ however, this may fail to be true even though the slice $\langle T, f, y \rangle$, as defined in [4, 4.3], is an analytic chain in M . For example if $f: \mathbb{R} \rightarrow \mathbb{R}$ is the identity map and

$$T = E^1 \llcorner \{x: x > 0\} - E^1 \llcorner \{x: x < 0\},$$

then

$$\langle \langle T, f, 0 \rangle \rangle^2 = (0)^2 \neq (\delta_0)^2 = \langle \langle T \rangle^2, f, 0 \rangle^2.$$

4.3. For S, f, Y as in the Slicing Modulo ν Theorem and $y \in Y$, we readily infer, by continuity and the definition of the slice [4, 4.3.1], the following four statements:

- (1) $\text{spt}^\nu \langle S, f, y \rangle^\nu \subset f^{-1}\{y\} \cap \text{spt}^\nu S$.
- (2) $\partial \langle S, f, y \rangle^\nu = (-1)^n \langle \partial S, f, y \rangle^\nu$ in case $k > n$.
- (3) $\langle S, f, y \rangle^\nu|_U = \langle S|_U, f|_U, y \rangle^\nu$ whenever U is an open subset of M .
- (4) $\langle S, b \circ f, b(y) \rangle^\nu = \langle S, f, y \rangle^\nu$ whenever b is an orientation-preserving analytic isomorphism of $\mathbb{R}^n$.

4.4. We will now discuss how *all of the statements and most of the proofs of [6, 4.4–5.11] carry over to the modulo ν case*. The wording of these propositions should be modified by replacing “chain, spt , $\mathcal{F}_*^{\text{loc}}(M)$, and $\langle \ , \ , \ \rangle$ ” to “chain modulo ν , spt^ν , $\mathcal{F}_*^{\text{loc}, \nu}$, and $\langle \ , \ , \ \rangle^\nu$ ”. We will number such generalizations by using the superscript ν above the number of the corresponding proposition of [6].

(4.4) $^\nu$ Here it is necessary to define $\langle f_\# T, g, y \rangle^\nu$ as $f_\# \langle \langle T, g \circ f, y \rangle^\nu \rangle$ and then observe by 4.2 and [6, 4.4] that this is consistent in case $f_\# T$ is an analytic chain modulo ν .

(4.5) $^\nu$ requires a new proof; we prove the first conclusion by considering two cases:

Case 1. $s = n + l$. Here, assuming the theorem false, we choose a point

$$x \in \text{spt}^\nu [\langle S, f \boxminus g, (a, b) \rangle^\nu - \langle \langle S, f, a \rangle^\nu, g, b \rangle^\nu],$$

then an open ball U about x in M and i and j in $\{0, 1, \dots, \nu - 1\}$ so that

$$U \cap \text{spt}^\nu \partial S = \emptyset, \quad (\text{Clos } U) \cap f^{-1}\{a\} \cap g^{-1}\{b\} \cap \text{spt}^\nu S = \{x\},$$

$$\langle S, f \boxminus g, (a, b) \rangle^\nu|_U = i(\delta_x)^\nu|_U \neq j(\delta_x)^\nu|_U = \langle \langle S, f, a \rangle^\nu, g, b \rangle^\nu|_U.$$

Since

$$\begin{aligned} 0 < \delta &= \inf \{ |(a, b) - (f \boxminus g)(w)| : w \in (\text{Bdry } U) \cap \text{spt}^\nu S \} \\ &\leq \inf \{ |b - g(w)| : w \in (\text{Bdry } U) \cap \text{spt}^\nu \langle S, f, a \rangle^\nu \}, \end{aligned}$$

$V = U \cap (f \boxminus g)^{-1}(U[(a, b), \delta])$ is open and the two functions, $(f|_V) \boxminus (g|_V)|_{\text{spt}^\nu S}$ and $(g|_V)|_{\text{spt}^\nu \langle S, f, a \rangle^\nu}$ are proper maps. Applying 3.2 (7) twice—with M, N, f , and T replaced:

first, by $V, U[(a, b), \delta], (f|_V) \boxminus (g|_V)$, and any analytic chain belonging to $S|_V$, and

second, by $V, U(b, \delta), g|_V$, and any analytic chain belonging to $\langle S, f, a \rangle^\nu|_V$, we choose open balls, W about a in $\mathbb{R}^n$ and Z about b in $\mathbb{R}^l$, and

$$\begin{aligned} \alpha &= f|_U \cap (f \boxminus g)^{-1}(W \times Z), & \beta &= g|_U \cap (f \boxminus g)^{-1}(W \times Z), \\ R &= S|_U \cap (f \boxminus g)^{-1}(W \times Z) \end{aligned}$$

such that $W \times Z \subset U[(a, b), \delta]$ and

$$(\alpha \boxminus \beta)_\# R = i(\mathbb{E}^n \times \mathbb{E}^l)^\nu|_{(W \times Z)}, \quad \beta_\# \langle R, \alpha, a \rangle^\nu = j(\mathbb{E}^l)^\nu|_Z.$$

Letting $q: W \times Z \rightarrow W$, $b: U \cap (f \boxminus g)^{-1}(W \times Z) \rightarrow W \times Z$ be given by $q(w, z) = w$, $b(u) = (a, \beta(u))$ for $(w, z) \in W \times Z$ and $u \in U \cap (f \boxminus g)^{-1}(W \times Z)$, we use (4.4) $^\nu$ and the modulo ν version of [4, 4.1.15] to compute

$$\begin{aligned} i(\delta_a \times E^l)^\nu | (W \times Z) &= \langle (\alpha \boxminus \beta)_\# R, q, a \rangle^\nu = \langle \alpha \boxminus \beta \rangle_\# \langle R, \alpha, a \rangle^\nu = b_\# \langle R, \alpha, a \rangle^\nu \\ &= [(\delta_a)^\nu | W] \times \beta_\# \langle R, \alpha, a \rangle^\nu = j(\delta_a \times E^l)^\nu | (W \times Z), \end{aligned}$$

hence $i = j$, a contradiction.

Case 2. $s > n + l$. Again assuming the theorem is false, we choose an open set U in $M \sim \text{spt}^\nu \partial S$ and an analytic map $p: U \rightarrow \mathbf{R}^{s-n-l}$ such that

$$B = U \cap \text{spt}^\nu \langle \langle S, f \boxminus g, (a, b) \rangle^\nu - \langle \langle S, f, a \rangle^\nu, g, b \rangle^\nu \rangle$$

is a nonempty connected analytic submanifold of M and $p|_B$ is an analytic isomorphism, let $w \in p(B)$, abbreviate $\bar{S} = S|_U$, $\bar{f} = f|_U$, $\bar{g} = g|_U$ and use 3.1, 3.2 (6) (1), Case 1, and 4.3 (4) to derive the contradiction

$$\begin{aligned} (0)^\nu &\neq \langle \bar{S}, \bar{f} \boxminus \bar{g}, (a, b) \rangle^\nu - \langle \bar{S}, \bar{f}, a \rangle^\nu, \bar{g}, b \rangle, p, w \rangle^\nu \\ &= \langle \bar{S}, \bar{f} \boxminus \bar{g}, (a, b) \rangle^\nu, p, w \rangle^\nu - \langle \bar{S}, \bar{f}, a \rangle^\nu, \bar{g} \boxminus p, (b, w) \rangle^\nu \\ &= \langle \bar{S}, (\bar{f} \boxminus \bar{g}) \boxminus p, ((a, b), w) \rangle^\nu - \langle \bar{S}, \bar{f} \boxminus (\bar{g} \boxminus p), (a, (b, w)) \rangle^\nu = (0)^\nu. \end{aligned}$$

The second conclusion of (4.5) $^\nu$ is readily obtained from the first as in [6, 4.5].

(4.6) $^\nu$ through (4.9) $^\nu$ now follows as in [6, 4.6–4.9].

In (§5) $^\nu$, M and N need only be orientable modulo ν with orienting modulo ν cycles $\mathfrak{M}, \mathfrak{N}$. The intersection modulo ν of Q and R , $Q \cap^\nu R \in \mathcal{F}_{q+r-m}^{\text{loc}, \nu}(M)$, is to be defined in a manner similar to [6, 5.0 (1)] where b now need only preserve orientation modulo ν ; also we make the obvious generalization to obtain the notion that $\{S, T\}$ intersect suitably modulo ν .

For (5.1) $^\nu$ we assume now that Q and R are *analytic chains modulo ν* satisfying

$$\begin{aligned} \dim(b^{-1}\{y\} \cap \text{spt}^\nu Q) &\leq q - k, & \dim(b^{-1}\{y\} \cap \text{spt}^\nu \partial Q) &\leq q - k - 1, \\ \dim(c^{-1}\{z\} \cap \text{spt}^\nu R) &\leq r - l, & \dim(c^{-1}\{z\} \cap \text{spt}^\nu \partial R) &\leq r - l - 1, \end{aligned}$$

and obtain the desired equations by continuity from [6, 5.1].

(5.2) $^\nu$ through (5.6) $^\nu$ may now be deduced as in [6, 5.2–5.6].

In (5.7) $^\nu$, L is an analytic chain modulo ν , $\mathfrak{M} = (E^m | M)^\nu$, and $\mathfrak{N} = (E^n | N)^\nu$, and we apply [6, 5.7] to any member T of L to infer that

$$\begin{aligned}
\tau_{\#} \langle \mathfrak{M} \times L, f \circ \sigma, 0 \rangle^{\nu} &= (\tau_{\#} \langle (E^m | M) \times T, f \circ \sigma, 0 \rangle)^{\nu} = (T)^{\nu} = L \\
&= (-1)^{(l+n)n} \langle \tilde{\sigma}_{\#} (T \times (E^n | N), \tilde{f} \circ \tilde{\tau}, 0) \rangle^{\nu} \\
&= (-1)^{(l+n)n} \tilde{\sigma}_{\#} \langle L \times \mathfrak{N}, \tilde{f} \circ \tilde{\tau}, 0 \rangle^{\nu}.
\end{aligned}$$

(5.8) $^{\nu}$ through (5.11) $^{\nu}$, Case 4, follow from the proofs of [6, 5.8–5.10, Case 4] by use of 3.2 in place of [4, 4.1.31] and [4, 4.2.28 (4.2.14) $^{\nu}$] in place of [4, 4.1.20].

To modify the proof of Case 5 it suffices to use 3.2 (7) to choose ρ and σ small enough so that

$$\begin{aligned}
(f | V)_{\#} [(S \times T) | V] &= (-1)^{(m-s)t_x} [E^m | U(0, \sigma)]^{\nu}, \\
\langle (S \times T) | V, f | V, 0 \rangle^{\nu} &= (-1)^{(m-s)t_x} (\delta_x | V)^{\nu}.
\end{aligned}$$

For Case 6, instead of using [6, 4.1], we observe that

If R is a k dimensional analytic chain modulo ν in an open subset M of $\mathbb{R}^m$ and $\langle R, p_{\lambda} | M, y \rangle^{\nu}$ is zero for all $\lambda \in \Lambda(m, k)$ and $\mathbb{Q}^k$ almost all y in $\mathbb{R}^k$, then R equals zero.

In fact, otherwise for any regular point x of $\text{spt}^{\nu} R \sim \text{spt}^{\nu} \partial R$ there is a $\lambda \in \Lambda(m, k)$ such that $\dim p_{\lambda} [\text{Tan}(\text{spt}^{\nu} R, x)] = k$, and, by [4, 3.1.18], an open neighborhood U of x in $M \sim \text{spt}^{\nu} \partial R$ such that $B = U \cap \text{spt}^{\nu} R$ is a connected analytic submanifold of M and $p_{\lambda} | B$ is an analytic isomorphism. Then by 3.1, $(p_{\lambda} | U)_{\#} (R | U)^{\nu}$ equals $[jE^k | p_{\lambda}(B)]^{\nu}$ for some $j \in \{1, 2, \dots, \nu - 1\}$; hence, by 3.2,

$$\langle R, p_{\lambda} | M, p_{\lambda}(x) \rangle^{\nu} | U = \langle R | U, p_{\lambda} | U, p_{\lambda}(x) \rangle^{\nu} = (j\delta_x | U)^{\nu} \neq (0)^{\nu}$$

whenever $x \in B$, a contradiction.

Finally Case 7 follows from Case 5, Case 6, and (5.11) $^{\nu}$ (3) (4) as before.

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