THE PERIODS OF EICHLER INTEGRALS FOR KLEINIAN GROUPS

BY

HIROKI SATO

ABSTRACT. We shall give period relations and inequalities for Eichler integrals for Kleinian groups Γ which have simply connected components of the region of discontinuity of Γ . These are a generalization of those for abelian integrals. By using the period inequality, we shall give an alternate proof of a result of Kra.

0. Introduction. Let Γ be a nonelementary finitely generated Kleinian group, and Δ_1 a simply connected component of the region of discontinuity Ω of Γ .

M. Eichler [4], L. V. Ahlfors [2], L. Bers [3] and I. Kra [5], [6] have represented periods of Eichler integrals as polynomials of degree at most $2q - 2$, $q \geq 2$ being an integer. By this method, however, period relations for Eichler integrals are very complicated even when Γ is a Fuchsian group of the first kind (Eichler [4]). On the other hand, G. Shimura [7] has regarded the periods as column number vectors of length $2q - 1$. In his paper he gave a certain period relation for Fuchsian groups.

By using Shimura's idea, we shall give period relations and inequalities for Eichler integrals for Kleinian groups. These are a generalization of those for abelian integrals. The main results in this paper are Theorems 1 and 2.

We shall state some notations in §1 and some lemmas in §2. In §3 we shall prove Theorem 1 and in §4 we shall state the period relations and inequalities, and give an alternate proof for the Kra result [5].

The author wishes to express his deep appreciation to Professor I. Kra, K. Mathumoto and K. Oikawa for encouragement and advice.

1. Notation. Throughout this paper Γ denotes a nonelementary finitely generated Kleinian group with a simply connected component Δ_1 of the region of discontinuity Ω of Γ . We denote by Λ the limit set, $\lambda(z)|dz|$ the Poincaré metric on Ω . Let $q \geq 1$ be an integer. Set $\Delta = \bigcup_{A \in \Gamma} A(\Delta_1)$. It is a well-known fact (cf. [1]) that Δ/Γ is a Riemann surface which is obtained from a compact Riemann surface, denoted by $\overline{\Delta/\Gamma}$, by deleting a finite number of points. It is

Received by the editors February 1, 1972 and, in revised form, January 16, 1973.

AMS (MOS) subject classifications (1970). Primary 30A58; Secondary 10D15, 20H10, 32N10.

Key words and phrases. Eichler integrals, Kleinian groups, cohomology group, potential, Beltrami coefficients, period relation and inequality.

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also known that Δ is a (disconnected) covering surface of Δ/Γ which ramifies over only a finite number of points.

We denote by R^n and C^n n -dimensional vector spaces over R and C , respectively, $n \geq 0$ being an integer. We regard an element in R^n (C^n) as a matrix with n rows and 1 column. We consider an element of Γ as a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc = 1$. We denote by $GL(m, C)$ the group of $m \times m$ invertible matrices over C . Let $\begin{pmatrix} u \\ v \end{pmatrix}$ be a vector in C^2 . For each $n = 2q - 2$, we denote by $\begin{pmatrix} u \\ v \end{pmatrix}^n$ the vector in C^{n+1} whose components are $u^n, u^{n-1}, \dots, uv^{n-1}, v^n$, where $\begin{pmatrix} u \\ v \end{pmatrix}^0 = 1$. For $A \in \Gamma$ we set $\begin{pmatrix} u_A \\ v_A \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix}$ and define $M(A) \in GL(n+1, C)$ by

$$\begin{pmatrix} u_A \\ v_A \end{pmatrix}^n = M(A) \begin{pmatrix} u \\ v \end{pmatrix}^n.$$

The following is due to Ahlfors [1]. Let $\Delta/\Gamma = S - \{p\}$ where S is a Riemann surface and $p \in S$. If there is a punctured neighborhood $M(p)$ of p such that π is unramified over $M(p)$, then there exists a parabolic transformation $A \in \Gamma$ with fixed point $s \in \Lambda$, and there is a Möbius transformation B with the following properties: (1) $B(\infty) = s$ and $B^{-1}AB(z) = z + 1$, $z \in C$, (2) $B^{-1}(\Delta_1)$ contains a half-plane $U_{B^{-1}AB} = \{z \in C | \operatorname{Im} z > c\}$, for some $c > 0$, (3) two points z_1 and z_2 of $B(U_{B^{-1}AB})$ are equivalent under Γ if and only if $z_2 = A^m(z_1)$ for some integer m , and (4) the image of $B(U_{B^{-1}AB})$ under π is a deleted neighborhood of p homeomorphic to a punctured disk. We call $W_A = B\{z \in C | 0 \leq \operatorname{Re} z < 1, \operatorname{Im} z > c\}$ a cusped region belonging to p .

A mapping $\chi: \Gamma \rightarrow C^{2q-1}$ is called a cocycle if $\chi_{AB} = {}^tM(B)\chi_A + \chi_B$ for $A, B \in \Gamma$, where ${}^tM(B)$ is the transposed matrix of $M(B)$. A cocycle $\chi: \Gamma \rightarrow C^{2q-1}$ is called a coboundary if there exists $V \in C^{2q-1}$ such that $\chi_A = {}^tM(A)V - V$ for any $\chi_A \in C^{2q-1}$, $A \in \Gamma$. Then the first cohomology group $H^1(\Gamma, C^{2q-1}, M)$ is the space of cocycles factored by the space of coboundaries. A cohomology class $P \in H^1(\Gamma, C^{2q-1}, M)$ is called Δ -parabolic if, for every parabolic transformation $B \in \Gamma$ corresponding to a puncture on Δ/Γ , there is a $V \in C^{2q-1}$ such that $P_B = {}^tM(B)V - V$ for some (and hence every) cocycle that represents P . The space of Δ -parabolic cohomology class is denoted by $PH_\Delta^1(\Gamma, C^{2q-1}, M)$.

For an $m \times n$ matrix $N = (a_{ij})$ ($i = 1, 2, \dots, n; j = 1, 2, \dots, m$) matrices $\bar{N}$ and $\tilde{N}$ are defined by $\bar{N} = (\bar{a}_{ij})$ and $\tilde{N} = (a_{m-i+1, n-j+1})$, respectively, where $\bar{a}_{ij}$ is the complex conjugate of a_{ij} .

A holomorphic function ϕ on Δ is called an automorphic form of weight $(-2q)$ on Δ , $q \geq 1$, if $\phi(Az)A'(z)^q = \phi(z)$ for all $A \in \Gamma$. For $q \geq 2$ an automorphic form ϕ of weight $(-2q)$ on Δ is called integrable if

$$\iint_{\Delta/\Gamma} \lambda(z)^{2-q} |\phi(z)| dx dy < \infty.$$

We denote the Banach space of integrable automorphic forms on Δ by $A_q(\Delta, \Gamma)$. The form ϕ is called bounded if

$$\sup\{\lambda(z)^{-q}|\phi(z)| \mid z \in \Delta\} < \infty.$$

The Banach space of bounded automorphic forms on Δ is denoted by $B_q(\Delta, \Gamma)$. For $\phi \in A_q(\Delta, \Gamma)$ and $\psi \in B_q(\Delta, \Gamma)$, we define the Petersson inner product by

$$(\phi, \psi) = \iint_{\Delta/\Gamma} \lambda(z)^{2-2q} \phi(z) \overline{\psi(z)} dx dy, \quad q \geq 2.$$

For $q = 1$ we shall interpret $A_1(\Delta, \Gamma)$ and $B_1(\Delta, \Gamma)$ as the Hilbert space of square integrable automorphic forms of weight (-2) with inner product defined by

$$(\phi, \psi) = \iint_{\Delta/\Gamma} \phi(z) \overline{\psi(z)} dx dy.$$

A holomorphic function E on Δ is called a holomorphic Eichler integral of order $(1-q)$ on Δ if $E(Az)A'(z)^{1-q} - E(z) \in \Pi_{2q-2}$ on Δ , for all $A \in \Gamma$, where Π_{2q-2} is the vector space of polynomials of degree at most $2q-2$. We shall say that an Eichler integral E of order $1-q$ is bounded if $\phi = D^{2q-1}E \in B_q(\Delta, \Gamma)$, where D means differentiation with respect to z . The projection of ϕ to Δ/Γ is then a meromorphic q -differential on Δ/Γ with order $\geq -(q-1)$ at the punctures on Δ/Γ . An Eichler integral E on Δ is called quasi-bounded if the projection of $D^{2q-1}E$ to Δ/Γ can be extended as a meromorphic q -differential to Δ/Γ whose order at a puncture is $\geq -q$. The space of bounded Eichler integrals modulo Π_{2q-2} will be denoted by $PE_{1-q}(\Delta, \Gamma)$. Similarly $E_{1-q}(\Delta, \Gamma)$ denotes the space of quasi-bounded Eichler integrals modulo Π_{2q-2} .

Let $f \in E_{1-q}(\Delta, \Gamma)$ and E a representative of f and set $D^{2q-1}E = \phi$. We set

$$f_{n-j}(z) = \sum_{k=0}^j ((-1)^k j! / (j-k)!) z^{j-k} D^{2q-2-k} E(z)$$

and set

$$(1) \quad \tilde{f}(z) = \begin{pmatrix} f_0(z) \\ f_1(z) \\ \vdots \\ f_n(z) \end{pmatrix}, \quad I' = \begin{pmatrix} 1 & & & 0 \\ -{}_nC_1 & & & \\ & {}_nC_2 & & \\ 0 & & \ddots & \\ & & & -{}_nC_{n-1} & 1 \end{pmatrix}$$

$\mathfrak{F}(z) = I' \tilde{f}(z)$ and $\omega(z) = \phi(z) \lambda_1^z dz$, where ${}_nC_i = n! / (n-i)! i!$. We call $\tilde{f}(z)$ and $\mathfrak{F}(z)$ column function vectors of length $n+1$ associated with E . For each $A \in \Gamma$ we define X_A and P_A by

$$X_A = \mathfrak{f}(Az) - M(A)\mathfrak{f}(z)$$

and

$$P_A = {}^tM(A)\mathfrak{F}(Az) - \mathfrak{F}(z)$$

and denote them by $\text{pd}_A(\mathfrak{f})$ and $\text{pd}_A(\mathfrak{F})$, respectively. We call X_A and P_A periods of $\mathfrak{f}$ and $\mathfrak{F}$ for $A \in \Gamma$, respectively. The mapping $A \mapsto P_A$ satisfies $P_{AB} = {}^tM(B)P_A + P_B$ for any $A, B \in \Gamma$, as is easily seen. Then a cohomology class is defined, which depends only on f and not E . We define by $E_{1-q}(\Delta, \Gamma, M)$ the space of all $\mathfrak{F}(z)$ modulo $\mathbb{C}^{2q-1}$. Similarly we define $PE_{1-q}(\Delta, \Gamma, M)$. Thus by the obvious way we may define a mapping

$$\alpha: E_{1-q}(\Delta, \Gamma, M) \rightarrow H^1(\Gamma, \mathbb{C}^{2q-1}, M)$$

and we know that $\alpha(PE_{1-q}(\Delta, \Gamma, M)) \subset PH^1(\Gamma, \mathbb{C}^{2q-1}, M)$ by the method similar to that of Kra [6].

If $a_1, a_2, \dots, a_{2q-1}$ are distinct points in Λ , and $\psi \in B_q(\Delta, \Gamma)$, then we call

$$\frac{(z - a_1) \cdots (z - a_{2q-1})}{2\pi i} \iint_{\Omega} \frac{\lambda(\zeta)^{2-2q} \overline{\psi(\zeta)} d\zeta \wedge d\bar{\zeta}}{(\zeta - z)(\zeta - a_1) \cdots (\zeta - a_{2q-1})},$$

$z \in \mathbb{C}$, $q \geq 2$, a potential for ψ , and denote it by $\text{Pot}(\psi)$. For $A \in \Gamma$, we define a period of potential of $\text{Pot}(\psi)$ by setting

$$\text{pd}_A \text{Pot}(\psi)(z) = \text{Pot}(\psi)(Az)A'(z)^{1-q} - \text{Pot}(\psi)(z), \quad z \in \mathbb{C}.$$

It is easily seen that $\text{Pot}(\psi)|_{\Omega - \Delta} \in PE_{1-q}(\Omega - \Delta, \Gamma)$ for $\psi \in B_q(\Delta, \Gamma)$. We set

$$g_{n-j}(z) = \sum_{k=0}^j ((-1)^k j! / (j-k)!) z^{j-k} D^{2q-2-k} \text{Pot}(\psi)(z), \quad z \in \Omega - \Delta.$$

We set

$$(2) \quad g(z) = \begin{pmatrix} g_0(z) \\ g_1(z) \\ \vdots \\ g_n(z) \end{pmatrix}$$

and set $\mathcal{G}(z) = l' \widetilde{g(z)}$. We call $g(z)$ and $\mathcal{G}(z)$ column function vectors of length $n+1$ associated with $\text{Pot}(\psi)$.

For each $A \in \Gamma$, we define Y_A and Q_A by

$$Y_A = g(Az) - M(A)g(z), \quad z \in \Omega - \Delta,$$

and

$$Q_A = {}^tM(A)\mathfrak{G}(Az) - \mathfrak{G}(z), \quad z \in \Omega - \Delta,$$

and denote them by $\text{pd}_A(g)$ and $\text{pd}(\mathfrak{G})$, respectively. We call Y_A and Q_A periods of g and $\mathfrak{G}$ for $A \in \Gamma$, respectively. The mapping $A \mapsto Q_A$ satisfies $Q_{AB} = {}^tM(B)Q_A + Q_B$, for any $A, B \in \Gamma$, as is easily seen. Then a cohomology class is defined, which depends only on ψ . The definition of Y_A , Q_A , etc., applies to the case $\Omega - \Delta \neq \emptyset$. These functions for the remaining case will be defined in the remark after Lemma 4. Similarly as above we define

$$\beta^*: B_q(\Delta, \Gamma) \rightarrow H^1(\Gamma, \mathbb{C}^{2q-1}, M).$$

Let Γ_1 be a subgroup of Γ which keeps Δ_1 invariant, and let b be a conformal mapping from Δ_1 on the upper half-plane U . Set $\Gamma'_1 = b\Gamma_1 b^{-1}$. Then Γ'_1 is a fuchsian group of the first kind. Let ω_0 be the fundamental region defined by Shimura [7] for Γ'_1 in U , then we let its boundary consist of sides $E_{A'_1}, E_{B'_1}, -A'_1(E_{A'_1}), -B'^{-1}_1(E_{B'_1}), \dots, E_{A'_g}, E_{B'_g}, -A'_g(E_{A'_g}), -B'^{-1}_g(E_{B'_g}), E_{C'_1}, -C'_1(E_{C'_1}), \dots, E_{C'_j}, -C'_j(E_{C'_j}), E_{D'_1}, -D'_1(E_{D'_1}), \dots, E_{D'_k}, -D'_k(E_{D'_k})$, where $A'_\lambda, B'_\lambda, C'_\mu$ and D'_ν are generators of Γ'_1 with relations $D'_k \dots D'_1 C'_j \dots C'_1 \prod_{\lambda=1}^g B'^{-1}_\lambda A'^{-1}_\lambda B'_\lambda A'_\lambda = 1$ and $C'^{\epsilon_\mu}_\mu = 1$ ($\mu = 1, 2, \dots, j$). If we set $A_\lambda = b^{-1}A'_\lambda b$, $B_\lambda = b^{-1}B'_\lambda b$, $C_\mu = b^{-1}C'_\mu b$ and $D_\nu = b^{-1}D'_\nu b$ ($\lambda = 1, \dots, g$; $\mu = 1, \dots, j$; $\nu = 1, \dots, k$), then Γ_1 is generated by $A_\lambda, B_\lambda, C_\mu$ and D_ν with relations $D_k \dots D_1 C_j \dots C_1 \prod_{\lambda=1}^g B^{-1}_\lambda A^{-1}_\lambda B_\lambda A_\lambda = 1$ and $C^{\epsilon_\mu}_\mu = 1$ ($\mu = 1, \dots, j$). We set $\omega_0 = b^{-1}(\omega'_0)$ and set $E_{A_\lambda} = b^{-1}(E_{A'_\lambda})$, $E_{B_\lambda} = b^{-1}(E_{B'_\lambda})$, $E_{C_\mu} = b^{-1}(E_{C'_\mu})$ and $E_{D_\nu} = b^{-1}(E_{D'_\nu})$. Then we easily see that ω_0 is a fundamental region for Γ_1 in Δ_1 whose boundary consists of $E_{A_1}, E_{B_1}, -A_1(E_{A_1}), -B_1^{-1}(E_{B_1}), \dots, E_{A_g}, E_{B_g}, -A_g(E_{A_g}), -B_g^{-1}(E_{B_g}), E_{C_1}, -C_1(E_{C_1}), \dots, E_j, -C_j(E_{C_j}), E_{D_1}, -D_1(E_{D_1}), \dots, E_{D_k}, -D_k(E_{D_k})$, since $-A_\lambda(E_{A_\lambda}) = b^{-1}(A'_\lambda(E_{A'_\lambda}))$, $-B_\lambda(E_{B_\lambda}) = b^{-1}(-B'_\lambda(E_{B'_\lambda}))$, $-C_\mu(E_{C_\mu}) = b^{-1}(-C'_\mu(E_{C'_\mu}))$ and $-D_\nu(E_{D_\nu}) = b^{-1}(-D'_\nu(E_{D'_\nu}))$. Since b is a conformal mapping, the sides of ω_0 consist of piecewise analytic arcs. We set $S_\lambda = B_\lambda^{-1}A_\lambda^{-1}B_\lambda A_\lambda$ and $T_\lambda = S_\lambda \dots S_1$ ($\lambda = 1, \dots, g$).

2. Lemmas. In this section we state some lemmas which are necessary to prove the subsequent theorems. Many of the properties in Lemmas 2, 3, 4 and 5 below can be summarized by saying that there is an isomorphism $\Pi_{2q-2} \rightarrow \mathbb{C}^{2q-1}$ which commutes with the action of Γ . However we shall state them for the sake of later use. For each $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, we denote by $A(z) = (az + b)/(cz + d)$. We set $n = 2q - 2$ once and for all.

Lemma 1. For $A \in \Gamma$

$$M(A) = I'^{-1}(\widetilde{{}^tM(A)})^{-1}I'.$$

Proof. We set $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $ad - bc = 1$. The (k, l) element of $M(A)$ is

$$\sum_{i+j=l-1} {}_{2q-1-k}C_i {}_{k-1}C_j a^{2q-k-i-1} c^{k-1-j} b^i d^j.$$

The (k, l) element of $M(A)$ is the (l, k) element of ${}^tM(A)$, which is the $(2q-l, 2q-k)$ element of ${}^tM(A)$. Hence the $(2q-l, 2q-k)$ element of ${}^tM(A)I'$ is

$$(3) \quad (-1)^{2q-k-1} {}_{2q-2}C_{2q-k-1} \sum_{i+j=l-1} {}_{2q-1-k}C_i {}_{k-1}C_j a^{2q-1-k-i} c^{k-1-j} b^i d^j.$$

On the other hand the $(2q-l, 2q-k)$ element of $I'M(A^{-1})$ is

$$(4) \quad (-1)^{2q-l-1} {}_{2q-2}C_{2q-l-1} \sum_{i+j=2q-1-k} {}_{l-1}C_i {}_{2q-l-1}C_j d^{l-1-i} (-c)^{2q-l-1-j} (-b)^i a^j.$$

We easily see that (3) and (4) are the same, that is,

$$(\widetilde{{}^tM(A)})I' = I'M(A^{-1}).$$

For the proof of some properties in Lemmas 2 through 4 below, see Shimura [7].

Lemma 2. For $A, B \in \Gamma$,

- (1) $(\begin{smallmatrix} Az \\ 1 \end{smallmatrix})^n A'(z)^{1-q} = M(A)(\begin{smallmatrix} z \\ 1 \end{smallmatrix})^n,$
- (2) $M(AB) = M(A)M(B),$
- (3) $M(A^{-1}) = M(A)^{-1}.$

Lemma 3. Let $f \in E_{1-q}(\Delta, \Gamma)$ and E a representative of f , and $\psi \in B_q(\Delta, \Gamma)$. Let $\tilde{f}(z)$, $\omega(z)$, $g(z)$, X_A , Y_A , P_A and Q_A be defined as in §1 with respect to E and ψ . Then, for $A, B \in \Gamma$,

- (1) $\omega(Az) = M(A)\omega(z),$
- (2) $d\tilde{f}(z) = \omega(z),$
- (3) $E(z) = (1/n!) {}^t\tilde{f}(z)I'(\begin{smallmatrix} 1 \\ z \end{smallmatrix})^n, \quad z \in \Delta,$
- (4) $E(Az)A'(z)^{1-q} - E(z) = (1/n!) {}^tP_A(\begin{smallmatrix} z \\ 1 \end{smallmatrix})^n, \quad z \in \Delta,$
- (5) $X_{AB} = X_A + M(A)X_B,$
- (6) $\text{Pot}(\psi)(z) = (1/n!) {}^t g(z)I'(\begin{smallmatrix} 1 \\ z \end{smallmatrix})^n, \quad z \in \mathbb{C},$
- (7) $\text{Pot}(\psi)(Az)A'(z)^{1-q} - \text{Pot}(\psi)(z) = (1/n!) {}^t Q_A(\begin{smallmatrix} z \\ 1 \end{smallmatrix})^n, \quad z \in \Omega - \Delta,$
- (8) $Y_{AB} = Y_A + M(A)Y_B,$
- (9) $P_A = I' \widetilde{M(A)^{-1}} \tilde{X}_A, \quad X_A = M(A)I'^{-1} \tilde{P}_A,$
- (10) $Q_A = I' \widetilde{M(A)^{-1}} \tilde{Y}_A, \quad Y_A = M(A)I'^{-1} \tilde{Q}_A.$

By (3) of Lemma 3 we have

Lemma 4. For each $A \in \Gamma$, X_A , P_A , Y_A and Q_A are all number vectors of length $2q - 1$.

This means that

$$\text{Pot}(\psi)(Az)A'(z)^{1-q} - \text{Pot}(\psi)(z) = (1/n!) {}^tQ_A \begin{pmatrix} z \\ 1 \end{pmatrix}^n$$

for $z \in \mathbb{C}$ and $A \in \Gamma$.

Remark. The above Q_A is defined in the case of $\Omega \neq \Delta$. However, since $\text{Pot}(\psi)(Az)A'(z)^{1-q} - \text{Pot}(\psi)(z) = v_A(z)$, $v_A \in \Pi_{2q-2}$, $A \in \Gamma$, $z \in \mathbb{C}$, we may define Q'_A by $v(A) = (1/n!) {}^tQ'_A \begin{pmatrix} z \\ 1 \end{pmatrix}^n$ with $Q'_A \in \mathbb{C}^{2q-1}$. Then we easily see that $Q'_{AB} = {}^tM(B)Q'_A + Q'_B$, $A, B \in \Gamma$. We set $Y'_A = M(A)I'^{-1}Q'_A$. Hereafter we take Q_A to be Q'_A and Y_A to be Y'_A , and note that these definitions agree with previous ones and are valid in the case $\Omega = \Delta$ as well.

Noting the fact pointed out in the first part of this section, we have the following from Kra's decomposition theorem [5], [6].

Lemma 5. (1) $H^1(\Gamma, \mathbb{C}^{2q-1}, M) = \alpha(E_{1-q}(\Delta, \Gamma, M) + \beta^*(B_q(\Delta, \Gamma)))$;
(2) $\beta^*(B_q(\Delta, \Gamma)) \subset PH^1_\Delta(\Gamma, \mathbb{C}^{2q-1}, M)$.

3. The main theorem.

Theorem 1. Let Γ be a nonelementary finitely generated Kleinian group and Δ_1 a simply connected component of Ω and set $\Delta = \bigcup_{A \in \Gamma} A(\Delta_1)$. Let $f \in E_{1-q}(\Delta, \Gamma)$, E an arbitrary representative of f and set $D^{2q-1}E = \phi$, $q \geq 2$. Let $\psi \in B_q(\Delta, \Gamma)$. Let $\mathfrak{f}(z)$ and $g(z)$ be column function vectors (1) and (2) associated with E and $\text{Pot}(\psi)$, respectively, and set $\mathfrak{G}(z) = I' \widehat{g(z)}$. Let $\text{pd}_A \mathfrak{f}(z) = X_A$ and $\text{pd}_A \mathfrak{G}(z) = Q_A$ for each $A \in \Gamma$. Then

$$\begin{aligned} 2in!(\phi, \psi) = & \sum_{\lambda=1}^g {}^tQ_{A_\lambda} [X_{A_\lambda^{-1}B_\lambda A_\lambda T_{\lambda-1}} - X_{T_{\lambda-1}}] \\ & + \sum_{\lambda=1}^g {}^tQ_{B_\lambda^{-1}} [X_{B_\lambda A_\lambda T_{\lambda-1}} - X_{A_\lambda^{-1}B_\lambda A_\lambda T_{\lambda-1}}] \\ & + \sum_{\mu=1}^j {}^tQ_{C_\mu} \left[e_\mu^{-1} \sum_{m=1}^{e_\mu-1} X_{C_\mu^m} - X_{C_{\mu-1} \cdots C_1 T_g} \right] \\ & - \sum_{\nu=1}^k {}^tQ_{D_\nu} [X_{D_{\nu-1} \cdots D_1 C_j \cdots C_1 T_g}] + \sum_{\nu=1}^k {}^tQ_{D_\nu} \mathfrak{f}(s_\nu), \end{aligned} \quad (I)$$

where s_ν is a cusp point of D_ν ($\nu = 1, 2, \dots, k$).

Proof. By (7) of Lemma 3 and Lemma 4, for every $A \in \Gamma$,

$$(5) \quad \text{Pot}(\psi)(Az)A'(z)^{1-q} - \text{Pot}(\psi)(z) = (1/n!) {}^tQ_A \begin{pmatrix} z \\ 1 \end{pmatrix}^n, \quad z \in \mathbb{C}.$$

Let $\eta(z)$ be a C^∞ -function on Δ defined by Kra [5], [6], that is, (1) $0 \leq \eta \leq 1$; (2) for each $z \in \Delta$, there is a neighborhood $U(z)$ of z and a finite subset J of Γ such that $\eta(A(U(z))) = 0$ for each $A \notin J$; (3) $\sum_{\gamma \in \Gamma} \eta(\gamma z) = 1$, $z \in \Delta$; and (4) if $U_A \in \omega_0$ is a cusped region belonging to a puncture on Δ/Γ and A is the corresponding transformation, then $\eta|_{B(U_A)} = 0$ all $B \in \Gamma - \{1, A\}$.

We set

$$\Theta(z) = \begin{pmatrix} \theta_0(z) \\ \theta_1(z) \\ \vdots \\ \theta_n(z) \end{pmatrix} = -\frac{1}{n!} \sum_{\gamma \in \Gamma} \eta(\gamma z) Q_\gamma + \frac{1}{n!} \sum_{\nu=1}^k \chi_\nu(z) \left\{ \sum_{\gamma \in \Gamma} \eta(\gamma z) {}^tM(\gamma) V_\nu \right\},$$

where V_ν is defined by ${}^tM(D_\nu)V_\nu - V_\nu = P_{D_\nu}$ ($\nu = 1, \dots, k$) and we let $\chi_\nu \in C^\infty(\Delta, \Gamma)$ be such that $0 \leq \chi_\nu \leq 1$, $\chi_\nu = 1$ in U_{D_ν} and $\chi_\nu = 0$ in $\bigcup_{i \neq \nu} U_{D_i}$ ($\nu = 1, \dots, k$). Then

$$\begin{aligned} & {}^tM(A)\Theta(Az) - \Theta(z) \\ &= -\frac{1}{n!} {}^tM(A) \sum_{\gamma \in \Gamma} \eta(\gamma Az) Q_\gamma + \frac{1}{n!} {}^tM(A) \sum_{\nu=1}^k \chi_\nu(Az) \left(\sum_{\gamma \in \Gamma} \eta(\gamma Az) {}^tM(\gamma) V_\nu \right) \\ & \quad + \frac{1}{n!} \sum_{\gamma \in \Gamma} \eta(\gamma z) Q_\gamma - \frac{1}{n!} \sum_{\nu=1}^k \chi_\nu(z) \left(\sum_{\gamma \in \Gamma} \eta(\gamma z) {}^tM(\gamma) V_\nu \right) \\ &= -\frac{1}{n!} \sum_{\gamma \in \Gamma} \eta(\gamma Az) (Q_{\gamma A} - Q_A) + \sum_{\gamma \in \Gamma} \frac{1}{n!} \eta(\gamma z) Q_\gamma \\ & \quad + \frac{1}{n!} \sum_{\nu=1}^k \chi_\nu(z) \left(\sum_{\gamma \in \Gamma} \eta(\gamma Az) {}^tM(\gamma A) V_\nu \right) \\ & \quad - \frac{1}{n!} \sum_{\nu=1}^k \chi_\nu(z) \left(\sum_{\gamma \in \Gamma} \eta(\gamma z) {}^tM(\gamma) V_\nu \right) \\ &= \frac{1}{n!} Q_A, \end{aligned}$$

for each $A \in \Gamma$.

We set

$$\mu(z) = \begin{pmatrix} \mu_0(z) \\ \mu_1(z) \\ \vdots \\ \mu_n(z) \end{pmatrix} = \begin{pmatrix} \partial\theta_0(z)/\partial\bar{z} \\ \partial\theta_1(z)/\partial\bar{z} \\ \vdots \\ \partial\theta_n(z)/\partial\bar{z} \end{pmatrix}.$$

Then we easily see that, for $A \in \Gamma$,

$${}^tM(A)\mu(Az)\overline{A'(z)} = \mu(z), \quad z \in \Delta.$$

Thus for $\phi = D^{2q-1}E$,

$$\iint_{\Delta/\Gamma} \phi(z) {}^t\mu(z) \begin{pmatrix} z \\ 1 \end{pmatrix}^n dz \wedge d\bar{z}$$

is well defined.

Next we show that

$$(6) \quad \iint_{\omega_0} \phi(z) {}^t\mu(z) \begin{pmatrix} z \\ 1 \end{pmatrix}^n dz \wedge d\bar{z} - \iint_{\omega_0} \phi(z) \lambda(z)^{2-2q} \overline{\psi(z)} dz \wedge d\bar{z} = 0.$$

For, since for $A \in \Gamma$,

$${}^t\Theta(Az) \begin{pmatrix} Az \\ 1 \end{pmatrix}^n A'(z)^{1-q} = {}^t\Theta(z) \begin{pmatrix} z \\ 1 \end{pmatrix}^n = \frac{1}{n!} {}^tQ_A \begin{pmatrix} z \\ 1 \end{pmatrix}^n, \quad z \in \Delta,$$

by (5) we have

$$(7) \quad \left\{ {}^t\Theta(Az) \begin{pmatrix} Az \\ 1 \end{pmatrix}^n - \text{Pot}(\psi)(Az) \right\} A'(z)^{1-q} = {}^t\Theta(z) \begin{pmatrix} z \\ 1 \end{pmatrix}^n - \text{Pot}(\psi)(z),$$

$z \in \Delta$. By using Stokes' theorem after Bers' trick [3], (6) is equal to

$$(8) \quad \int_{\partial\omega_0} \left\{ {}^t\Theta(z) \begin{pmatrix} z \\ 1 \end{pmatrix}^n - \text{Pot}(\psi)(z) \right\} \phi(z) dz.$$

This is equal to zero, in fact, since its integrals along two identified sides cancel each other and, therefore, (8) is equal to zero, that is,

$$-2i(\phi, \psi) = \iint_{\omega_0} \phi(z) {}^t\mu(z) \begin{pmatrix} z \\ 1 \end{pmatrix}^n dz \wedge d\bar{z}.$$

On the other hand,

$$\iint_{\omega_0} {}^t\mu(z)\phi(z)\begin{pmatrix} z \\ 1 \end{pmatrix}^n dz \wedge d\bar{z} = \int_{\partial\omega_0} {}^t\Theta(z)\begin{pmatrix} z \\ 1 \end{pmatrix}^n \phi(z) dz = \int_{\partial\omega_0} {}^t\Theta(z)\omega(z),$$

where $\omega(z) = \phi(z)\begin{pmatrix} z \\ 1 \end{pmatrix}^n dz$. Then

$$\begin{aligned} \int_{\partial\omega_0} {}^t\Theta(z)\omega(z) &= \int_{\partial\omega_0} {}^t\Theta(z) d\tilde{\Gamma}(z) \\ &= \sum_{\lambda=1}^g \left(\int_{E_{A_\lambda}} {}^t\Theta(z) d\tilde{\Gamma}(z) - \int_{A_\lambda(E_{A_\lambda})} {}^t\Theta(z) d\tilde{\Gamma}(z) \right) \\ &\quad + \sum_{\lambda=1}^g \left(\int_{E_{B_\lambda}} {}^t\Theta(z) d\tilde{\Gamma}(z) - \int_{B_\lambda^{-1}(E_{B_\lambda})} {}^t\Theta(z) d\tilde{\Gamma}(z) \right) \\ &\quad + \sum_{\mu=1}^j \left(\int_{E_{C_\mu}} {}^t\Theta(z) d\tilde{\Gamma}(z) - \int_{C_\mu(E_{C_\mu})} {}^t\Theta(z) d\tilde{\Gamma}(z) \right) \\ &\quad + \sum_{\nu=1}^k \left(\int_{E_{D_\nu}} {}^t\Theta(z) d\tilde{\Gamma}(z) - \int_{D_\nu(E_{D_\nu})} {}^t\Theta(z) d\tilde{\Gamma}(z) \right). \end{aligned}$$

Now for any element A of $\{A_\lambda, B_\lambda^{-1}, C_\mu, D_\nu \mid \lambda = 1, \dots, g; \mu = 1, \dots, j; \nu = 1, \dots, k\}$,

$$\begin{aligned} \int_{A(E_A)} {}^t\Theta(z) d\tilde{\Gamma}(z) &= \int_{E_A} {}^t\Theta(A(z)) d\tilde{\Gamma}(A(z)) \\ &= \int_{E_A} \left\{ {}^t({}^tM(A^{-1})\Theta(z)) + \frac{1}{n!} {}^t({}^tM(A^{-1})Q_A) d(M(A)\tilde{\Gamma}(z) + X_A) \right\} \\ &= \int_{E_A} {}^t\Theta(z) d\tilde{\Gamma}(z) + \int_{E_A} \frac{1}{n!} {}^tQ_A d\tilde{\Gamma}(z), \end{aligned}$$

so that

$$\begin{aligned} \int_{\partial\omega_0} {}^t\Theta(z)\omega(z) &= - \left\{ \sum_{\lambda} \left(\int_{E_{A_\lambda}} \frac{1}{n!} {}^tQ_{A_\lambda} d\tilde{\Gamma}(z) + \int_{E_{B_\lambda}} \frac{1}{n!} {}^tQ_{B_\lambda^{-1}} d\tilde{\Gamma}(z) \right) \right. \\ &\quad \left. + \sum_{\mu} \int_{E_{C_\mu}} \frac{1}{n!} {}^tQ_{C_\mu} d\tilde{\Gamma}(z) + \sum_{\nu} \int_{E_{D_\nu}} \frac{1}{n!} {}^tQ_{D_\nu} d\tilde{\Gamma}(z) \right\}. \end{aligned}$$

Denote by u_0 the starting point of E_{A_1} . Then

$$\begin{aligned}
 2in!(\phi, \psi) &= \sum_{\lambda=1}^g {}^tQ_{A_\lambda} [M(A_\lambda^{-1} B_\lambda A_\lambda T_{\lambda-1}) - M(T_{\lambda-1})] \tilde{f}(u_0) \\
 &+ \sum_{\lambda=1}^g {}^tQ_{B_\lambda^{-1}} [M(B_\lambda A_\lambda T_{\lambda-1}) - M(A_\lambda^{-1} B_\lambda A_\lambda T_{\lambda-1})] \tilde{f}(u_0) \\
 &+ \sum_{\mu=1}^j {}^tQ_{C_\mu} [M(C_{\mu-1} \cdots C_1 T_g)] \tilde{f}(u_0) \\
 (9) \quad &- \sum_{\nu=1}^k {}^tQ_{D_\nu} [M(D_{\nu-1} \cdots D_1 C_j \cdots C_1 T_g)] \tilde{f}(u_0) \\
 &+ \sum_{\lambda=1}^g {}^tQ_{A_\lambda} [X_{A_\lambda^{-1} B_\lambda A_\lambda T_{\lambda-1}} - X_{T_{\lambda-1}}] \\
 &+ \sum_{\lambda=1}^g {}^tQ_{B_\lambda^{-1}} [X_{B_\lambda A_\lambda T_{\lambda-1}} - X_{A_\lambda^{-1} B_\lambda A_\lambda T_{\lambda-1}}] \\
 &+ \sum_{\mu=1}^j {}^tQ_{C_\mu} \left[e_\mu^{-1} \sum_{m=1}^{e_\mu-1} X_{C_\mu^m} - X_{C_{\mu-1} \cdots C_1 T_g} \right] \\
 &- \sum_{\nu=1}^k {}^tQ_{D_\nu} [X_{D_{\nu-1} \cdots D_1 C_j \cdots C_1 T_g}] + \sum_{\nu=1}^k {}^tQ_{D_\nu} \tilde{f}(s_\nu).
 \end{aligned}$$

For, let t_μ be a fixed point for C_μ ($\mu = 1, \dots, j$) in $\bar{\omega}_0$. Then

$$\tilde{f}(t_\mu) = \tilde{f}(C_\mu^m(t_\mu)) = M(C_\mu^m) \tilde{f}(t_\mu) + X_{C_\mu^m}$$

and

$$e_\mu \tilde{f}(t_\mu) = \sum_{m=0}^{e_\mu-1} M(C_\mu^m) \tilde{f}(t_\mu) + \sum_{m=0}^{e_\mu-1} X_{C_\mu^m}.$$

On the other hand

$$0 = Q_{C_\mu^{e_\mu}} = \sum_{m=0}^{e_\mu-1} {}^tM(C_\mu^m) Q_{C_\mu}.$$

Hence

$$\begin{aligned}
\sum_{\mu} {}^tQ_{C_{\mu}} \mathfrak{f}(t_{\mu}) &= \sum_{\mu} {}^tQ_{C_{\mu}} e_{\mu}^{-1} \left\{ \sum_{m=0}^{e_{\mu}-1} M(C_{\mu}^m) \mathfrak{f}(t_{\mu}) + \sum_{m=0}^{e_{\mu}-1} X_{C_{\mu}^m} \right\} \\
&= \sum_{\mu} e_{\mu}^{-1} \left\{ \sum_{m=0}^{e_{\mu}-1} {}^tQ_{C_{\mu}} M(C_{\mu}^m) \mathfrak{f}(t_{\mu}) + {}^tQ_{C_{\mu}} \sum_{m=0}^{e_{\mu}-1} X_{C_{\mu}^m} \right\} \\
&= \sum_{\mu} {}^tQ_{C_{\mu}} \left(\sum_{m=0}^{e_{\mu}-1} e_{\mu}^{-1} X_{C_{\mu}^m} \right) + \sum_{\mu} e_{\mu}^{-1} \left(\sum_{m=0}^{e_{\mu}-1} {}^tQ_{C_{\mu}} M(C_{\mu}^m) \mathfrak{f}(t_{\mu}) \right) \\
&= \sum_{\mu} {}^tQ_{C_{\mu}} e_{\mu}^{-1} \left(\sum_{m=0}^{e_{\mu}-1} X_{C_{\mu}^m} \right).
\end{aligned}$$

We have

$$\begin{aligned}
&\sum_{\mu} {}^tQ_{C_{\mu}} \mathfrak{f}(C_{\mu-1} \cdots C_1 T_g(u_0)) \\
&= \sum_{\mu} {}^tQ_{C_{\mu}} M(C_{\mu-1} \cdots C_1 T_g) \mathfrak{f}(u_0) + \sum_{\mu} {}^tQ_{C_{\mu}} X_{C_{\mu-1} \cdots C_1 T_g}.
\end{aligned}$$

Furthermore we have

$$\begin{aligned}
&\sum_{\nu} {}^tQ_{D_{\nu}} (\mathfrak{f}(s_{\nu}) - \mathfrak{f}(D_{\nu-1} \cdots D_1 C_j \cdots C_1 T_g(u_0))) \\
&= \sum_{\nu} {}^tQ_{D_{\nu}} \mathfrak{f}(s_{\nu}) - \sum_{\nu} {}^tQ_{D_{\nu}} M(D_{\nu-1} \cdots D_1 C_j \cdots C_1 T_g) \mathfrak{f}(u_0) \\
&\quad - \sum_{\nu} {}^tQ_{D_{\nu}} X_{D_{\nu-1} \cdots D_1 C_j \cdots C_1 T_g}.
\end{aligned}$$

We denote by Ψ the sum of the first four terms of (9). Set

$$\begin{aligned}
N_1 &= \sum_{\lambda=1}^g (M(A_{\lambda}^{-1} B_{\lambda} A_{\lambda} T_{\lambda-1}) - M(T_{\lambda-1})) Q_{A_{\lambda}} \\
&\quad + \sum_{\lambda=1}^g (M(B_{\lambda} A_{\lambda} T_{\lambda-1}) - M(A_{\lambda}^{-1} B_{\lambda} A_{\lambda} T_{\lambda-1})) Q_{B_{\lambda}^{-1}}, \\
N_2 &= \sum_{\mu=1}^j {}^tM(C_{\mu-1} \cdots C_1 T_g) Q_{C_{\mu}}, \quad \text{and} \\
N_3 &= \sum_{\nu=1}^k {}^tM(D_{\nu-1} \cdots D_1 C_j \cdots C_1 T_g) Q_{D_{\nu}}.
\end{aligned}$$

Then we have

$$\begin{aligned}
N_1 &= \sum_{\lambda=1}^g (Q_{T_{\lambda-1}} - Q_{T_\lambda}), \\
N_2 &= \sum_{\mu=1}^j (Q_{C_\mu C_{\mu-1} \dots C_1 T_g} - Q_{C_{\mu-1} \dots C_1 T_g}), \text{ and} \\
N_3 &= \sum_{\nu=1}^k (Q_{D_\nu \dots D_1 C_j \dots C_1 T_g} - Q_{D_{\nu-1} \dots D_1 C_j \dots C_1 T_g}),
\end{aligned}$$

hence $N_1 - N_2 - N_3 = 0$, since $D_k \dots D_1 C_j \dots C_1 T_g = 1$, that is, $Q_{D_k \dots D_1 C_j \dots C_1 T_g} = 0$. Thus we have the desired result. Our proof is now complete.

Corollary 1 (Period inequality). *Under the same assumptions as in Theorem 1 (let $\phi = \psi$),*

$$\begin{aligned}
\frac{1}{2i} \left\{ \sum_{\lambda=1}^g {}^t Q_{A_\lambda} [X_{A_\lambda^{-1} B_\lambda A_\lambda T_{\lambda-1}} - X_{T_{\lambda-1}}] + \sum_{\lambda=1}^g {}^t Q_{B_\lambda^{-1}} [X_{B_\lambda A_\lambda T_{\lambda-1}} - X_{A_\lambda^{-1} B_\lambda A_\lambda T_{\lambda-1}}] \right. \\
+ \sum_{\mu=1}^j {}^t Q_{C_\mu} \left[e_\mu^{-1} \sum_{m=1}^{e_\mu-1} X_{C_\mu^m} - X_{C_{\mu-1} \dots C_1 T_g} \right] \\
\left. - \sum_{\nu=1}^k {}^t Q_{D_\nu} [X_{D_{\nu-1} \dots D_1 C_j \dots C_1 T_g}] + \sum_{\nu=1}^k {}^t Q_{D_\nu} [\zeta_\nu] \right\} > 0.
\end{aligned}$$

Corollary 2. *Let Γ be a fuchsian group of the first kind, and let $\Delta_1 = U$. Let $f \in PE_{1-q}(U, \Gamma)$, $f^* \in E_{1-q}(U, \Gamma)$, $q \geq 2$, and E, E^* arbitrary representatives of f and f^* , respectively. Set $D^{2q-1}E = \phi$ and $D^{2q-1}E^* = \phi^*$. Let $\mathfrak{f}$ and $\mathfrak{f}^*$ be column vectors of length $2q-1$ of the form (1) associated with E and E^* , respectively, and set $\text{pd}_A \mathfrak{f}(z) = X_A$ and $\text{pd}_A \mathfrak{f}^*(z) = X_A^*$ for each $A \in \Gamma$. Then*

$$\begin{aligned}
(\phi^*, \phi) &= \frac{(-1)^{q-1}}{2i} \left\{ \sum_{\lambda=1}^g {}^t \widetilde{X}_{A_\lambda} I' [X_{A_\lambda^{-1} B_\lambda A_\lambda T_{\lambda-1}}^* - X_{T_{\lambda-1}}^*] \right. \\
&\quad + \sum_{\lambda=1}^g {}^t \widetilde{X}_{B_\lambda} I' [X_{B_\lambda A_\lambda T_{\lambda-1}}^* - X_{A_\lambda^{-1} B_\lambda A_\lambda T_{\lambda-1}}^*] \\
&\quad + \sum_{\mu=1}^j {}^t \widetilde{X}_{C_\mu} I' \left[e_\mu^{-1} \sum_{m=1}^{e_\mu-1} X_{C_\mu^m}^* - X_{C_{\mu-1} \dots C_1 T_g}^* \right] \\
&\quad \left. - \sum_{\nu=1}^k {}^t \widetilde{X}_{D_\nu} I' [X_{D_{\nu-1} \dots D_1 C_j \dots C_1 T_g}^*] + \sum_{\nu=1}^k {}^t \widetilde{X}_{D_\nu} I' [\zeta_\nu] \right\}.
\end{aligned}$$

Proof. If we set

$$\psi(z) = \frac{(2q-1)!}{2\pi i} \iint_U \frac{(\zeta - \bar{\zeta}/2i)^{2q-2} \overline{\phi(\zeta)} d\zeta \wedge d\bar{\zeta}}{(\zeta - z)^{2q}}, \quad z \in L,$$

then $\psi \in B_q(L, \Gamma)$ (Bers [3]), where L is the lower half-plane. Set $E_1(z) \triangleq \text{Pot}(\phi)(z)$, $z \in L$. Then by Bers [3], $D^{2q-1}E_1 = \psi$ on L . If for $z \in U$, we set $E_2(z) = \overline{E_1(\bar{z})}$ and $D^{2q-1}E_2(z) = \psi_1(z)$; then $\psi_1(z) = \overline{\psi(\bar{z})}$, $\psi_1 \in B_q(U, \Gamma)$ and $E_2 \in E_{1-q}(U, \Gamma)$. Then $\overline{\psi(\bar{z})} = c_q \phi(z)$, $z \in U$ (see Kra [5, p. 554]), so that $\psi_1(z) = c_q \phi(z)$, $z \in U$, where $c_q = (-1)^{q-1}(2q-2)!$. Hence $E_2 = c_q E$.

On the other hand if for each $A \in \Gamma$, we set $\text{pd}_A(\text{Pot}(\phi)) = Q_A$ and $\text{pd}_A(\mathfrak{F}) = P_A$, then

$$\begin{aligned} E_2(Az)A'(z)^{1-q} - E_2(z) &= \overline{E_1(\overline{Az})A'(\bar{z})^{1-q} - E_1(\bar{z})} \\ &= \frac{1}{n!} {}^t\bar{Q}_A \begin{pmatrix} z \\ 1 \end{pmatrix}^n, \quad z \in U, \end{aligned}$$

and

$$E(Az)A'(z)^{1-q} - E(z) = \frac{1}{n!} {}^tP_A \begin{pmatrix} z \\ 1 \end{pmatrix}^n, \quad z \in U,$$

where $\mathfrak{F}(z) = I' \widetilde{\mathfrak{F}}(z)$. Hence $\bar{Q}_A = c_q P_A$. Using Lemma 3, we have

$$\begin{aligned} {}^tP_A &= {}^t(I' \widetilde{M(A^{-1})} \widetilde{X}_A) \\ &= {}^t\widetilde{X}_A {}^t\widetilde{M(A)}^{-1} I' \\ &= -{}^t\widetilde{X}_{A^{-1}} I'. \end{aligned}$$

Substituting these into (I), we have the desired result. Our proof is now complete.

Remark. By using the same method as in the proof of Theorem 2 below, we can show that even in the case of $q = 1$, the above corollary remains valid.

4. Period relations and inequalities. By modifying Shimura's method [7], we can prove the following theorem from Lemmas 1, 2, 3 and 4. The proof is omitted here.

Theorem 2 (Period relation). Let Γ be the same group as in Theorem 1. Let $f_1, f_2 \in E_{1-q}(\Delta, \Gamma, M)$, $q \geq 1$ and $\mathfrak{F}_1, \mathfrak{F}_2$ arbitrary representatives of f_1 and f_2 , respectively. Set $\text{pd}_A \mathfrak{F}_1 = P_A^{(1)}$ and $\text{pd}_A \mathfrak{F}_2 = P_A^{(2)}$ for each $A \in \Gamma$. Then

$$\begin{aligned}
& \sum_{\lambda=1}^g {}^tP_{A_\lambda}^{(1)} [M(A_\lambda^{-1} B_\lambda A_\lambda T_{\lambda-1}) I'^{-1} \tilde{P}_{A_\lambda^{-1} B_\lambda A_\lambda T_{\lambda-1}}^{(2)} - M(T_{\lambda-1}) I'^{-1} \tilde{P}_{T_{\lambda-1}}^{(2)}] \\
& + \sum_{\lambda=1}^g {}^tP_{B_\lambda}^{(1)} [M(B_\lambda A_\lambda T_{\lambda-1}) I'^{-1} \tilde{P}_{B_\lambda A_\lambda T_{\lambda-1}}^{(2)} - M(A_\lambda^{-1} B_\lambda A_\lambda T_{\lambda-1}) I'^{-1} \tilde{P}_{A_\lambda^{-1} B_\lambda A_\lambda T_{\lambda-1}}^{(2)}] \\
& + \sum_{\mu=1}^j {}^tP_{C_\mu}^{(1)} \left[e_\mu^{-1} \sum_{m=1}^{e_\mu-1} M(C_\mu^m) I'^{-1} \tilde{P}_{C_\mu^m}^{(2)} - M(C_{\mu-1} \dots C_1 T_g) I'^{-1} \tilde{P}_{C_{\mu-1} \dots C_1 T_g}^{(2)} \right] \\
& - \sum_{\nu=1}^k {}^tP_{D_\nu}^{(1)} [M(D_{\nu-1} \dots D_1 C_j \dots C_1 T_g) I'^{-1} \tilde{P}_{D_{\nu-1} \dots D_1 C_j \dots C_1 T_g}^{(2)}] \\
& + \sum_{\nu=1}^k {}^tP_{D_\nu}^{(1)} \tilde{f}(s_\nu) = 0.
\end{aligned}$$

Corollary 1 (Period relation for a fuchsian group). *Let Γ be a fuchsian group of the first kind, and let $\Delta_1 = U$. Let $f_1, f_2 \in E_{1-q}(\Delta, \Gamma, M)$, $q \geq 1$, and $\mathfrak{F}_1, \mathfrak{F}_2$ arbitrary representatives of f_1 and f_2 , respectively and set $\tilde{\mathfrak{F}}_1 = I' \tilde{f}_1$ and $\tilde{\mathfrak{F}}_2 = I' \tilde{f}_2$. Set $\text{pd}_A \tilde{f}_1 = X_A^{(1)}$ and $\text{pd}_A \tilde{f}_2 = X_A^{(2)}$ for every $A \in \Gamma$. Then*

$$\begin{aligned}
& \sum_{\lambda=1}^g {}^t\tilde{X}_{A_\lambda}^{(1)} I' [X_{A_\lambda^{-1} B_\lambda A_\lambda T_{\lambda-1}}^{(2)} - X_{T_{\lambda-1}}^{(2)}] \\
\text{(III)} \quad & + \sum_{\lambda=1}^g {}^t\tilde{X}_{B_\lambda}^{(1)} I' [X_{B_\lambda A_\lambda T_{\lambda-1}}^{(2)} - X_{A_\lambda^{-1} B_\lambda A_\lambda T_{\lambda-1}}^{(2)}] \\
& + \sum_{\mu=1}^j {}^t\tilde{X}_{C_\mu}^{(1)} I' \left[e_\mu^{-1} \sum_{m=1}^{e_\mu-1} X_{C_\mu^m}^{(2)} - X_{C_{\mu-1} \dots C_1 T_g}^{(2)} \right] \\
& - \sum_{\nu=1}^k {}^t\tilde{X}_{D_\nu}^{(1)} I' [X_{D_{\nu-1} \dots D_1 C_j \dots C_1 T_g}^{(2)}] + \sum_{\nu=1}^k {}^t\tilde{X}_{D_\nu}^{(1)} \tilde{f}(s_\nu) = 0.
\end{aligned}$$

Remark. Especially, when $q = 1$,

$$\sum_{\lambda=1}^g (X_{B_\lambda}^{(1)} X_{A_\lambda}^{(2)} - X_{A_\lambda}^{(1)} X_{B_\lambda}^{(2)}) = 0.$$

For $I' = 1$ and $X_{A^{-1}}^{(i)} = -X_A^{(i)}$ ($i = 1, 2$) for all $A \in \Gamma$. This is the period relation for abelian integrals. We set $X_A = x_A + iy_A$, where x_A and y_A are real number vectors.

Corollary 2 (Period inequality). *Let Γ be the same group as in the above corollary. Let $f \in E_{1-q}(U, \Gamma, M)$ and $\mathfrak{F}$ be representative of f , $q \geq 1$. Set*

$\mathfrak{F} = I' \mathfrak{F}, (1/n!) D^{2q-1} \mathfrak{F}(z) \binom{z}{1}^n = \phi(z)$ and $\text{pd}_A \mathfrak{F}(z) = x_A + iy_A$ for every $A \in \Gamma$.
If $\phi \neq 0$, then

$$\begin{aligned} (-1)^q & \left[\sum_{\lambda=1}^g \tilde{y}_{A_{\lambda}^{-1}} I' (x_{A_{\lambda}^{-1} B_{\lambda} A_{\lambda} T_{\lambda-1}} - x_{T_{\lambda-1}}) \right. \\ & + \sum_{\lambda=1}^g \tilde{y}_{B_{\lambda}} I' (x_{B_{\lambda} A_{\lambda} T_{\lambda-1}} - x_{A_{\lambda}^{-1} B_{\lambda} A_{\lambda} T_{\lambda-1}}) + \sum_{\mu=1}^j \tilde{y}_{C_{\mu}^{-1}} I' \left[e_{\mu}^{-1} \sum_{m=1}^{\epsilon_{\mu}-1} x_{C_{\mu}^m} - x_{C_{\mu-1}} \cdots C_1 T_g \right] \\ & \left. - \sum_{\nu=1}^k \tilde{y}_{D_{\nu}^{-1}} I' [x_{D_{\nu-1}} \cdots D_1 C_j \cdots C_1 T_g] + \sum_{\nu=1}^k \tilde{y}_{D_{\nu}^{-1}} I' (\text{Re } \mathfrak{F}(s_{\nu})) \right] > 0. \end{aligned}$$

Proof. Set

$$\begin{aligned} \Phi_1 &= \sum_{\lambda=1}^g \tilde{x}_{A_{\lambda}^{-1}} I' [x_{A_{\lambda}^{-1} B_{\lambda} A_{\lambda} T_{\lambda-1}} - x_{T_{\lambda-1}}] + \sum_{\lambda=1}^g \tilde{x}_{B_{\lambda}} I' [x_{B_{\lambda} A_{\lambda} T_{\lambda-1}} - x_{A_{\lambda}^{-1} B_{\lambda} A_{\lambda} T_{\lambda-1}}] \\ &+ \sum_{\mu=1}^j \tilde{x}_{C_{\mu}^{-1}} I' \left[e_{\mu}^{-1} \sum_{m=1}^{\epsilon_{\mu}-1} x_{C_{\mu}^m} - x_{C_{\mu-1}} \cdots C_1 T_g \right] - \sum_{\nu=1}^k \tilde{x}_{D_{\nu}^{-1}} I' [x_{D_{\nu-1}} \cdots D_1 C_j \cdots C_1 T_g] \\ &+ \sum_{\nu=1}^k \tilde{x}_{D_{\nu}^{-1}} I' \text{Re } \mathfrak{F}(s_{\nu}), \end{aligned}$$

$$\begin{aligned} \Phi_2 &= \sum_{\lambda=1}^g \tilde{x}_{A_{\lambda}^{-1}} I' [y_{A_{\lambda}^{-1} B_{\lambda} A_{\lambda} T_{\lambda-1}} - y_{T_{\lambda-1}}] + \sum_{\lambda=1}^g \tilde{x}_{B_{\lambda}} I' [y_{B_{\lambda} A_{\lambda} T_{\lambda-1}} - y_{A_{\lambda}^{-1} B_{\lambda} A_{\lambda} T_{\lambda-1}}] \\ &+ \sum_{\mu=1}^j \tilde{x}_{C_{\mu}^{-1}} I' \left[e_{\mu}^{-1} \sum_{m=1}^{\epsilon_{\mu}-1} y_{C_{\mu}^m} - y_{C_{\mu-1}} \cdots C_1 T_g \right] - \sum_{\nu=1}^k \tilde{x}_{D_{\nu}^{-1}} I' [y_{D_{\nu-1}} \cdots D_1 C_j \cdots C_1 T_g] \\ &+ \sum_{\nu=1}^k \tilde{x}_{D_{\nu}^{-1}} I' \text{Im } \mathfrak{F}(s_{\nu}), \end{aligned}$$

$$\begin{aligned} \Phi_3 &= \sum_{\lambda=1}^g \tilde{y}_{A_{\lambda}^{-1}} I' [x_{A_{\lambda}^{-1} B_{\lambda} A_{\lambda} T_{\lambda-1}} - x_{T_{\lambda-1}}] + \sum_{\lambda=1}^g \tilde{y}_{B_{\lambda}} I' [x_{B_{\lambda} A_{\lambda} T_{\lambda-1}} - x_{A_{\lambda}^{-1} B_{\lambda} A_{\lambda} T_{\lambda-1}}] \\ &+ \sum_{\mu=1}^j \tilde{y}_{C_{\mu}^{-1}} I' \left[e_{\mu}^{-1} \sum_{m=1}^{\epsilon_{\mu}-1} x_{C_{\mu}^m} - x_{C_{\mu-1}} \cdots C_1 T_g \right] - \sum_{\nu=1}^k \tilde{y}_{D_{\nu}^{-1}} I' [x_{D_{\nu-1}} \cdots D_1 C_j \cdots C_1 T_g] \\ &+ \sum_{\nu=1}^k \tilde{y}_{D_{\nu}^{-1}} I' \text{Re } \mathfrak{F}(s_{\nu}), \end{aligned}$$

$$\begin{aligned}\Phi_4 = & \sum_{\lambda=1}^g \tilde{\gamma}_{A_\lambda^{-1}}^{l'} [y_{A_\lambda^{-1} B_\lambda A_\lambda T_{\lambda-1}} - y_{T_{\lambda-1}}] + \sum_{\lambda=1}^g \tilde{\gamma}_{B_\lambda}^{l'} [y_{B_\lambda A_\lambda T_{\lambda-1}} - y_{A_\lambda^{-1} B_\lambda A_\lambda T_{\lambda-1}}] \\ & + \sum_{\mu=1}^j \tilde{\gamma}_{C_\mu^{-1}}^{l'} \left[e_\mu^{-1} \sum_{m=1}^{e_\mu-1} y_{C_\mu^m} - y_{C_{\mu-1} \cdots C_1 T_g} \right] - \sum_{\nu=1}^k \tilde{\gamma}_{D_\nu^{-1}}^{l'} [y_{D_\nu^{-1} \cdots D_1 C_j \cdots C_1 T_g}] \\ & + \sum_{\nu=1}^k \tilde{\gamma}_{D_\nu^{-1}}^{l'} \operatorname{Im} \{s_\nu\}.\end{aligned}$$

Combining the equations (II) and (III) in the case of $\phi = \psi \neq 0$, we have

$$\Phi_1 + i\Phi_2 - i\Phi_3 + \Phi_4 = 2i(-1)^{q-1} \|\phi\|^2$$

and

$$\Phi_1 + i\Phi_2 + i\Phi_3 - \Phi_4 = 0.$$

Thus $\Phi_1 = \Phi_4 = 0$ and $\Phi_3 = -\Phi_2 = (-1)^q \|\phi\|^2$. Since $(-1)^q \Phi_3 > 0$, we have the desired result.

Remark. Especially, when $q = 1$,

$$(-1) \sum_{\lambda=1}^g (y_{B_\lambda} x_{A_\lambda} - y_{A_\lambda} x_{B_\lambda}) > 0.$$

This is the period inequality for abelian integrals.

The following result of Kra [5] is obtained from the above corollary.

Corollary 3. Let Γ be the same group as in the above corollary. If X_A is real for every $A \in \Gamma$, then $X_A = 0$.

Proof. In Corollary 2 to Theorem 2, we have $y_{A_\lambda} = 0$, $y_{B_\lambda} = 0$ ($\lambda = 1, \dots, g$), $y_{C_\mu} = 0$ ($\mu = 1, \dots, j$) and $y_{D_\nu} = 0$ ($\nu = 1, \dots, k$), and so ϕ must be zero. Hence $X_A = 0$.

Finally we consider meromorphic Eichler integrals. We denote by $M_{1-q}(\Delta, \Gamma)$ the space of identified meromorphic Eichler integrals. Then we have the following:

Theorem 3. Let Γ be the same group as in Theorem 1. Assume, for the sake of simplicity, that the group has neither parabolic nor elliptic elements. Let $f \in M_{1-q}(\Delta, \Gamma)$, $q \geq 1$, and E an arbitrary representative of f . Let E have only one pole at u_1 in ω_0 with principal part $(1/z^m)$, $m \geq 1$. Let E^* be an arbitrary representative of f^* , $f^* \in E_{1-q}(\Delta, \Gamma)$ such that $D^{2q-1}E^* = \phi^* \in B_q(\Delta, \Gamma)$ has the representation $\phi^*(z) = (c_0 + c_1 z + \dots) dz^q$ about u_1 . Let $\operatorname{pd}_A f = X_A$ and

$\text{pd}_A \tilde{\Gamma}^* = X_A^*$ for each $A \in \Gamma$, where $\tilde{\Gamma}$ and $\tilde{\Gamma}^*$ are column vectors of length $2q - 1$ of the form (1) associated with E and E^* , respectively. Then

$$\sum_{\lambda=1}^g [({}^t\tilde{X}_{A_{\lambda}^{-1}} I' X_{B_{\lambda}}^* - {}^t\tilde{X}_{B_{\lambda}^{-1}} I' X_{A_{\lambda}}^*) + ({}^t\tilde{X}_{A_{\lambda}} - \tilde{X}_{B_{\lambda}^{-1}}) I' M(A_{\lambda}) X_{T_{\lambda-1}}^* \\ + ({}^t\tilde{X}_{A_{\lambda}^{-1}} - \tilde{X}_{B_{\lambda}}) I' M(B_{\lambda}) X_{T_{\lambda}}^*] - 2\pi i n l c_{m-1}.$$

Proof. We easily see that

$$\int {}^t\tilde{\Gamma}(z) I' \left(\frac{1}{z}\right)^n \phi^*(z) dz = n! \int E(z) \phi^*(z) dz = 2\pi i n l c_{m-1}.$$

By using the same way as in Theorem 2, we see that the left-side hand is equal to

$$\sum_{\lambda=1}^g [({}^t\tilde{X}_{A_{\lambda}^{-1}} I' X_{B_{\lambda}}^* - {}^t\tilde{X}_{B_{\lambda}^{-1}} I' X_{A_{\lambda}}^*) + ({}^t\tilde{X}_{A_{\lambda}} - \tilde{X}_{B_{\lambda}^{-1}}) I' M(A_{\lambda}) X_{T_{\lambda-1}}^* \\ + ({}^t\tilde{X}_{A_{\lambda}^{-1}} - \tilde{X}_{B_{\lambda}}) I' M(B_{\lambda}) X_{T_{\lambda}}^*].$$

Remark. Let $\tilde{\Delta}$ be a finite sum of nonequivalent simply connected components $\Delta_1, \Delta_2, \dots, \Delta_t$, that is, $\tilde{\Delta} = \bigcup_{i=1}^t \Delta_i$, where $\Delta_i \neq A(\Delta_j)$ ($i \neq j$) for any $A \in \Gamma$. Then we obtain similar results as above.

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DEPARTMENT OF MATHEMATICS, SHIZUOKA UNIVERSITY, OHYA SHIZUOKA, JAPAN