TOPOLOGICAL ENTROPY FOR NONCOMPACT SETS

BY

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ABSTRACT. For $f: X \rightarrow X$ continuous and $Y \subset X$ a topological entropy $h(f, Y)$ is defined. For X compact one obtains results generalizing known theorems about entropy for compact Y and about Hausdorff dimension for certain $Y \subset X = S^1$. A notion of entropy-conjugacy is proposed for homeomorphisms.

The topological entropy of a continuous map on a compact space was defined by Adler, Konheim and McAndrew [1]. In the present paper we will define entropy for subsets of compact spaces in a way which resembles Hausdorff dimension. This will be used to generalize known results about the Hausdorff dimension of the quasiregular points of certain measures and to define a notion of conjugacy that is a cross between the topological and measure theoretic ones.

In [5] we gave a definition of entropy for uniformly continuous maps on metric spaces. That definition was motivated by different examples (linear maps on R^n and calculating entropy on T^n) and it sometimes differs from the definition given here.

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1. The definition. Let $f: X \rightarrow X$ be continuous and $Y \subset X$. The topological entropy $h(f, Y)$ will be defined much like Hausdorff dimension, with the "size" of a set reflecting how f acts on it rather than its diameter. Let $\mathcal{Q}$ be a finite open cover of X . We write $E < \mathcal{Q}$ if E is contained in some member of $\mathcal{Q}$ and $\{E_i\} < \mathcal{Q}$ if every $E_i < \mathcal{Q}$. Let $n_{f, \mathcal{Q}}(E)$ be the biggest nonnegative integer such that

$$f^k E < \mathcal{Q} \text{ for all } k \in [0, n_{f, \mathcal{Q}}(E));$$

$n_{f, \mathcal{Q}}(E) = 0$ if $E \not< \mathcal{Q}$ and $n_{f, \mathcal{Q}}(E) = +\infty$ if all $f^k E < \mathcal{Q}$. Now set

$$D_{\mathcal{Q}}(E) = \exp(-n_{f, \mathcal{Q}}(E)) \quad \text{and} \quad D_{\mathcal{Q}}(\mathcal{E}, \lambda) = \sum_{i=1}^{\infty} D_{\mathcal{Q}}(E_i)^{\lambda}$$

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for $\mathcal{E} = \{E_i\}_{i=1}^\infty$ and $\lambda \in R$. We define a measure $m_{\mathcal{Q}, \lambda}$ by

$$m_{\mathcal{Q}, \lambda}(Y) = \lim_{\epsilon \rightarrow 0} \inf \left\{ D_{\mathcal{Q}}(\mathcal{E}, \lambda) : \bigcup E_i \supset Y \text{ and } D_{\mathcal{Q}}(E_i) < \epsilon \right\}.$$

Notice that $m_{\mathcal{Q}, \lambda}(Y) \leq m_{\mathcal{Q}, \lambda'}(Y)$ for $\lambda > \lambda'$ and $m_{\mathcal{Q}, \lambda}(Y) \notin \{0, +\infty\}$ for at most one λ . Define

$$h_{\mathcal{Q}}(f, Y) = \inf \{ \lambda : m_{\mathcal{Q}, \lambda}(Y) = 0 \} \text{ and finally } h(f, Y) = \sup_{\mathcal{Q}} h_{\mathcal{Q}}(f, Y)$$

where $\mathcal{Q}$ ranges over all finite open covers of X . For $Y = X$ we write $h(f) = h(f, X)$.

Remark. The number $h(f, Y) = h_X(f, Y)$ depends very much on which space X we consider the domain of f . For instance, $f(x) = x + 1$ defines a homeomorphism of R which can be extended to a homeomorphism of S^1 . By Proposition 1 below $h_{S^1}(f, S^1)$ is just the usual entropy of the homeomorphism $f: S^1 \rightarrow S^1$ and thus equals 0 [1, p. 315]; for $Y \subset S^1$ we have $0 \leq h_{S^1}(f, Y) \leq h_{S^1}(f, S^1)$ and so $h_{S^1}(f, Y) = 0$. On the other hand suppose $Y = \bigcup_{n=-\infty}^{+\infty} (n + A)$ where $A \subset (0, 1)$ is a Cantor set. Since Y is closed in R , one can prove $h_Y(f, Y) = h_R(f, Y)$. For any homeomorphism $g: A \rightarrow A$, $\pi: Y \rightarrow A$ defined by $\pi(n + a) = g^n(a)$ displays g as a quotient of $f|Y$. From this one can conclude that $h(g) \leq h(f|Y)$; as $h(g)$ can be made large, $h(f|Y) = +\infty$. Then $h_R(f, Y) = +\infty$ but $h_{S^1}(f, Y) = 0$. This example was suggested to us by L. Goodwyn.

Proposition 1. *If X is compact, then $h(f)$ equals the usual topological entropy.*

Proof. First let us recall the usual definition of entropy for compact X [1]. Let $\mathcal{Q}_{f,n} = \{A_{i_0} \cap f^{-1}A_{i_1} \cap \dots \cap f^{-n+1}A_{i_{n-1}} : A_{i_k} \in \mathcal{Q}\}$ for an open cover $\mathcal{Q}$ of X . If $N(\mathcal{B})$ denotes the smallest cardinality of any subcover of the open $\mathcal{B}$, then

$$\underline{h}(f, \mathcal{Q}) = \lim_{n \rightarrow \infty} \frac{1}{n} \log N(\mathcal{Q}_{f,n})$$

exists and the topological entropy is defined by

$$\underline{h}(f) = \sup_{\mathcal{Q}} \underline{h}(f, \mathcal{Q})$$

where $\mathcal{Q}$ runs over all finite open covers of X . Letting $\mathcal{E}_n$ be a subcover with $N(\mathcal{Q}_{f,n})$ members

$$D_{\mathcal{Q}}(\mathcal{E}_n, \lambda) \leq N(\mathcal{Q}_{f,n})e^{-n\lambda}$$

and

$$m_{\mathcal{Q}, \lambda}(X) \leq \lim_{n \rightarrow \infty} [\exp(-\lambda + n^{-1} \log N(\mathcal{Q}_{f,n}))]^n.$$

For $\lambda > \underline{h}(f, \mathcal{Q})$ we get $m_{\mathcal{Q}, \lambda}(X) = 0$. Hence $b_{\mathcal{Q}}(f, X) \leq \underline{h}(f, \mathcal{Q})$.

We prove $b_{\mathcal{Q}}(f, X) \geq \underline{h}(f, \mathcal{Q})$ by showing $\underline{h}(f, \mathcal{Q}) \leq \lambda$ whenever $m_{\mathcal{Q}, \lambda}(X) = 0$. For such a λ there is a countable covering $\tilde{\mathcal{E}} = \{E_i\}$ of X so that $D_{\mathcal{Q}}(\tilde{\mathcal{E}}, \lambda) < 1$. If $n_{f, \mathcal{Q}}(E_i) < \infty$, we may assume E_i is open (there is an open $F_i \supset E_i$ with $D_{\mathcal{Q}}(F_i) = D_{\mathcal{Q}}(E_i)$). The E_i 's with $n_{f, \mathcal{Q}}(E_i) = \infty$ may be replaced by open sets so that $D_{\mathcal{Q}}(\tilde{\mathcal{E}}, \lambda)$ is still less than 1 (though it may increase). As X is compact, the open cover $\tilde{\mathcal{E}}$ now has a finite subcover $\mathcal{D} = \{D_1, \dots, D_m\}$. Then

$$\sum_{s=1}^{\infty} \sum_{j_1, \dots, j_s} \exp(-\lambda n_{f, \mathcal{Q}}(D_{j_1}, \dots, D_{j_s})) = \sum_{k=1}^{\infty} D_{\mathcal{Q}}(\mathcal{D}, \lambda)^k < \infty$$

where $n_{f, \mathcal{Q}}(D_{j_1}, \dots, D_{j_s}) = \sum_{r=1}^s n_{f, \mathcal{Q}}(D_{j_r})$.

Let

$$C(D_{j_1}, \dots, D_{j_s}) = \{x \in X: f^{t_r} x \in D_{j_r} \text{ for each } r \in [1, s]\}$$

$$\text{where } t_r = n_{f, \mathcal{Q}}(D_{j_1}) + \dots + n_{f, \mathcal{Q}}(D_{j_{r-1}}).$$

Then $C(D_{j_1}, \dots, D_{j_s}) \subset \mathcal{Q}_{f, n}$ for $n \leq n_{f, \mathcal{Q}}(D_{j_1}, \dots, D_{j_s})$. If $M = \max_i n_{f, \mathcal{Q}}(D_i)$, then $\{C(D_{j_1}, \dots, D_{j_s}): s \geq 1, n_{f, \mathcal{Q}}(D_{j_1}, \dots, D_{j_s}) \in [n, n+M]\}$ is a cover of X subordinate to $\mathcal{Q}_{f, n}$. Hence

$$\begin{aligned} & N(\mathcal{Q}_{f, n}) e^{-\lambda n} \\ & \leq e^{M\lambda} \sum \{\exp(-\lambda n_{f, \mathcal{Q}}(D_{j_1}, \dots, D_{j_s})): n_{f, \mathcal{Q}}(D_{j_1}, \dots, D_{j_s}) \in [n, n+M]\}^\lambda. \end{aligned}$$

As the right side is bounded in n , $b(f, \mathcal{Q}) \leq \lambda$.

This proof is almost identical with Furstenberg [10, Proposition III.1] and resembles the proof of a well-known theorem of information theory [19].

We now state (without proof) some basic facts.

Proposition 2. (a) If $f_1: X_1 \rightarrow X_1$ and $f_2: X_2 \rightarrow X_2$ are topologically conjugate (i.e., there is a homeomorphism $\pi: X_1 \rightarrow X_2$ with $\pi f_1 = f_2 \pi$), then

$$b(f_1, Y_1) = b(f_2, \pi(Y_1)) \text{ for } Y_1 \subset X_1.$$

$$(b) \quad b(f, f(Y)) = b(f, Y).$$

$$(c) \quad b(f, \bigcup_{i=1}^{\infty} Y_i) = \sup_i b(f, Y_i).$$

$$(d) \quad b(f^m, Y) = m b(f, Y) \text{ for } m > 0.$$

We now give an example which motivated this paper. Define $f: S^1 \rightarrow S^1$ by $f(z) = z^n$. If $Y \subset S^1$ is closed and $f(Y) \subset Y$ then the Hausdorff dimension of Y satisfies $bd(Y) = b(f|Y)/\log n$. This was proved by Furstenberg [10, Proposition III.1]. For an ergodic f -invariant probability measure μ on S^1 , it is known that (Colebrook [7]; see also [3] and [9]) $bd(G(\mu)) = b_\mu(f)/\log n$ where $G(\mu)$ denotes

the set of generic points of μ . The above two formulas suggest that one might have $b_\mu(f) = b(f, G(\mu))$ if the right side is correctly defined for the noncompact set $G(\mu)$. The intermediate Hausdorff dimension of course motivated our definition of entropy; Theorem 3 shows that the hoped for formula holds for any continuous map on a compact metric space. We mention that another aspect of Colebrook's paper [7] has been generalized by K. Sigmund [20].

2. Goodwyn's theorem. In this section we will generalize a theorem of Goodwyn [13]. For a continuous map $f: X \rightarrow X$ let $M(f)$ be the set of all f -invariant Borel probability measures on X . We refer the reader to [4] or [14] for the definition of $b_\mu(f)$.

Theorem 1. *Let $f: X \rightarrow X$ be a continuous map of a compact metric space and $\mu \in M(f)$. If $Y \subset X$ and $\mu(Y) = 1$, then $b_\mu(f) \leq b(f, Y)$.*

Lemma 1. *Let α be a finite Borel partition of X such that every $x \in X$ is in the closures of at most M sets of α . Then*

$$b_\mu(f, \alpha) \leq b(f, Y) + \log M.$$

Proof. For each $x \in X$ let $I_n(x) = -\log \mu(A)$ where $A \in \alpha_{f,n}$ contains x . The Shannon-McMillian-Breiman theorem [14] says that for some μ -integrable function $I(x)$ one has $I_n(x)/n \rightarrow I(x)$ a.e. and $a = \int I(x) d\mu = b_\mu(f, \alpha)$. For $\delta > 0$ the set $Y_\delta = \{y \in Y: I(y) \geq a - \delta\}$ has positive measure. By Egorov's theorem there is an N so that

$$Y_{\delta,N} = \{y \in Y_\delta: I_n(y)/n \geq a - 2\delta \forall n \geq N\}$$

has positive measure.

Let $\mathcal{B}$ be a finite open cover of X each member of which intersects at most M members of α . Suppose $\mathcal{E} = \{E_i\}$ covers Y and $D_{\mathcal{B}}(E_i) \leq e^{-N}$. If $\beta \in \alpha_{f,n,\mathcal{Q}}(E_i)$ intersects $Y_{\delta,N}$, then $\mu(\beta) \leq \exp((-a + 2\delta)n_{f,\mathcal{Q}}(E_i))$. Since $E_i \cap Y_{\delta,N}$ is covered by at most $M^{n_{f,\mathcal{Q}}(E_i)}$ such β 's,

$$\mu(E_i \cap Y_{\delta,N}) \leq \exp(n_{f,\mathcal{Q}}(E_i)(\log M - a + 2\delta)).$$

For $\lambda = -\log M + a - 2\delta$ we have

$$D_{\mathcal{Q}}(\mathcal{E}, \lambda) = \sum_i \exp(-\lambda n_{f,\mathcal{Q}}(E_i)) \geq \sum_i \mu(E_i \cap Y_{\delta,N}) \geq \mu(Y_{\delta,N}).$$

Letting $\mathcal{E}$ vary, $m_{\mathcal{Q},\lambda}(Y) \geq \mu(Y_{\delta,N}) > 0$. Hence $b(f, Y) \geq b_{\mathcal{Q}}(f, Y) \geq \lambda = -\log M + a - 2\delta$. Letting $\delta \rightarrow 0$ we have our result.

Lemma 2. *Let $\mathcal{Q}$ be a finite open cover of X . For each $n > 0$ there is a finite Borel partition α_n of X such that $f^k \alpha_n \prec \mathcal{Q}$ for all $k \in [0, n]$ and at most $n \text{ card } \mathcal{Q}$ sets in α_n can have a point in all their closures.*

Proof. This idea for this lemma is from Goodwyn [13] and the statement as above is in [15]. Let $\mathcal{Q} = \{A_1, \dots, A_m\}$ and $g_1, \dots, g_m$ be a partition of unity subordinate to $\mathcal{Q}$. Then $G = (g_1, \dots, g_m): X \rightarrow s_{m-1} \subset R^m$ where s_{m-1} is an $m-1$ dimensional simplex. Now $\{U_1, \dots, U_m\}$ is an open cover of s_{m-1} where $U_i = \{x \in s_{m-1}: x_i > 0\}$ and $G^{-1}U_i \subset A_i$. As $(s_{m-1})^n$ is $nm-n$ dimensional, there is a finite Borel partition α_n^* of s_{m-1}^n with at most nm members having a point in all their closures and such that each member of α_n^* lies in some $U_{i_1} \times \dots \times U_{i_n}$. Then $\alpha_n = L^{-1}\alpha_n^*$ works where $L = (G, G \circ f, \dots, G \circ f^{n-1}): X \rightarrow s_{m-1}^n$.

Lemma 3. *Given a finite Borel partition β and $\epsilon > 0$ there is an open cover $\mathcal{Q}$ so that $H_\mu(\beta|\alpha) < \epsilon$ whenever α is a finite Borel partition with $\alpha < \mathcal{Q}$.*

Proof. Let $\beta = \{B_1, \dots, B_m\}$. There is a $\delta > 0$ so that the following is true:

$$H_\mu(\beta|\alpha) < \epsilon \text{ if there is a Borel partition } \{C_1, \dots, C_m\} \\ \text{with each } C_i \text{ a union of members of } \alpha \text{ and } \sum_{i \neq j} P(B_i \cap C_j) < \delta$$

(see [4, Theorem 6.2]). Choose compact sets $K_i \subset B_i$ so that $\mu(B_i \setminus K_i) < \delta/m$. Let $\mathcal{Q}$ be an open cover each member of which intersects at most one K_i . For $\alpha < \mathcal{Q}$ put $A \in \alpha$ in C_i if $A \cap K_i \neq \emptyset$, and in any C_j if $A \cap \bigcup_j K_j = \emptyset$. Then $C_j \cap K_i = \emptyset$ for $i \neq j$ and so

$$\sum_{i \neq j} P(B_i \cap C_j) \leq \sum_i P(B_i \setminus K_i) < \delta.$$

Proof of Theorem 1. Let β be a finite Borel partition of X and $\epsilon > 0$. Let $\mathcal{Q}$ be as in Lemma 3 and α_n as in Lemma 2. Then

$$\begin{aligned} h_\mu(f, \beta) &= n^{-1} h_\mu(f^n, \beta_{f,n}) \leq n^{-1} h_\mu(f^n, \alpha_n) + n^{-1} H_\mu(\beta_{f,n} | \alpha_n) \\ &\leq n^{-1} [h(f^n, Y) + \log(n \text{ card } \mathcal{Q})] + n^{-1} \sum_{k=0}^{n-1} H_\mu(f^{-k}\beta | \alpha_n) \\ &\leq h(f, Y) + n^{-1} \log(n \text{ card } \mathcal{Q}) + n^{-1} \sum_{k=0}^{n-1} H_\mu(\beta | f^k \alpha_n) \\ &\leq h(f, Y) + n^{-1} \log(n \text{ card } \mathcal{Q}) + \epsilon. \end{aligned}$$

Here we used Lemmas 1 and 2 and some general facts:

$$\begin{aligned}
b_\mu(f, \eta) &\leq b_\mu(f, \xi) + H_\mu(\eta | \xi), \\
H_\mu(\eta \vee \gamma | \xi) &\leq H_\mu(\eta | \xi) + H(\gamma | \xi), \\
H_\mu(f^{-1}\eta | f^{-1}\xi) &= H_\mu(\eta | \xi).
\end{aligned}$$

Proofs of these are in [4] and [14]. Finally, let $n \rightarrow \infty$ and then let $\epsilon \rightarrow 0$. The proof is finished.

3. Generic points. For X a compact metric space, the set $M(X)$ of all Borel probability measures on X with the weak topology is a compact metrizable space [18]. $\mu_n \rightarrow \mu$ implies that for $V \supset K$ with V open and K compact one has $\liminf \mu_n(V) \geq \mu(K)$. For $x \in X$ let μ_x denote the unit measure concentrated on x . If a continuous $f: X \rightarrow X$ is given, define

$$\mu_{x,n} = n^{-1}(\mu_x + \mu_{fx} + \cdots + \mu_{f^{n-1}x}).$$

Let $V_f(x)$ be the set of all limit points in $M(X)$ of the sequence $\mu_{x,n}$. Then $V_f(x) \neq \emptyset$ and one checks that $V_f(x) \subset M(f)$. x is a *generic point* for μ if $V_f(x) = \{\mu\}$. Our main result is that $b(f, G(\mu)) = b_\mu(f)$ for μ ergodic where $G(\mu)$ is the set of generic points for μ .

$p = (p_1, \dots, p_N)$ is an N -distribution if $\sum_1^N p_i = 1$ and $p_i \geq 0$; we set $H(p) = -\sum_i p_i \log p_i$. If $a = (a_1, \dots, a_m) \in \{1, \dots, N\}^m$, then $\text{dist } a = (p_1, \dots, p_N)$ where $p_i = m^{-1}$ (number of j with $a_j = i$). If p and q are N -distributions, then $|p - q| = \max_i |p_i - q_i|$.

Lemma 4. *Let*

$$R(N, m, t) = \{a \in \{1, \dots, N\}^m: H(\text{dist } a) \leq t\}.$$

Then, fixing N and t ,

$$\limsup_{m \rightarrow \infty} \frac{1}{m} \log \text{card } R(N, m, t) \leq t.$$

Proof. For an N -distribution q and $\alpha \in (0, 1)$ consider $R_m(q) = \{a \in \{1, \dots, N\}^m: |q - \text{dist } a| < \alpha\}$. Let μ be the measure on $\Sigma_N = \{1, \dots, N\}^{\mathbb{Z}}$ for the Bernoulli shift with distribution $q' = (1 - \alpha)q + \alpha(1/N, \dots, 1/N)$. Each $a \in R_m(q)$ corresponds to a cylinder set $C_a \subset \Sigma_N$. Since $|q - \text{dist } a| < \alpha$, the number of i 's occurring in a is at most $(q_i + \alpha)m$. As the symbol i has probability $q'_i = (1 - \alpha)q_i + \alpha/N$,

$$\mu(C_a) \leq \prod_{i=1}^N q_i^{(q_i + \alpha)m}.$$

Since the C_α 's are disjoint and have total μ -measure at most 1,

$$1 \geq \text{card } R_m(q) \prod_i q_i^{(q_i + \alpha)m}.$$

Taking logarithms we get

$$\begin{aligned} \frac{1}{m} \log \text{card } R_m(q) &\leq \sum_i -(q_i + \alpha) \log q_i' \\ &\leq H(q') + \sum_i (|q_i' - q_i| + \alpha) |\log q_i'|. \end{aligned}$$

As $q_i' \geq \alpha/N$, $|\log q_i'| \leq \log N - \log \alpha$; also $|q_i' - q_i| = |\alpha/N - \alpha q_i| \leq 2\alpha$. So

$$m^{-1} \log \text{card } R_m(q) \leq H(q') + 3\alpha N(\log N - \log \alpha).$$

Now $H(q)$ is uniformly continuous in q and $\log \alpha \rightarrow 0$ as $\alpha \rightarrow 0$. Hence given any $\epsilon > 0$, for small α one has

$$m^{-1} \log \text{card } R_m(q) \leq H(q) + \epsilon$$

for all m and q .

Once an α is chosen one can find a finite set Q of N -distributions so that

(a) $H(q) \leq t$ for $q \in Q$ and

(b) if $H(q^*) \leq t$, then $|q^* - q| < \alpha$ for some $q \in Q$.

Then $R(N, m, t) \subset \bigcup_{q \in Q} R_m(q)$.

$$m^{-1} \log \text{card } R(N, m, t) \leq m^{-1} \log \text{card } Q + (t + \epsilon).$$

Letting $m \rightarrow \infty$ and then $\epsilon \rightarrow 0$ we get our result.

Now suppose $\beta = \{B_1, \dots, B_N\}$ is a cover of X . An n -choice for x (with respect to β and f) is a $\underline{B} = (B_{i_0}, \dots, B_{i_{n-1}}) \in \beta^n$ with $f^k(x) \in B_{i_k}$ for $k \in [0, n)$. An n -choice gives an N -distribution $q(\underline{B}) = \text{dist}(i_0, \dots, i_{n-1})$. The set of such distributions for the various n -choices for x we denote by $\text{Dist}_\beta(x, n)$.

Lemma 5. Suppose $f: X \rightarrow X$ is a continuous map of a topological space, $\mathcal{B}$ an open cover of X , β a finite cover of X and M a positive integer so that $f^k \beta \prec \mathcal{B}$ for all $k \in [0, M)$. For $t \geq 0$ define

$$Q(t, \beta) = \left\{ x \in X : \liminf_{n \rightarrow \infty} (\inf \{H(q) : q \in \text{Dist}_\beta(x, n)\}) \leq t \right\}.$$

Then $h_{\mathcal{B}}(f, Q(t, \beta)) \leq t/M$.

Proof. Let $N = \text{card } \beta$ and $\epsilon > 0$. By Lemma 4 there is an m_ϵ so that

$$\text{card } R(N, m, t + \epsilon) \leq e^{m(t+2\epsilon)}$$

for all $m \geq m_\epsilon$. As $\text{Dist}_\beta(x, n)$ depends only slightly on the last few $f^i(x)$ when n is large and $H(q)$ is continuous in q , one has

$$\liminf_{m \rightarrow \infty} (\inf \{H(q): q \in \text{Dist}_\beta(x, mM)\}) \leq t$$

for $x \in Q(t, \beta)$. Let $\underline{B}_n(x) = (B_{i_0}, \dots, B_{i_{n-1}})$ be an n -choice with distribution $q(x, n)$ minimizing $H(q)$ over $\text{Dist}_\beta(x, n)$. For $k \in [0, M]$ define

$$q_k(x, m) = \text{dist} \{i_{k+rM}: r \in [0, m)\}.$$

Then $q(x, mM) = (1/M) \sum_k q_k(x, m)$. By the concavity of $H(q)$ in q one has $H(q_k(x, m)) \leq H(q(x, mM))$ for some k (depending on x and m).

Fix now any $m_0 \geq m_\epsilon$. For $m \geq m_0$ and $k \in [0, M]$ define

$$S(m, k) = \{x \in X: H(q_k(x, m)) \leq t + \epsilon\}.$$

Then $Q(t, \beta) \subset \bigcup \{S(m, k): m \geq m_0, k \in [0, M]\}$. Assume $x \in S(m, k)$; $a(x) = (B_{i_k}, B_{i_{k+M}}, \dots, B_{i_{k+(m-1)M}})$ is in $R(N, m, t + \epsilon)$. Define

$$A_k(x, m) = \{y \in X: f^j y \in B_{i_j} \text{ for } j \in [0, k) \text{ and}$$

$$f^{k+rM} y \in B_{i_{k+rM}} \text{ for } r \in [0, m)\}.$$

Now $f^j A_k(x, m)$ is contained in some member of β for each $j \in [0, mM)$. Hence $D_{\mathfrak{B}} A_k(x, m) \leq e^{-mM}$. Let $\tilde{G}(m_0) = \{A_k(x, m): x \in S(m, k), m \geq m_0, k \in [0, M]\}$.

Then $\tilde{G}(m_0)$ covers $Q(t, \beta)$. Since there are at most $(\text{card } \beta)^k \cdot \text{card } R(N, m, t + \epsilon)$ different $A_k(x, m)$ with $x \in S(m, k)$,

$$\begin{aligned} D_{\mathfrak{B}}(\tilde{G}(m_0), (t + 3\epsilon)/M) &\leq \sum_{\substack{k \in [0, M) \\ m \geq m_0}} (\text{card } \beta)^k \text{card } R(N, m, t + \epsilon) e^{-m(t+3\epsilon)} \\ &\leq (\text{card } \beta)^{M-1} \sum_{m \geq m_0} e^{-m\epsilon}. \end{aligned}$$

As this quantity approaches 0 as $m_0 \rightarrow \infty$, $m_{\mathfrak{B}, (t+3\epsilon)/M}(Q(t, \beta)) = 0$ and $h_{\mathfrak{B}}(f, Q(t, \beta)) \leq (t + 3\epsilon)/M$. Now let $\epsilon \rightarrow 0$.

Theorem 2. Let $f: X \rightarrow X$ be a continuous map on a compact metric space. Set

$$QR(t) = \{x \in X: \exists \mu \in V_f(x) \text{ with } h_\mu(f) \leq t\}.$$

Then $h(f, QR(t)) \leq t$.

Proof. Let $\mathfrak{B}$ be a finite open cover of X and α a Borel partition of X with the closures of members of α contained in members of $\mathfrak{B}$. Fix $\epsilon > 0$ and let

$$W_\epsilon(M) = \{x \in X: \exists \mu \in V_f(x) \text{ with } (1/M)H_\mu(\alpha_{f,M}) < t + \epsilon\}.$$

If $b_\mu(f) \leq t$, then

$$\lim_{M \rightarrow \infty} \frac{1}{M} H_\mu(\alpha_{f,M}) = b_\mu(f, \alpha) \leq b_\mu(f)$$

implies that $(1/M)H_\mu(\alpha_{f,M}) < t + \epsilon$ for some M . Hence $QR(t) \subset \bigcup_M W_\epsilon(M)$.

Now fix an M and let $\alpha_{f,M} = \{E_1, \dots, E_N\}$. Pick $U_i \supset E_i$ open so that $f^k U_i \subset \mathcal{B}$ for $k \in [0, M]$; set $\beta = \{U_1, \dots, U_N\}$. We will show $W_\epsilon(M) \subset Q(M(t + 2\epsilon), \beta)$. Consider $x \in W_\epsilon(M)$ and $\mu \in V_f(x)$ with $(1/M)H(\alpha_{f,M}) < t + \epsilon$. Let $q' = (\mu(E_1), \dots, \mu(E_N))$ and pick $\delta > 0$ so that

$$|q - q'| \leq \delta \text{ implies } H(q) \leq M(t + 2\epsilon).$$

Now choose compact $K_i \subset E_i$ so that $\mu(E_i \setminus K_i) < \delta/2N$ and disjoint open V_i 's with $U_i \supset V_i \supset K_i$. Let $\underline{B}_n(x) \in \beta^n$ be an n -choice for x so that $B_{i_k} = U_j$ whenever $f^k x \in V_j$. Since $\mu \in V_f(x)$, $\mu_{x,n_j} \rightarrow \mu$ for some $n_j \rightarrow \infty$. For large j one has

$$\mu_{x,n_j}(V_i) \geq \mu(K_i) - \delta/2N$$

for all i . If $q^j = \text{dist } \underline{B}_n(x) = (q^j_1, \dots, q^j_N)$, it follows that $q^j_i \geq \mu(K_i) - \delta/2N \geq \mu(E_i) - \delta/N$. We get $|q^j - q'| \leq \delta$ and $H(q^j) \leq M(t + 2\epsilon)$. Hence $x \in Q(M(t + 2\epsilon), \beta)$.

Lemma 5 now gives us $b_{\mathcal{B}}(f, W_\epsilon(M)) \leq t + 2\epsilon$. By Proposition 2(d) we get $b_{\mathcal{B}}(f, QR(t)) \leq t + 2\epsilon$. Letting $\epsilon \rightarrow 0$ and varying $\mathcal{B}$ we are done.

Corollary. Let $f: X \rightarrow X$ be a continuous map of a compact metric space. Then

$$b(f) = \sup_{\mu \in M(f)} b_\mu(f).$$

Proof. Let $t = \sup_\mu b_\mu(f)$. As $V_f(x) \neq \emptyset$ for $x \in X$, $X \subset QR(t)$ and $b(f) = b(f, X) \leq t$. On the other hand $b(f) \geq t$ by Goodwyn's theorem (Theorem 1).

Remark. This result is already known; see [8] for the finite dimensional metric case and [12] for compact Hausdorff spaces.

Theorem 3. Let f be a continuous map on a compact metric space and $\mu \in M(f)$ be ergodic. Let $G(\mu)$ be the set of generic points of μ , i.e.,

$$G(\mu) = \{x: V_f(x) = \{\mu\}\}.$$

Then $b(f, G(\mu)) = b_\mu(f)$.

Proof. By the ergodic theorem, one has $\mu(G(\mu)) = 1$. Theorem 1 then gives $H(f, G(\mu)) \geq b_\mu(f)$. As $G(\mu) \subset QR(b_\mu(f))$, Theorem 2 gives the reverse inequality.

4. A type of conjugacy. We will call two homeomorphisms $f: X \rightarrow X$ and $g: Y \rightarrow Y$ *entropy conjugate* if there are $X' \subset X$ and $Y' \subset Y$ such that

- (i) X' and Y' are Borel sets,
- (ii) $f(X') \subset X'$, $g(Y') \subset Y'$,
- (iii) $h(f, X \setminus X') < h(f)$, $h(g, Y \setminus Y') < h(g)$, and
- (iv) $f|_{X'}$ and $g|_{Y'}$ are topologically conjugate.

Unfortunately this does not seem to be an equivalence relation.

Proposition 3. *If f and g are entropy-conjugate homeomorphisms of compact metric spaces, then $h(f) = h(g)$.*

Proof. Suppose $\mu \in M(f)$ and $h_\mu(f) > h(f, X \setminus X')$. Since μ is f -invariant and $f(X') \subset X'$, one can find $B \subset X \setminus X'$ with $\mu(B) = \mu(X \setminus X')$ and $f(B) = B$. By Theorem 1, $\mu(B) < 1$. Define $\mu_{X'}(E) = \mu(E \cap X')/\mu(X')$. Then $\mu_{X'} = \mu$ (if $\mu(X') = 1$) or $\mu = \mu(X')\mu_{X'} + \mu(B)\mu_B$. In the second case $\mu_{X'}, \mu_B \in M(f)$ and

$$h_\mu(f) = \mu(X')h_{\mu_{X'}}(f) + \mu(B)h_{\mu_B}(f).$$

By Theorem 1 we have $h_{\mu_B}(f) \leq h(f, X \setminus X') < h_\mu(f)$ and so $h_{\mu_{X'}}(f) \geq h_\mu(f)$. If $\mu_{X'} = \mu$, we of course also have $h_{\mu_{X'}}(f) \geq h_\mu(f)$. Since $\mu_{X'}(X') = 1$, the topological conjugacy of $f|_{X'}$ and $g|_{Y'}$ gives us a measure ν on Y' with (g, ν) conjugate to $(f, \mu_{X'})$; in particular

$$h_\nu(g) = h_{\mu_{X'}}(f) \geq h_\mu(f).$$

By Goodwyn's theorem $h(g) \geq h_\nu(g)$. Using the Dinaburg-Goodman theorem (corollary to Theorem 2) one can make $h_\mu(f)$ arbitrarily close to $h(f)$ (and so satisfy $h_\mu(f) > h(f, X \setminus X')$). One gets $h(g) \geq h(f)$. By symmetry one likewise has $h(g) \leq h(f)$.

There is a natural class of homeomorphisms for which the converse of Proposition 3 may hold. Let $\Sigma_n = \prod_{\mathbb{Z}} \{1, \dots, n\}$ and define the shift $\sigma_n: \Sigma_n \rightarrow \Sigma_n$ by

$$(\sigma_n x)_i = x_{i+1} \quad \text{for } x = (x_i).$$

σ_n is a homeomorphism of the compact metrizable space Σ_n . For A an $n \times n$ matrix of 0's and 1's define

$$\Sigma_n(A) = \{(X_i) \in \Sigma_n : A_{x_i x_{i+1}} = 1 \ \forall i\}.$$

Then $\sigma_n|_{\Sigma_n(A)}$ is a homeomorphism of a compact space.

Conjecture. Suppose $\sigma_n|_{\Sigma_n(A)}$ and $\sigma_m|_{\Sigma_m(B)}$ are topologically mixing and have the same topological entropy. Then they are entropy conjugate.

This conjecture is related to the symbolic dynamics of diffeomorphisms [2],

[16], [6]. From [6] it follows that the nonwandering set of an Axiom A diffeomorphism is entropy conjugate to some $\Sigma_n(A)$ (called a subshift of finite type). The codings of [2] show that the conjecture is true for the subshifts of finite type that arise from hyperbolic automorphisms of T^2 . The codings were used in [2] to prove that entropy classifies such maps on T^2 up to measure theoretic conjugacy; Friedman and Ornstein [11] now supplant these codes for this purpose. The notion of entropy conjugacy attempts to clarify the topological content of the Adler-Weiss codings (see problem 3 of [21]).

Proposition 4. *Suppose f and g are entropy-conjugate homeomorphisms of compact metric spaces. Then f is intrinsically ergodic iff g is.*

Proof. Intrinsic ergodicity [17] means there is a unique $\mu \in M(f)$ with $h_\mu(f) = h(f)$. The proof is like that of Proposition 3.

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