RESTRICTING A SCHAUDER BASIS TO A SET OF POSITIVE MEASURE

BY

JAMES SHIREY

ABSTRACT. Let $\{f_n\}$ be an orthonormal system of functions on $[0, 1]$ containing a subsystem $\{f_{n_k}\}$ for which (a) $f_{n_k} \rightarrow 0$ weakly in L_2 , and (b) given $E \subset [0, 1]$, $|E| > 0$, $\liminf \int_E |f_{n_k}(x)| dx > 0$. There then exists a subsystem $\{g_n\}$ of $\{f_n\}$ such that for any set E as above, the linear span of $\{g_n\}$ in $L_1(E)$ is not dense.

For every set E as above, there is an element of $L_p(E)$, $1 < p < \infty$, whose Walsh series expansion converges conditionally and an element of $L_1(E)$ whose Haar series expansion converges conditionally.

1. Gaposhkin, in a discussion of certain properties of "lacunary" systems of functions, see [G], makes extensive use of the following notion: A sequence $\{f_n\}$ of real-valued functions on $[0, 1]$ is a "Riesz system" if the following estimates hold:

$$A_1 \left(\sum_1^N C_n^2 \right)^{1/2} \leq \int_0^1 \left| \sum_1^N C_n f_n(x) \right| dx \leq A_2 \left(\sum_1^N C_n^2 \right)^{1/2}, \quad N \geq 1,$$

where the constants $0 < A_1 \leq A_2$ are independent of the choice of $\{C_n\}$ and of N . An inequality of this type is used as a definition of the term "lacunary" in [KP] in an examination of complemented subspaces of the L_p spaces. Moreover, the classical definition of a lacunary trigonometric system is used primarily to insure that such an inequality obtains [Zy, p. 203]. One difficulty in the application of this notion is finding enough Riesz systems in context. To this end, a sequence $\{f_n\}$ is said to have "property (B)" if there is a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \rightarrow 0$ weakly in L_2 and $\liminf \int_0^1 |f_{n_k}(x)| dx > 0$.

A sequence $\{f_n\}$ is said to have "property (B')" if there is a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \rightarrow 0$ weakly in L_2 and $\liminf \int_E |f_{n_k}(x)| dx > 0$ whenever $E \subset (0, 1)$ and $|E| > 0$. The Lemma 1.2.6 of [G] shows that any sequence of functions having property (B) contains a subsequence that is a Riesz system. Lemma 1.2.6' of [G] shows that for any sequence of functions $\{f_n\}$ having property (B') there is a subsequence $\{f_{n_k}\}$ for which the following obtains: given $E \subset [0, 1]$ and $|E| > 0$, there is a $k_0 = k_0(E)$ such that $\{f_{n_k}\}_{k > k_0}$ is a Riesz system when the f_{n_k} are restricted to the set E .

Received by the editors October 23, 1972.

AMS (MOS) subject classifications (1970). Primary 42A64, 42A44.

Copyright © 1974, American Mathematical Society

In [PZ] Price and Zink, by taking advantage of the symmetry properties of the Rademacher functions, conclude that for any set $E \subset (0, 1)$, $|E| > 0$, the closed linear span of the Rademacher functions in $L_2(E)$ is not all of $L_2(E)$. We offer another explanation for this phenomenon by noting that the conclusion of Lemma 1.2.6' (cited above) applies to the Rademacher system. We also obtain a generalized form of the Khintchine inequality [K, p. 130] that is of use in the next section.

Let $\|f\| = \int_0^1 |f(t)| dt$ and $\|f\|_E = \int_E |f(t)| dt$ when $|E| > 0$. For each f in $L_1(0, 1)$ define

$$f^E(t) = \begin{cases} f(t) & \text{if } t \text{ is in } E, \\ 0 & \text{otherwise.} \end{cases}$$

For $X \subset L_1(0, 1)$ let X^E denote the closure in $L_1(E)$ of $\{f^E: f \text{ is in } X\}$.

A set $S \subset L_1(0, 1)$ is "complete on the set E " where $|E| > 0$ provided that $S^E = L_1(E)$. Otherwise, S is "incomplete on the set E ."

Lemma 1. *Let X be a subspace of $L_1(0, 1)$ having a separable topological dual X' and for which X^E has finite codimension in $L_1(E)$, $|E| > 0$.*

There then exists an f in X^E such that if $\{f_n\} \subset X$ and $\|f - f_n\|_E = o(1)$, then $\|f_n\| \rightarrow \infty$. Such functions comprise all of X^E except for a set of the first category in X^E .

Proof. There can be no constant $M > 0$ such that given an $\epsilon > 0$ and an f in X^E , there exists a g in X for which $\|f - g\|_E$ and $\|g\| \leq M\|f\|_E$, where M is independent of ϵ and f . For, suppose that such an M exists. Let

$$R = \{f^E: f \in X, \|f\|_E \leq 1, \text{ and } \|f\| \leq M\}, \quad S = \{f: f \in X^E, \text{ and } \|f\|_E \leq 1\},$$

and let $T \in (X^E)'$. T can be extended to an element of $L_1'(0, 1)$, and so there exists a bounded measurable function b for which $T(f) = \int_E f(t)b(t)dt$.

Let $\|T\|$ denote the norm of T as an operator on X^E , and let $N(T)$ denote the norm of T as an operator on X . Since R is by assumption dense in S ,

$$MN(T) \geq \sup \{|T(f)|: f \in R\} = \sup \{|T(f)|: f \in S\} = \|T\|.$$

The nonseparability of $(X^E)'$ now implies the nonseparability of X' , a contradiction.

Now let $K_n = \{f \in X^E: \|f\|_E \leq 1 \text{ and there is a sequence } \{f_m\} \subset X \text{ for which } \|f_m - f\|_E = o(1) \text{ and } \|f_m\| \leq n, \text{ for all } m\}$.

Each K_n is closed, convex, symmetric with respect to the origin in X^E . From what has been shown above, K_n is not all of the unit ball in X^E and is consequently nowhere dense. Thus $\bigcup_n K_n$ is of the first category, which is precisely the conclusion desired.

A set $S \subset L_1(0, 1)$ is "complete on a set E " where $|E| > 0$ provided that for any f in $L_1(0, 1)$ and any $\epsilon > 0$, there is a g in S for which $\|f - g\|_E < \epsilon$.

Theorem 2. Let $\{f_n\}$ be an orthonormal sequence of functions in $L_1(0, 1)$, $E \subset (0, 1)$, and $|E| > 0$, and let X denote the closed linear span of $\{f_n\}$ in $L_1(0, 1)$.

If X' is separable and $\{f_n^E\}$ is a Riesz system, then $\{f_n\}$ is incomplete on the set E .

Proof. Suppose $\{f_n\}$ is complete on the set E . Let f be as in the conclusion of Lemma 1 and let $\{g_n\} \subset X$ be such that $\|f - g_n\|_E = o(1)$. Lemma 1 implies that $\|g_n\| \rightarrow \infty$ but the hypothesized properties of $\{f_n\}$ imply that $\{\|g_n\|\}$ is bounded, a contradiction.

Corollary 3. Every orthonormal system $\{f_n\}$ satisfying property (B') contains a subsystem that is incomplete on every set of positive measure.

Proof. Lemma 1.2.6' of [G] assures the existence of a subsystem $\{f_{nk}\}$ having the following property. Given $E \subset (0, 1)$, $|E| > 0$, there is a $k_0 = k_0(E)$ such that $\{f_{nk}\}_{k > k_0}$ and $\{f_{nk}^E\}_{k > k_0}$ are both Riesz systems. Theorem 2 now implies that $\{f_{nk}\}$ is incomplete on the set E , proving the corollary.

As a special case, we consider the Walsh system [K, p. 132]. This system is easily seen to satisfy property (B').

Lemma 4. Let $\{w_n\}$ be a subsystem of the orthonormal system of Walsh possessing the property (*):

(*) if $b_1, \dots, b_k, i_1, \dots, i_k$ denote positive integers, then

$$\int_0^1 w_{i_1}^{b_1}(t) \dots w_{i_k}^{b_k}(t) dt = \begin{cases} 1, & \text{when all } b_i \text{ are even,} \\ 0, & \text{otherwise.} \end{cases}$$

Let $1 \leq p \leq 2$, $E \subset [0, 1]$, $|E| > 0$, $0 < \epsilon < |E|/5$, and let $K \subset E$ with $|E \setminus K| < \epsilon$. Then there are constants A, B , and N depending only on E such that for any $n > m > N$, and any sequence $a_m \dots a_n$ of real numbers

$$A \left(\sum_m^n a_i^2 \right)^{1/2} \leq \left(\int_K \left| \sum_m^n a_i w_i(t) \right|^p dt \right)^{1/p} \leq B \left(\sum_m^n a_i^2 \right)^{1/2}.$$

Proof. For any measurable set $S \subset [0, 1]$, define $S_{ij} \equiv \int_S w_i(t) w_j(t) dt$. Note that $\{w_i w_j\}_{i < j}$ is a subsystem of the Walsh system and that S_{ij} is the i, j th Walsh-Fourier coefficient of the characteristic function of S . Bessel's inequality implies that $\sum_{i < j} S_{ij}^2 \leq |S|$.

Since $\|\cdot\|_1 \leq \|\cdot\|_p \leq \|\cdot\|_2$, it is sufficient to establish the conclusion of the lemma for $p = 1$ and $p = 2$. For $p = 2$

$$\begin{aligned} \int_K \left(\sum_m^n a_i w_i(t) \right)^2 dt &= \int_K \left(\sum_m^n a_i^2 + 2 \sum_{m \leq i < j \leq n} a_i a_j w_i(t) w_j(t) \right) dt \\ &\leq |K| \sum_m^n a_i^2 + 2 \left(\sum_{m \leq i < j \leq n} a_i^2 a_j^2 \right)^{1/2} \left(\sum_{m \leq i < j \leq n} K_{ij}^2 \right)^{1/2}. \end{aligned}$$

A direct calculation yields $(\sum_{m \leq i < j \leq n} a_i^2 a_j^2)^{1/2} = \sum_m^n a_i^2$. Also,

$$\begin{aligned} \left(\sum_{m \leq i < j \leq n} K_{ij}^2 \right)^{1/2} &\leq \left(\sum_{m \leq i < j \leq n} E_{ij}^2 \right)^{1/2} + \left(\sum_{m \leq i < j \leq n} (E \sim K)_{ij}^2 \right)^{1/2} \\ &\leq \left(\sum_{m \leq i < j \leq n} E_{ij}^2 \right)^{1/2} + \epsilon. \end{aligned}$$

Thus, if N is chosen so that $\sum_N^\infty E_{ij}^2 < \epsilon$, and $n > m > N$, then

$$\begin{aligned} \int_K \left(\sum_m^n a_i w_i(t) \right)^2 dt &\leq |E| \sum_m^n a_i^2 + 2 \left(\sum_m^n a_i^2 \right) (2\epsilon) \\ &= (|E| + 4\epsilon) \left(\sum_m^n a_i^2 \right). \end{aligned}$$

Similar considerations will establish the other half of the desired inequality. One obtains

$$\int_K \left(\sum_m^n a_i w_i(t) \right)^2 dt \geq (|E| - 5\epsilon) \left(\sum_m^n a_i^2 \right),$$

and the case $p = 2$ is established.

For $p = 1$

$$\begin{aligned} \int_K \left| \sum_m^n a_i w_i(t) \right| dt &\leq |E|^{1/2} \left(\int_K \left(\sum_m^n a_i w_i(t) \right)^2 dt \right)^{1/2} \\ &\leq |E|^{1/2} (|E| + 4\epsilon)^{1/2} \left(\sum_m^n a_i^2 \right)^{1/2} \end{aligned}$$

provided that $n > m > N(\epsilon)$.

For the left-hand side we have:

$$\begin{aligned} \left(\sum_m^n a_k^2 \right)^{1/2} &\leq (|E| - 5\epsilon)^{-1/2} \left(\int_K \left(\sum_m^n a_i w_i(t) \right)^2 dt \right)^{1/2} \\ &= A \left(\int_K \left(\sum_m^n a_i w_i(t) \right)^{2/3} \left(\sum_m^n a_i w_i(t) \right)^{4/3} dt \right)^{1/2}. \end{aligned}$$

An application of Hölder's inequality with $p = 3/2$ and $q = 3$ yields

$$(+)\quad \left(\sum_m^n a_k^2 \right)^{1/2} \leq A \left(\int_K \left| \sum_m^n a_i w_i(t) \right| dt \right)^{1/3} \left(\int_K \left(\sum_m^n a_i w_i(t) \right)^4 dt \right)^{1/6}.$$

Observe that the Khintchine inequality [K, p. 130] is valid for any system of Walsh functions which possesses property (*). Taking this into account, set $p = 4$ in the Khintchine inequality. The last term in line (+) above can now be replaced by $B(\sum_m^n a_i^2)^{1/3}$, where $B > 0$ depends only on the choice of $p = 4$, without disturbing the sense of the inequality. The desired result is obtained when both sides of the resultant expression are cubed.

Corollary 5. *No system $\{w_k\}$ of Walsh functions possessing property (*) of Lemma 4 can be complete on any set of positive measure.*

Proof. Let $E \subset (0, 1)$, $|E| > 0$. Lemma 4 implies the existence of $N = N(E)$ for which $\{w_n\}_{n > N}$ and $\{w_n^E\}_{n > N}$ are both Riesz systems. Theorem 2 now implies that $\{w_n\}$ is incomplete on the set E .

A "quasi-basis" for a Banach space $(X, \|\cdot\|)$ is a double sequence $\{x_n, X_n\}$ of elements of X and of continuous linear functionals, respectively, such that the series $\sum_1^\infty X_n(x)x_n$ converges in norm to x for each x in X . Such a structure may arise in the following way.

Suppose $\{y_n, Y_n\}$ is a Schauder basis for $L_p(0, 1)$, $1 \leq p < \infty$. Let $|E| > 0$ and define for all z in $L_p(0, 1)$

$$z^E(t) = \begin{cases} z(t), & t \text{ in } E, \\ 0, & \text{otherwise.} \end{cases}$$

For each given ϕ in $L_p'(0, 1)$, define $\phi^E(z) = \phi(z^E)$ for any z in $L_p(0, 1)$.

It is easily verified that $\{y_n^E, Y_n^E\}$ is a quasi-basis for $L_p(E)$. This double sequence shall be called the "restriction of $\{y_n, Y_n\}$ to the set E ."

The quasi-basis $\{x_n, X_n\}$ is "unconditional" if $\sum_1^\infty X_n(x)x_n$ converges unconditionally for every x in X . Otherwise, the quasi-basis is "conditional." The restriction of an unconditional basis for $L_p(0, 1)$ to any set E , $|E| > 0$, is clearly an unconditional quasi-basis for $L_p(E)$. A problem that arises in this context is the determination of the sets E , if there are any, for which a given conditional basis for $L_p(0, 1)$ restricts to an unconditional quasi-basis for

$L_p(E)$. Two special cases are considered below: it turns out that no such sets exist for the Walsh system [K, p. 132] in $L_p(0, 1)$, $1 < p < \infty$, or for the Haar system [K, p. 120] in $L_1(0, 1)$.

Let $(X, \|\cdot\|)$ denote a Banach space of equivalence classes of Lebesgue measurable functions on $[0, 1]$ (see [L, p. 66] and [Z, Chapter 15]). Such spaces may be defined by specifying a certain class H of elements of $L_1(0, 1)$ and defining

$$\|x\| = \sup \left\{ \int_0^1 |x(t)b(t)| dt : b \in H \right\} \quad \text{and} \quad X = \{x : \|x\| < \infty\}.$$

Various conditions may be imposed on H in order that $(X, \|\cdot\|)$ turns out as a Banach space.

Theorem 6. *Given a quasi-basis $\{x_n, X_n\}$ for $(X, \|\cdot\|)$. Let $\{r_n\}$ denote the Rademacher system and define*

$$C = \left\{ \theta : \sum_1^\infty r_k(\theta) X_k(x) x_k \text{ converges for all } x \text{ in } X \right\}.$$

Let $Z_N = \{x \in X : X_1(x) = \dots = X_N(x) = 0\}$ and let $G(x) = \|(\sum_1^\infty X_n^2(x) x_n^2)^{1/2}\|$.

Then C is a Borel set, and if $|C| > 0$ there is an N for which the identity mapping $(Z_N, \|\cdot\|) \rightarrow (Z_N, G(\cdot))$ is continuous.

Proof. Let $x(n, m, \theta, t) = \sum_1^m r_k(\theta) X_k(x) x_k(t)$. For each (n, m) , $\|x(n, m, \theta, \cdot)\|$ is a step function of θ and

$$C = \left\{ \theta : \limsup_{n, m \rightarrow \infty} \|x(n, m, \theta, \cdot)\| = 0 \right\}.$$

C is therefore a Borel set.

For each θ in C , the sequence of linear operators $T_{n\theta}(\cdot)$ defined by

$$T_{n\theta}(x) = \sum_1^n r_k(\theta) X_k(x) x_k$$

is uniformly bounded by virtue of the Banach-Steinhaus theorem. The set $S_M \equiv \{\theta : \|T_{n\theta}\| \leq M \text{ for all } n\}$ is clearly a Borel set, and $\bigcup (S_M \cap C) = C$. Since $|C| > 0$, there is a constant $K > 0$ for which $|S_K \cap C| > 0$. Define $S \equiv S_K \cap C$, and for each θ in S let $T_\theta(x) \equiv \lim_n T_{n\theta}(x)$. Then,

$$(1) \quad \|T_\theta(x)\| = \lim_n \|T_{n\theta}(x)\| \leq M \|x\| \quad \text{for all } \theta \in S,$$

and S is seen to be a Borel set of positive measure

For any $\epsilon > 0$ there is, by definition of $\|\cdot\|$, an b in H such that

$$(2) \quad G(x) \leq \int_0^1 \left(\sum_1^\infty X_k^2(x) x_k^2(t) \right)^{1/2} |b(t)| dt + \epsilon.$$

By Lemma 4, if $F \subset S$ is of sufficiently large measure, there is a $B > 0$ and an integer $N > 0$, depending only on S , such that

$$(3) \quad B \left(\sum_N^n X_k^2(x) x_k^2(t) \right)^{1/2} |b(t)| \leq \int_F |x(N, n, \theta, t) b(t)| d\theta$$

for every x in X , and for every $n > N$.

For each θ in S ,

$$(4) \quad \lim_n \int_0^1 |x(N, n, \theta, t) b(t)| dt = \int_0^1 |x(n, \infty, \theta, t) b(t)| dt.$$

By Egoroff's theorem there is $F \subset S$ in which the convergence in (4) is uniform, and having measure large enough so that line (3) obtains. We may integrate (4) with respect to θ over the set F , and the order of integration on the left side may be reversed. This yields

$$(5) \quad \lim_n \int_0^1 \left[\int_F |x(N, n, \theta, t) b(t)| d\theta \right] dt = \int_F \left[\int_0^1 |x(N, \infty, \theta, t) b(t)| dt \right] d\theta.$$

Assume now that x is in Z_N , and combine (1), (2), (3), and (5):

$$G(x) \leq \frac{1}{B} \int_F \left[\int_0^1 |x(N, \infty, \theta, t) b(t)| dt \right] d\theta + \epsilon \leq \frac{1}{B} \int_F \|T_\theta(x)\| d\theta + \epsilon \leq \frac{M}{B} \|x\| + \epsilon$$

for all x in Z_N , and for any $\epsilon > 0$.

The norm $G(\cdot)$ and unconditional convergence have been discussed previously in [0], and in the case that $\{x_n, X_n\}$ is an unconditional basis it has been shown in [SZ] that the norms $G(\cdot)$ and $\|\cdot\|$ are equivalent.

Theorem 6 can be used to partially extend Lemma 4 to all of the spaces $L_p(0, 1)$, $1 < p < \infty$:

Corollary 7. *Let $\{w_n\}$ be a subsystem of the Walsh system possessing the property (*) described in Lemma 4, and let $1 \leq p < \infty$, $|E| > 0$. Then there are constants A, B , and N depending only on E such that for any $n > m > N$ and any sequence $a_m \dots a_n$ of real numbers*

$$A \left(\sum_m^n a_i^2 \right)^{1/2} \leq \left(\int_E \left| \sum_m^n a_i w_i(t) \right|^p dt \right)^{1/p} \leq B \left(\sum_m^n a_i^2 \right)^{1/2}.$$

Proof. For $2 \leq p < \infty$, let N be determined by the set E as in Lemma 4, and let Y be the closed linear span of $\{w_i^E\}_{i \geq N}$ in $L_p(E)$. If continuous linear functionals W_i on Y can be found for which $\{w_i^E, W_i\}_{i \geq N}$ is an unconditional basis for Y , the desired conclusion will then follow from Lemma 4, Theorem 6 and the fact that $\|\cdot\|_p \geq \|\cdot\|_2$.

Lemma 4 implies that for each x in Y , there is a unique sequence $\{a_i\}$ in l_2 for which the series $\sum_{i \geq N} a_i w_i^E$ converges unconditionally to x in the norm of $L_2(0, 1)$. The Khintchine inequality with the functions w_i substituted for Rademacher's functions [see proof of Lemma 4 above] implies that $\sum_{i \geq N} a_i w_i \equiv \bar{x}$ converges unconditionally in the norm of $L_p(0, 1)$ and that the mapping $x \rightarrow \bar{x}$ taking $Y \rightarrow L_p(0, 1)$ is continuous in that norm.

Clearly, the functionals W_i defined by $W_i(x) \equiv \int_0^1 w_i(t) \bar{x}(t) dt$ are as desired.

The case $1 \leq p \leq 2$ is a special case of Lemma 1, proving the corollary.

It is known that the Walsh system is a conditional basis for $L_p(0, 1)$, $1 < p < \infty$, $p \neq 2$, [O] and [P]. Also, the Haar system is a conditional basis for $L_1(0, 1)$, [D] and [M]. The conditionality of these systems is preserved under restriction to a set of positive measure.

Theorem 8. Let $\{w_n, W_n\}$ denote the quasi-basis for $L_p(E)$, $1 < p < \infty$, obtained by restricting the Walsh system to a set E , $|E| > 0$.

If $p \neq 2$, then $\{w_n, W_n\}$ is a conditional quasi-basis for $L_p(E)$. If $p = 2$, then this quasi-basis is unconditional.

Proof. If $p = 2$, the Walsh system is an unconditional basis for $L_p(0, 1)$, and so $\{w_n, W_n\}$ is an unconditional quasi-basis for $L_p(E)$.

The remaining cases $1 < p < 2$ and $2 < p < \infty$ are considered separately. For $p < 2$, let $x \in L_p(E) \sim L_2(E)$. Then

$$G(x) = \left\| \left(\sum_1^\infty W_n^2(x) w_n^2 \right)^{1/2} \right\|_p = \left(\sum_1^\infty W_n^2(x) \right)^{1/2} |E| = \infty.$$

It follows that the series $\sum_1^\infty W_n(x) w_n$ is conditionally convergent [O].

For $p > 2$, let $y \in L_q(E)$ where $1/p + 1/q = 1$. Were $\{w_n, W_n\}$ an unconditional quasi-basis for $L_p(E)$, then

$$\epsilon(x) \equiv \text{L. lim}_n \sum_1^n W_k(x) \epsilon_k w_k$$

would exist for every x in $L_p(E)$ and for every sequence $\epsilon \equiv \{\epsilon_k\}$ where $\epsilon_k = \pm 1$. Then

$$(y, \epsilon(x)) = \sum_1^{\infty} W_k(x) \epsilon_k(w_k, y) = \sum_1^{\infty} W_k(y) \epsilon_k(w_k, x)$$

would converge for all x and ϵ .

The series $\sum_1^{\infty} W_k(y) w_k$ would then be subseries convergent in the weak topology on $L_q(E)$ and hence subseries convergent in the norm topology [D, p. 60]. $\{w_k, W_k\}$ would then be an unconditional quasi-basis for $L_q(E)$, which is impossible since $1 < q < 2$.

Theorem 9. Let $\{b_{np}^E, H_{np}^E\}$ denote the quasi-basis for $L_1(E)$ obtained by restricting the Haar system to a set E , $|E| > 0$.

Then this system is a conditional quasi-basis for $L_1(E)$.

Proof. Theorem 6 implies that if $\{b_{np}^E, H_{np}^E\}$ were an unconditional quasi-basis for $L_1(E)$, then the norm $\|\cdot\|$ would dominate the norm $G(\cdot)$ on some space $Z_N = \{x \in L_1(E) : H_{np}(x) = 0 \text{ for } n \leq N, p = 0, 1, \dots, 2^n - 1\}$. But this cannot be the case: a sequence $\{y_p\}$ is constructed for which

- (A) $\{y_p\} \subset Z_N$,
- (B) $\{y_p\}$ is bounded in $(L_1(E), \|\cdot\|)$,
- (C) $\{y_p\}$ is unbounded in $(L_1(E), G(\cdot))$.

Let p be a fixed positive integer, $p > 4$, and let $\epsilon < 1/p$. Since almost every point of E is a point of metric density 1, there exists a sequence $\{I_k\}_{k=0}^p$ of dyadic intervals, each of which is the support of a Haar function which shall be denoted by b_k , and for which

- I_{k+1} is the left half of I_k ,
- $|I_k| = 2^{-n-k}$, $k = 0, 1, \dots, p$,
- $0 \leq 1 - |E \cap I_k|/|I_k| < \epsilon$, $k = 0, \dots, p$.

Let $E_k = E \cap I_k$ and split E_k into a left half L_k and a right half R_k of equal measure. Define, for $k = 0, 1, \dots, p$,

$$Y_k(t) = \begin{cases} \sqrt{2^{n+k}} & \text{if } t \text{ is in } L_k, \\ -\sqrt{2^{n+k}} & \text{if } t \text{ is in } R_k, \\ 0 & \text{otherwise,} \end{cases}$$

and let $y_p = \sum_{k=0}^p \sqrt{2^{n+k}} Y_k$.

The proof shall be completed when it is demonstrated that the sequence $\{y_p\}$ possesses properties (A), (B) and (C) above. This necessitates estimates on $H_i^E(y_p) = \sum_{k=0}^p \sqrt{2^{n+k}} H_i^E(Y_k)$.

The function Y_k resembles the Haar function b_k restricted to the set E . In particular, if e_k is defined by the formula

$$Y_k(t) = b_k^E(t) + e_k(t)$$

then

$$\begin{aligned} |\text{supp } e_k| &\leq \frac{1}{2}(|I_k| - |E_k|) \leq \epsilon 2^{-n-k-1}, \\ |e_k(t)| &\leq 2\sqrt{2^{n+k}}. \end{aligned}$$

Estimation of $H_i^E(Y_k)$ for $k \geq i$:

$$|H_i^E(b_i)| = 2^{n+i}|E_i|$$

and

$$|H_i^E(e_i)| \leq \sqrt{2^{n+i}} \max |e_i(t)| |\text{supp } e_i| = 2^{n+i+1} |\text{supp } e_i| \leq \epsilon.$$

Thus,

$$|H_i^E(Y_i)| \geq 2^{n+i}|E_i| - \epsilon \geq 1 - 3\epsilon.$$

If $k > i$, then $\text{supp } Y_k$ is contained in that part of E_i on which b_i is of constant sign. In this case, $H_i^E(Y_k) = 0$.

Estimation of $H_i^E(Y_k)$ for $k < i$:

$$\begin{aligned} |H_i^E(b_k)| &\leq \sqrt{2^{n+i}} \sqrt{2^{n+k}} (|I_i| - |E_i|) \leq \epsilon \sqrt{2^{n+i}} \sqrt{2^{n+k}} 2^{-n-i} = \epsilon \sqrt{2^{k-i}}, \\ |H_i^E(e_k)| &\leq 2\sqrt{2^{n+i}} \sqrt{2^{n+k}} |\text{supp } e_k \cap E_i|. \end{aligned}$$

If $k < i-1$, $\text{supp } e_k \cap E_i = \emptyset$ and then $H_i^E(e_k) = 0$. Otherwise, $|H_i^E(e_{i-1})| \leq 2\sqrt{2^{n+i}} \sqrt{2^{n+i-1}} 2^{-n-i} \epsilon = \epsilon \sqrt{2}$. This gives

$$|H_i^E(Y_k)| \leq \epsilon \sqrt{2^{k-i}} \quad \text{if } k < i-1,$$

$$|H_i^E(Y_{i-1})| \leq \epsilon \sqrt{2} + \epsilon/\sqrt{2} \leq 3\epsilon.$$

Upon combining the above estimates in $H_i^E(Y_k)$,

$$\begin{aligned} |H_i^E(y_p)| &= \left| \sum_{k=0}^p \sqrt{2^{n+k}} H_i^E(Y_k) \right| \\ &\geq |\sqrt{2^{n+i}} H_i^E(Y_i)| - \sqrt{2^{n+i-1}} |H_i^E(Y_{i-1})| - \sum_{k=0}^{i-2} \sqrt{2^{n+k}} |H_i^E(Y_k)| \\ &\geq \sqrt{2^{n+i}} (1 - 3\epsilon) - 3\epsilon \sqrt{2^{n+i-1}} - \epsilon \sum_{k=0}^{i-2} \sqrt{2^{n+k}} \sqrt{2^{k-i}} \\ &\geq \sqrt{2^{n+i}} (1 - 6\epsilon) - \epsilon \sqrt{2^{n-i}} \sum_{k=0}^{i-2} 2^k \\ &\geq \sqrt{2^{n+i}} (1 - 6\epsilon) - \epsilon \sqrt{2^{n+i}} = \sqrt{2^{n+i}} (1 - 7\epsilon). \end{aligned}$$

Let $\|\cdot\|_E$ denote the L_1 -norm of functions restricted to E . An application of the estimate above then yields

$$\begin{aligned} G(y_p) &\geq \left\| \left(\sum_{i=0}^p [H_i^E(y_p)]^2 b_i^2 \right)^{1/2} \right\|_E \geq A \left\| \left(\sum_{i=0}^p 2^{n+i} b_i^2 \right)^{1/2} \right\|_E \\ &\geq B \sum_{i=0}^p 2^{-n-i} \left(\sum_{k=0}^i 2^{2(n+k)} \right)^{1/2} \geq Cp. \end{aligned}$$

Statement (C) is thereby established. Moreover,

$$\|y_p\| \leq \left\| \sum_{k=0}^p \sqrt{2^{n+k}} b_k \right\| + \left\| \sum_{k=0}^p \sqrt{2^{n+k}} e_k \right\| \leq 2 + \sum_{k=0}^p \sqrt{2^{n+k}} \|e_k\| \leq 2 + p\epsilon \leq 3.$$

This establishes statement (B). Given any N , it is clear that the intervals I_k can be so chosen that the functions y_k are all elements of Z_N . Then y_p will also be an element of Z_N , and (A) is established.

REFERENCES

- [D] M. M. Day, *Normed linear spaces*, 2nd rev. ed., *Ergebnisse der Mathematik und ihrer Grenzgebiete*, N. F., Heft 21, Academic Press, New York; Springer-Verlag, Berlin, 1962. MR 26 #2847.
- [G] V. F. Gapoškin, *Lacunary series and independent functions*, *Uspehi Mat. Nauk* 21 (1966), no. 6 (132), 3–82 = *Russian Math. Surveys* 21 (1966), no. 6, 1–82. MR 34 #6374.
- [K] S. Kaczmarz and H. Steinhaus, *Theorie der Orthogonalreihen*, reprint, Chelsea, New York, 1951. MR 20 #1148.
- [KP] M. I. Kadec and A. Pełczyński, *Bases, lacunary sequences, and complemented subspaces in the spaces L_p* , *Studia Math.* 21 (1961/62), 161–176. MR 27 #2851.
- [L] G. G. Lorentz, *Bernstein polynomials*, *Mathematical Expositions*, no. 8, Univ. of Toronto Press, Toronto, 1953. MR 15, 217.
- [M] Jürg T. Marti, *Introduction to the theory of bases*, Springer-Verlag, New York, 1959.
- [O] W. Orlicz, *Über unbedingte Convergenz in Funktionenräumen. I*, *Studia Math.* 4 (1933), 33–37.
- [P] R. E. A. C. Paley, *A remarkable series of orthogonal functions*, *Proc. London Math. Soc.* 34 (1931), 241–264.
- [PZ] J. J. Price and R. E. Zink, *On sets of completeness for families of Haar functions*, *Trans. Amer. Math. Soc.* 119 (1965), 262–269. MR 32 #1499.
- [SZ] J. Shirey and R. E. Zink, *On unconditional bases in certain Banach function spaces*, *Studia Math.* 36 (1970), 169–175. MR 43 #899.
- [Z] A. C. Zaanen, *An introduction to the theory of integration*, rev. ed., North-Holland, Amsterdam; Interscience, New York, 1967. MR 36 #5286.
- [Zy] A. Zygmund, *Trigonometrical series*, 2nd ed., reprinted with corrections and some additions, Cambridge Univ. Press, New York, 1968. MR 38 #4882.