STABILITY PROPERTIES OF A CLASS OF ATTRACTORS

BY
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ABSTRACT. Let A be an attractor of an analytical dynamical system defined in $R^n \times R$. The class of attractors considered in this paper consists of those A which remain stable as invariant subsets of the complex extension of the flow to $C^n \times R$. If A is a critical point or a closed orbit, these are the elementary or generic attractors. It is shown that such an A is always a submanifold of R^n and that there exists a Lie group acting on A and containing the given flow as a one parameter dense subgroup; as a consequence, some necessary and sufficient conditions for an analytical dynamical system to have an attracting generic periodic motion are given.

It is also shown that for any flow C^1 -close to the given one, there is a unique retraction of a neighbourhood of A onto a submanifold of R^n homeomorphic to A that commutes with the flow.

1. In this paper we study a family of attractors of an analytic flow taking place in R^n whose properties are, in a certain sense, similar to those of the simplest attractors.

Let Ω be an open connected subset of R^n and $X: \Omega \rightarrow R^n$ an analytic vector field defining the analytic flow ϕ on an open subset of $\Omega \times R$. A compact ϕ -invariant set $A \subset \Omega$ is an attractor if:

- (1) for any neighbourhood $U \subset \Omega$ of A , there is another neighbourhood V , such that $x \in V$ implies $\phi(x, t) \in U$ for $t \geq 0$, and
- (2) there is a neighbourhood of A with the property that if x belongs to it, $\lim_{t \rightarrow \infty} \text{dist}(\phi(x, t), A) = 0$.

Since X is analytic there is a complex extension of X to a C^n -neighbourhood W of A . We consider here the attractors A which are stable as invariant subsets of the flow generated by this extension of X , or, in other words, we consider those A that satisfy property (1) with respect to the analytic extension of ϕ to an open subset of $W \times R$. (1) For brevity, let us denote also by ϕ this extension and call C -generic these attractors. It is not reasonable to expect an attractor A to be also an attractor of the extended flow: if A contains a periodic motion x , say of minimum period $\tau > 0$, then for small real s , $\phi(x, si)$ lies close to A but not on it and $\phi(\phi(x, si), \tau) = \phi(x, \tau + si) = \phi(x, si)$ since the equality $\phi(x, t + \tau) = \phi(x, t)$ holds for real t ; hence, condition (2) is not satisfied. One of the motivations for studying this class of attractors comes from the fact that in the simplest cases the

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- (1) The parameter t of the flow ϕ will always be assumed to be real unless otherwise stated.

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C -generic attractors are exactly the generic attractors defined in [5, §2, Theorem 1].

We prove, among other things, that a C -generic attractor is always an analytical manifold, and construct a compact Lie group acting on it and containing the given flow as a one parameter dense subgroup. As a corollary we get some characterizations of analytic dynamical systems possessing a generic attracting periodic motion. Also, we get several results on the perturbation theory of C -generic attractors by studying the action of the flow on a Banach manifold of retractions of Ω .

Finally, perhaps it is convenient to say that most of the results can be extended in several ways, for instance replacing R^n by a differentiable manifold and the analytic flow by a C^1 -flow, but since we believe that these generalizations will not add anything of a conceptual character, we prefer to keep in the above-mentioned context, where some of the results are richer and the techniques simpler.

2. We begin by studying the simplest attractors, i.e., equilibrium points and closed orbits.

Theorem 1. *Let A be an attractor for an analytic flow in R^n and assume that A is a periodic motion or a critical point. Then, A is generic [5] if and only if it is C -generic.*

Proof. We give the proof for A a periodic motion; the case of a critical point can be handled with similar arguments.

We assume, as we may, that A goes through the origin 0 in R^n and that the vector field X takes at this point the value $X(0) = (0, \dots, a)$ where $a \neq 0$. Let $\tau > 0$ be the fundamental period of A and $\phi = (\phi_1 \dots \phi_n)$ the flow defined by X . Consider the equation

$$(1) \quad \phi_n(x, t) = 0$$

for x in a complex neighbourhood of 0 and t in a complex neighbourhood of τ . Applying the implicit function theorem we get another complex neighbourhood of 0 and a positive number ε such that, for x in that neighbourhood, equation (1) has exactly one complex root $t(x)$ satisfying $|t(x) - \tau| < \varepsilon$. For points $x = (x_1, \dots, x_n)$ close to 0 and lying on the hyperplane $H = \{x \in C^n: x_n = 0\}$, we consider the complex analytic Poincaré mapping $\psi: H \rightarrow H$ defined by $\psi(x) = \phi(x, t(x))$.

To prove the necessity, call $T: H \rightarrow H$ the linear part of ψ at the origin which, by assumption, has proper values $\lambda_i: i = 1, 2, \dots, n-1$, satisfying the inequalities $0 < |\lambda_i| < 1$, and take a norm $\|\cdot\|$ for H , such that the corresponding $\|T\| < 1$ (see [5, p. 102]).

If we write $\psi(x) = Tx + \psi_0(x)$, then

$$\lim_{\|x\| \rightarrow 0} \frac{\psi_0(x)}{\|x\|} = 0$$

and we may choose a complex neighbourhood of 0 in H such that if x belongs to it, $\|\psi_0(x)\| < \frac{1}{2}(1 - \|T\|)\|x\|$. Thus for these x , $\|\psi(x)\| \leq (\|T\| + \frac{1}{2}(1 - \|T\|)) \cdot \|x\| = \mu\|x\|$, where $0 < \mu < 1$. Therefore if x is small enough, the n -iterate of ψ , ψ^n , is defined at x for $n = 1, 2, \dots$, and $\|\psi^n(x)\| \leq \mu^n\|x\|$.

To prove that for real $t \geq 0$, $\phi(x, t)$ is close to $\phi(0, t)$, let us write

$$\begin{aligned} \phi(x, n\tau) &= \phi\left(\phi\left(x, \sum_1^n t(\psi^{i-1}(x))\right), \sum_1^n (\tau - t(\psi^{i-1}(x)))\right) \\ &= \phi\left(\psi^n(x), \sum_1^n (\tau - t(\psi^{i-1}(x)))\right), \end{aligned}$$

and notice that, since $t(x)$ depends analytically on x ,

$$\left| \sum_1^n (\tau - t(\psi^{i-1}(x))) \right| \leq \sum_1^n M\|\psi^{i-1}(x)\| \leq M\|x\| \sum_1^n \mu^{i-1}.$$

From these remarks follows easily the asserted stability of the periodic motion as an invariant subset of the extended flow.

Now we prove the sufficiency. The functions ϕ_t , $\phi_t(x) = \phi(x, t)$, are defined, analytic and uniformly bounded for $t \geq 0$ on a suitable C^n -neighbourhood of 0; using Cauchy's formula we may show that the derivatives $(\partial\phi_t/\partial x_j)(x, t)$, $i = 1, \dots, n$, $j = 1, \dots, n$, are also uniformly bounded for $t \geq 0$ in a smaller C^n -neighbourhood of 0. This implies, in particular, the positive Lyapunov stability of the periodic motion. Let U be a C^n -neighbourhood of 0 and ε a positive real number so that for $x \in U$, $\phi_n(x, t) = 0$ has only one root $t(x)$ with $|t(x) - \tau| < \varepsilon$. Because of the Lyapunov stability, and since $\phi(0, n\tau) = 0$, $n \geq 0$, we may choose a neighbourhood of 0 in H , say V , such that $\phi(x, n\tau) \in U$ for $x \in V$, $n = 0, 1, 2, \dots$. Consequently, there are complex numbers z_n , $n = 1, 2, \dots$, such that for these n , $\phi(x, n\tau + z_n) \in H$, and $|z_n| < \varepsilon$. It is clear then that if ε is small enough, $\psi^n(x) = \phi(x, n\tau + z_n)$, and therefore 0 is a positively stable fixed point of the complex Poincaré diffeomorphism ψ .

Let V be a neighbourhood of 0 in the complex subspace H such that the ψ^n , $n = 1, 2, \dots$, are defined and equiuniformly bounded on V ; since the ψ^n constitute there a normal family, there is a subsequence ψ^{n_k} that converges uniformly on V to an analytic function. It follows easily from the assumptions that if x is real and sufficiently small, $\lim_{n \rightarrow \infty} \psi^n(x) = 0$, which implies readily that $\psi^{n_k} \rightarrow 0$ uniformly on V . Consequently, there is a $\delta > 0$ and a positive integer N so that if $x \in H$, $\|x\| \leq \delta$, then $\|\psi^N(x)\| < \delta$, and $\psi^N(x) \in V$, $n = 1, 2, \dots, N$. This situation cannot be changed by small analytic perturbations of ψ , i.e. if ψ' is close enough to ψ on V and $\psi'(0) = 0$, ψ'^N will have the

same property as ψ with respect to the complex ball $\{x \in H: \|x\| \leq \delta\}$.

On the other hand, this property implies that all the iterates ψ^n , $n = 1, 2, \dots$, are equiuniformly bounded on that ball, and as we remarked before, this in turn implies the positive stability of 0 under the discrete flow defined by ψ . Then if the linear part of ψ had some proper value of absolute value 1, we could add to ψ a small linear transformation of H to get an analytic diffeomorphism ψ' close to ψ and having 0 as an unstable fixed point [5]. This completes the proof.

3. In this section we prove several results of a topological character concerning general C -generic attractors.

Let $U \subset R^n$ be a compact neighbourhood of the C -generic attractor A with the property that for $x \in U$, $\lim_{t \rightarrow \infty} \text{dist}(\phi(x, t), A) = 0$, and such that there exists W , a complex C^n -neighbourhood of U , having all its iterates $\phi(W, t)$, $t \geq 0$, uniformly bounded. The analytic functions ϕ_t , $t \geq 0$, constitute a normal family on W .

Lemma 1. *For every $\varepsilon > 0$ there exists $\delta > 0$ such that, for $x, y \in A$, $\|x - y\| \leq \delta$ implies $\|\phi(x, t), \phi(y, t)\| \leq \varepsilon$ for $|t| < \infty$.*

Proof. As a consequence of Cauchy's formula and the compactness of U we have that for every $\varepsilon > 0$ there exists $\delta > 0$ such that $x, y \in U$ and $\|x - y\| \leq \delta$ imply $\|\phi(x, t) - \phi(y, t)\| \leq \varepsilon$ for $t \geq 0$. On the other hand, it is clear that if $x = \lim(\phi(x_0, t_n))$, $y = \lim(\phi(y_0, t_n))$, $\lim t_n = +\infty$, and $\|\phi(x_0, t) - \phi(y_0, t)\| \leq \varepsilon$ for $t \geq 0$, then, for each $t \in R$, $\|\phi(x, t) - \phi(y, t)\| \leq \varepsilon$.

Now choose a sequence ϕ_{t_n} , $t_n \rightarrow \infty$, such that ϕ_{t_n} converges uniformly to a continuous mapping $\theta: U \rightarrow U$. Because of the way U was chosen, $\theta(U) \subset A$; on the other hand it follows from the compactness of U , the fact that, for each n , $\phi_{t_n}(U) \supset A$ and the uniform convergence of the ϕ_{t_n} , that $\theta(U) \supset A$.

Thus $\theta(U) = A$. Then for each $x \in A$, $\theta^{-1}(x)$ is a compact nonvoid subset of U , and it is easy to show that if V is any neighbourhood of $\theta^{-1}(x)$, $\theta(V)$ is an A -neighbourhood of x . It follows from this and the previous remarks that given $\varepsilon > 0$ and $x \in A$ there exists an A -neighbourhood of x such that if y and z belong to it, $\|\phi(y, t) - \phi(z, t)\| \leq \varepsilon$, $t \in R$, and the thesis of the lemma is now an immediate consequence of the compactness of A .

Let V be an open positively invariant neighbourhood of A , $V \subset U$.

Lemma 2. *There exists one and only one analytic retraction α of V onto A that commutes with the flow, i.e. such that $\alpha'_x X_x = X_{\alpha(x)}$.*

Proof. Let t_n be a sequence such that $t_n \rightarrow \infty$, $t_n - t_{n-1} \rightarrow \infty$ and ϕ_{t_n} converges uniformly on U . Then, if s_k is a subsequence of $t_n - t_{n-1}$ such that ϕ_{s_k} converges uniformly on V to an analytic function α , we have clearly that $\alpha(V) = A$, and since for n large enough $\|\phi_{t_{n+1}}(x) - \phi_{t_n}(x)\|$ is arbitrarily small, uniformly for all $x \in A$, we get from Lemma 1 that $\|\phi_{s_k}(x) - x\|$ is also uniformly small for k large.

Therefore $\alpha(x) = x$ for $x \in A$ and α is an analytic retraction of U onto A . Obviously, α commutes with each ϕ_t .

Let $x \in V$, then $\|\phi(x, t) - \phi(\alpha(x), t)\| = \|\phi(x, t) - \alpha(\phi(x, t))\|$ and as $\lim_{t \rightarrow \infty} \text{dist}(\phi(x, t), A) = 0$ and α is a retraction onto A , it follows that $\lim_{t \rightarrow \infty} \|\phi(x, t) - \phi(\alpha(x), t)\| = 0$.

Now we prove the uniqueness of α by showing that $\alpha(x)$ is the only point in A with that property. In fact, the existence of another point in A with the same property would contradict Lemma 1.

Lemma 3. *Let $V \subset R^n$ be an open set, A a compact subset of V and α an analytic retraction of V onto A . Then A is an analytic submanifold of R^n .*

Proof. Let $k = \max_{x \in V} \{\text{rank } \alpha'_x\}$; as the case $k = 0$ is trivial, we may assume $k > 0$. Take $x \in V$ such that $\text{rank } \alpha'_x = k$; since $\alpha \circ \alpha = \alpha$, $\alpha'_{\alpha(x)} \circ \alpha'_x = \alpha'_x$, and it follows that α' has maximum rank at $\alpha(x)$ and hence in a neighbourhood of it. From well-known results in calculus we get that there exists an R^n -neighbourhood V of $\alpha(x)$ such that $\alpha(V)$, as a subspace of R^n , is analytically homeomorphic to an open subset of R^k . On the other hand $V \cap A = \alpha(V \cap A) \subset \alpha(V)$, and since $\alpha(V) \subset A$, $V \cap A = V \cap A \cap \alpha(V) = V \cap \alpha(V)$. Consequently, $\alpha(x)$ has an A -neighbourhood analytically homeomorphic to an open subset of R^k .

The set $B = \{y \in A : \text{rank } \alpha'_y = k\}$ is open in A ; let us show that B is also closed. If x_n is a sequence in B converging to $x \in A$, and $\{u_n^1, \dots, u_n^k\}$ an orthonormal system in the range of α'_{x_n} , $n = 1, 2, \dots$, such that $u_n^i \rightarrow u^i$, $i = 1, \dots, k$, we have, since for all n , $\alpha'_{x_n} u_n^i = u_n^i$, that α' has maximum rank at x as we wanted to show. Since A is connected, $B = A$, and on account of the previous remark, this completes the proof.

Theorem 2. *A compact connected subset of R^n is a C -generic attractor of some analytic flow in R^n if and only if it is an analytic submanifold of R^n .*

Proof. The necessity follows from the lemmas. For the sufficiency, consider an analytic submanifold $A \subset R^n$ and the function $r: R^n \rightarrow A$ defined by $r(x)$, where

$$\|x - r(x)\| = \min_{y \in A} \|x - y\|.$$

If x is close enough to A , it is known (see [2, p. 38]) that $r(x)$ is well defined and the function r is analytic in a suitable R^n -neighbourhood of A . Moreover, if $x \in A$ and if u is small and normal to A at x , $r(x + u) = x$. Consequently the analytic vector field $x \rightarrow r(x) - x$ defined on that neighbourhood determines an analytic flow ϕ , such that $\phi(x, t) = r(x) + (x - r(x))e^{-t}$. From this formula it is easy to obtain that A is stable as an invariant subset of the C^n -extension of ϕ .

4. Now we are going to study the flow on C -generic attractors and to associate to each a Lie group acting on it.

Let A be a C -generic attractor; it follows easily from Lemma 1 that if $x \in A$, the closure of the trajectory through x is a minimal set M_x . So A is a union of

disjoint minimal sets. Let G be the closure of the set of analytic homeomorphisms $\{\phi_t/A: t \in R\}$ considered as a subspace of the space of mappings from A into A with the topology of uniform convergence on A . Since there exists a sequence $t_n \rightarrow \infty$ such that ϕ_{t_n}/A converges uniformly to the identity (Lemma 2), G is also the closure of the set $\{\phi_t/A: t \geq 0\}$ and inasmuch as the ϕ_t , $t \geq 0$, constitute a normal family when restricted to a suitable C^n -neighbourhood of A , we may deduce easily that each element of G is analytic and that $(G, \circ)$ with the relative topology is a compact connected abelian group such that each M_x is homeomorphic to a factor group of G .

Theorem 3. *Let k be the highest of the dimensions of the minimal sets contained in A . Then G is a k -dimensional torus.*

Proof. Since G is a group acting effectively by analytic mappings on the manifold A , it is a Lie group (see [3, p. 208]) and, being abelian compact and connected, it is a torus. To prove that G is k -dimensional, choose x_0 such that $\dim M_{x_0} = k$ and call G_{x_0} the subgroup of G , $G_{x_0} = \{\alpha \in G: \alpha(x_0) = x_0\}$. Clearly, $G_{x_0} = G_x$ for each $x \in M_{x_0}$. The mapping $[\gamma] \rightarrow \gamma(x_0)$ from G/G_{x_0} to M_{x_0} is a homeomorphism and M_{x_0} gets in this way a Lie group structure. Let U be a neighbourhood of x_0 in M_{x_0} such that it contains no nontrivial subgroups. We may choose $\delta > 0$ so small that if $x \in A$, $\gamma \in G$, $\|x - x_0\| \leq \delta$ and $\gamma(x) = x$, then $\gamma^n(x_0) \in U$ for every integer n , and therefore $\gamma(x_0) = x_0$. Consequently if $\|x - x_0\| \leq \delta$, $G_x \subset G_{x_0}$; hence $\dim M_x = k$, and it follows that G_{x_0}/G_x is finite. Thus for each $x \in A$ with $\|x - x_0\| \leq \delta$ there is a positive integer $n(x)$ such that $\gamma \in G$ and $\gamma(x_0) = x_0$ imply $\gamma^{n(x)}(x) = x$. By means of a simple category argument we may now show that there is an open subset of A , say D , and a positive integer N such that $\gamma^N(x) = x$ for every $x \in D$, provided that $\gamma(x_0) = x_0$. Since each $\gamma \in G$ is analytic, it follows that if for some $\gamma \in G$, $\gamma(x_0) = x_0$, then γ^N is the identity mapping of A . Thus, being a Lie group, G_{x_0} has to be finite, which implies clearly that $\dim G = k$.

Let a, b be real numbers and let r and w stand respectively for $x_1^2 + x_2^2$ and $(r - 1)^2 + x_3^2$; the vector field

$$\begin{aligned} & \frac{1}{2}(-2bx_2 - ax_1x_3r^{-1} - x_1r^{-1}(r-1)(w-\frac{1}{4}))\partial/\partial x_1 \\ & + \frac{1}{2}(2bx_1 - ax_3x_2r^{-1} - (r-1)r^{-1}x_2(w-\frac{1}{4}))\partial/\partial x_2 \\ & + (a(r-1) - x_3(w-\frac{1}{4}))\partial/\partial x_3 \end{aligned}$$

is defined in some open subset of R^3 and admits the torus $w = \frac{1}{4}$ as a C -generic attractor. The flow ϕ can be given explicitly:

$$\begin{aligned} \phi(x, t) = & ((r(t))^{1/2} \cos(bt + \psi_0), (r(t))^{1/2} \sin(bt + \psi_0), \\ & w^{1/2} \sin(at + \Phi_0)(4w + (1 - 4w)\exp -\frac{1}{2}t)^{-1/2}); \end{aligned}$$

here $r(t) = 1 + (w^{1/2} \cos(at + \Phi_0)(4w + (1 - 4w)\exp -\frac{1}{2}t)^{-1/2}$ and $\cos \Phi_0 = (r - 1)/w^{1/2}$, $\sin \Phi_0 = x_3/w^{1/2}$, $\cos \psi_0 = x_1/r^{1/2}$ and $\sin \psi_0 = x_2/r^{1/2}$. If a/b is irrational the group G associated to the attractor is of dimension 2.

The vector field $(-x_2 - x_1\rho)\partial/\partial x_1 + (x_1 - x_2\rho)\partial/\partial x_2 - x_3\rho\partial/\partial x_3$, where $\rho = x_1^2 + x_2^2 + x_3^2 - 1$, has the unit sphere of R^3 as a C -generic attractor, and the group G associated to it is isomorphic to R/Z (Z denotes the integers). All the motions on $\rho = 0$ are periodic and except for the poles all have fundamental period 1. If we consider the product of the flow defined by this vector field and a flow in R^2 having a periodic motion of fundamental period $\frac{3}{4}$ as a C -generic attractor, we get a product flow that has a C -generic attractor with a one-dimensional group G . All the minimal sets on the attractor are one-dimensional and in some of them G does not act effectively.

Theorem 4. *Let A be a C -generic attractor of the flow ϕ defined by the analytic vector field X such that the group G associated to A and X has dimension greater than one. Then there is a C^n -neighbourhood U of A such that for every $\epsilon > 0$ there exists another analytic vector field X' defined in U with the following properties:*

- (1) $\|X_x - X'_x\| \leq \epsilon$ for every $x \in U$.
- (2) A is a C -generic attractor for X' and the retractions for X and X' defined by Lemma 2 are the same when restricted to U .
- (3) The group G' associated to A and X' is one dimensional.

Proof. The retraction α defined for X by Lemma 2 may be extended to a small C^n -neighbourhood W of A ; let us call β this extension. The set $B = \{x \in W; \beta(x) = x\}$ satisfies $B \cap R^n = A$ since $\beta/R^n = \alpha$.

As for $x \in A$, $R(\alpha'_x) \cap N(\alpha'_x) = \{0\}$ ($R(\alpha'_x)$ and $N(\alpha'_x)$ denote respectively the range and null-space of α'_x), we have that for x on a certain C^n -neighbourhood $V \subset W$ of A , the linear subspaces of C^n , $R(\beta'_x)$ and $N(\beta'_x)$, have also a trivial intersection and therefore, for each u tangent to B at $\beta(x)$, the intersection

$$\{\beta'^{-1}_x(u)\} \cap \{v \in C^n: v = X_x + r, r \in R(\beta'_x)\}$$

consists of exactly one point. It is easy to show that if $\bar{X}$ is an analytic vector field on $B \cap V$, the vector field X' defined on $\{\beta^{-1}(B \cap V)\} \cap V$ by $X'_x \in \{\beta'^{-1}_x(\bar{X}_{\beta(x)})\} \cap \{v \in C^n: v = X_x + r, r \in R(\beta'_x)\}$ is also analytic and $X'/A = \bar{X}/A$.

Take $U \subset V$ to be a compact C^n -neighbourhood of A so that $\beta(U)$ is invariant under the complex extension of ϕ , $\|\phi_t(x) - \beta \circ \phi_t(x)\|$ converges uniformly to 0 on U , and that, for some $\delta > 0$,

$$U = \{\beta^{-1}(\beta(U))\} \cap \{x \in C^n: \|\beta(x) - x\| \leq \delta\}.$$

Let $T > 0$ be such that for $x \in U$, $\|\phi_T(x) - \beta \circ \phi_T(x)\| \leq \delta/2$. If X' is another analytic vector field on U that commutes with β and such that $\beta(U)$ is also an invariant subset of the flow ϕ' defined by X' , then if X' is chosen so close

to X that $\|\phi'_T(x) - \beta \circ \phi'_T(x)\| \leq \delta$, we get that $\phi'_T(U) \subset U$ for, on the other hand, $\beta \circ \phi'_T(U) = \phi'_T(\beta\{U\}) = \beta(U)$. Hence, X' has a C -generic attractor contained in U and containing A .

Choose in G a one parameter subgroup G' isomorphic to R/Z and close enough to $\{\phi_t: |t| < \infty\}$. G' defines a flow and an analytic vector field $\bar{X}$ on A . If we define X' on U as above we have that X' is close to X on U and that every compact X invariant subset of U is also X' invariant. Therefore the previous remarks apply to X' and we get that X' has a C -generic attractor A' , $A \subset A' \subset U$. Since $A \neq A'$ implies $\dim A' > \dim A$, our result follows easily from the next lemma and the fact that if X' is close enough to X there exists an X and X' positively invariant neighbourhood of A and A' .

Lemma 4. *Let A be a C -generic attractor of the analytic flow ϕ . Then A is a deformation retract of each sufficiently small positively invariant R^n -neighbourhood of it.*

Proof. Let U be as above and α the retraction of $U \cap R^n$ onto A . Take $V \subset U \cap R^n$ to be a positively invariant neighbourhood of A and assume that the sequence $\{\phi_{t_n}; n = 1, 2, \dots\}$ converges uniformly to α on U .

Let $\varepsilon > 0$ be small enough that for each $x \in A$, the ε -ball centered at x lies in V , and choose a positive integer N with the property that $\|\phi_{t_k}(x) - \alpha(x)\| \leq \varepsilon$ for each $x \in V$ and each $k \geq N$. Now we define the retraction $H: V \times [0, 1] \rightarrow V$ by

- (i) $H(x, u) = \phi(x, u/(1 - u))$ for $x \in V$, $0 \leq u \leq t_N/(1 + t_N)$.
- (ii) $H(x, u) = (1 - s_k(u))\phi(x, t_k) + s_k(u)\phi(x, t_{k+1})$ for $x \in V$, $t_k/(1 + t_k) \leq u \leq t_{k+1}/(1 + t_{k+1})$, $k \geq N$. Here

$$s_k(u) = \left(u - \frac{t_k}{1 + t_k}\right) \left(\frac{t_{k+1}}{1 + t_{k+1}} - \frac{t_k}{1 + t_k}\right)^{-1}.$$

- (iii) $H(x, 1) = \alpha(x)$, $x \in V$.

5. Let X be an analytic vector field defined on an open subset $\Omega \subset R^n$. To state shortly some corollaries of the previous results, let us call an open subset $D \subset \Omega$ admissible if

- (i) D is positively invariant under the flow of ϕ defined by X ;
- (ii) $X_x \neq 0$ for every $x \in D$;
- (iii) the integral homology groups of D are those of S^1 .

It follows from the previous results that X has a periodic motion as a generic attractor if and only if there exists an admissible D such that the trajectories of the complex extension of ϕ issuing from a suitable C^n -neighbourhood of D remain uniformly bounded in the future. As a consequence, we also get the following corollary.

Corollary 1. *A necessary and sufficient condition for X to have a periodic motion as a generic attractor is the existence of a differentiable function $\lambda: N \rightarrow R$ (N a C^n*

open set), $\lambda(x) = 0$ for $x \in N \cap R^n$, such that:

(1) The vector field $(1 + \lambda i)X$ is defined on N and there exists a compact $2n$ -dimensional differentiable submanifold N with boundary $M \subset N$ such that at each point of the boundary of M , the vector field $(1 + \lambda i)X$ points to the interior of M .

(2) $M \cap R^n$ contains an admissible set D .

(3) For some $k > 0$, $\dot{\lambda} \leq -k\lambda^2$ for $x \in N$, where

$$\dot{\lambda}(x) = (1 + \lambda(x)i)X_x(\lambda) = \lim_{t \rightarrow 0} \frac{1}{t}(\lambda(x + t(1 + \lambda(x)i)X_x) - \lambda(x)).$$

Proof. Sufficiency. Because of (1) the trajectories of $(1 + \lambda i)X$ are bounded in the future. On the other hand it follows easily from (3) that if $x(t)$ is a trajectory of $(1 + \lambda i)X$, $\lambda^2(x(t)) = \mu(t)$ satisfies $\mu(t) \leq \mu(0)\exp(-2kt)$ and hence $|\lambda(x(t))| \leq |\lambda(x(0))|\exp(-kt)$. Consider the function $s(t) = \int_0^t \lambda(x(u))du$; clearly $|s(t)| \leq |\lambda(x(0))|/k$ for $t \geq 0$.

The differential equation $\dot{s} = \lambda(\phi(x, t + is))$ is defined for small t and s and if $s(t)$ is its solution with $s(0) = 0$ it is easy to check that $\phi(x, t + is(t))$ is the trajectory $x(t)$ of $(1 + \lambda i)X$ with the initial condition $x(0) = x$. So, $s(t) = \int_0^t \lambda(x(u))du$ and thus $|s(t)| \leq |\lambda(x)|/k$.

If $x \in M$ and is close enough to R^n , $\phi(x, t)$ cannot get far from M ; otherwise, $x(t)$ would also be far from it, which is absurd on account of (1). Therefore the $\phi(x, t)$ are uniformly bounded on a C^n -neighbourhood of D and in view of previous remarks this completes the proof of the sufficiency.

Necessity. Let A be the periodic motion and U a C^n -neighbourhood of A such that for each $x \in U$, $\beta(x)$ (β denotes the complex extension of the retraction α ; see the proof of Theorem 4) can be written $\beta(x) = \phi(\bar{x}, t + is)$ where $\bar{x} \in A$ and s is a real number. Clearly s is uniquely determined and we define $\lambda(x)$ as $-s(\beta(x))$. Then, the trajectories of $(1 + \lambda i)X$ are $\phi(x, t + i\lambda(x)(1 - e^{-t}))$ as it is easy to prove owing to the commutativity between β and the flow. Since $\lambda(\phi(x, t + i\lambda(x)(1 - e^{-t}))) = \lambda(x)e^{-t}$ it follows that $\dot{\lambda}(x) = -\lambda(x)$ or $\dot{\lambda} \leq -\lambda^2$ which implies (3). On the other hand, (1) is a consequence of the fact that $\phi(x, t + i\lambda(x)(1 - e^{-t}))$ converges to A uniformly on a suitable C^n -neighbourhood of A . (See, for instance, [1].)

Other corollaries can also be obtained using the well-known results of Montel on normal families. For instance,

Corollary 2. Let a_i , $i = 1, 2, \dots, n$ be positive real numbers. Then a necessary and sufficient condition for X to have a periodic motion as a generic attractor is the existence of an admissible subset D such that all the trajectories of the vector field $(\prod_{i=1}^n (x_i^2 + a_i))X$ issuing from some C^n -neighbourhood of D can be defined in the future.

Proof. Since on R^n the flow defined by $(\prod_{i=1}^n (x_i^2 + a_i))X$ is just a reparametrization of ϕ , the necessity follows at once from Theorem 1.

For the sufficiency, let us denote by $\bar{\phi}$ the flow defined by $(\prod_1^n (x_i^2 + a_i))X$. If U is a C^n -neighbourhood of D such that the $\bar{\phi}_t$ are defined on it for all t and such that no point with some $x_i = \pm i(a_i)^{1/2}$ belongs to it, then, for no t , $\bar{\phi}_t(U)$ intersects the hyperplanes $x_i = \pm i(a_i)^{1/2}$ that consist only of critical points of the vector field $(\prod_1^n (x_i^2 + a_i))X$. Therefore by Montel's theorem the $\bar{\phi}_t$ constitute a normal family and being uniformly bounded on D they are uniformly bounded on a suitable C^n -neighbourhood of it. Since, on R^n , ϕ is a reparametrization of $\bar{\phi}$, the result follows again from Theorem 1.

6. Let X be an analytic vector field, A a C -generic attractor of the flow associated to X , U an open and bounded positively invariant R^n -neighbourhood of A , and let α be the retraction of U onto A defined previously. In this and the following section we show, among other things, that if Y is a vector field close to X on U (in a sense to be made precise later), then there exists a unique retraction γ , close to α , of U onto a compact submanifold of U homeomorphic to A that commutes with Y . These results are connected, in particular, with the perturbation theory of invariant manifolds. (See, for instance, [4].)

First, we study some geometrical properties of small functions defined on $R(\alpha)$ (R stands for range). The discussion that follows until and including Lemma 8 concerns those properties. Let $\delta_0 > 0$ be such that $B = \{x \in R^n: \|x - y\| \leq \delta_0, \text{ for some } y \in R(\alpha)\}$ is contained in U , and let $M > 1$ be an upper bound for the norm of the matrix $((\partial \alpha_i / \partial x_j)(x_{i,j}))$ where $x_{i,j} \in B$, $i, j = 1, \dots, n$. Choose $\varepsilon > 0$ such that $a = 1 - \varepsilon(2 + M) > 0$, and $\delta_1 > 0$ such that if $x \in R(\alpha)$ and $\|x_{i,j} - x\| \leq \delta_1$, $i, j = 1, \dots, n$, then $\|\alpha'_x - ((\partial \alpha_i / \partial x_j)(x_{i,j}))\| \leq \varepsilon$. Also, take $\delta_2 = \delta_1/4$ and δ_3 such that $M\delta_3 < \delta_2/3$.

Lemma 5. *Let $f: R(\alpha) \rightarrow R^n$ be a continuous function such that $\|f\| = \sup_{x \in R(\alpha)} \|f(x)\| \leq \delta_3$ and that for $x, y \in R(\alpha)$ and $\|x - y\| \leq \delta_2$, $\|f(x) - f(y)\| \leq \varepsilon\|x - y\|$. Then the function $H_f: R(\alpha) \rightarrow R(\alpha)$, $H_f(x) = \alpha(x + f(x))$ is a homeomorphism onto $R(\alpha)$ and $\|H_f(x) - H_f(y)\| \geq a\|x - y\|$ for $x, y \in R(\alpha)$ and $\|x - y\| \leq \delta_2$.*

Proof. We have that

$$H_f(x) - H_f(y) = \alpha(x + f(x)) - \alpha(y + f(y)) = \alpha'_*(x - y + f(x) - f(y)),$$

where $(\alpha'_* = (\partial \alpha_i / \partial x_j)(x_{i,j}))$ for some $x_{i,j} \in U$, $\|x_{i,j} - x\| \leq \delta_1$, $i, j = 1, \dots, n$. On the other hand, since $\alpha(x) - \alpha(y) = x - y$, we have that $\alpha'_{**}(x - y) = x - y$ where $\|\alpha'_{**} - \alpha'_x\| \leq \varepsilon$. Thus $\alpha'_x(x - y) = x - y + (\alpha'_x - \alpha'_{**})(x - y)$, and therefore

$$\begin{aligned} H_f(x) - H_f(y) &= \alpha'_x(x - y) + (\alpha'_* - \alpha'_x)(x - y) + \alpha'_*(f(x) - f(y)) \\ &= x - y + (\alpha'_x - \alpha'_{**})(x - y) + (\alpha'_* - \alpha'_x)(x - y) + \alpha'_*(f(x) - f(y)), \end{aligned}$$

and

$$\begin{aligned}\|H_f(x) - H_f(y)\| &\geq \|x - y\| - \varepsilon\|x - y\| - \varepsilon\|x - y\| - M\varepsilon\|x - y\| \\ &= a\|x - y\|.\end{aligned}$$

To prove that H_f is a homeomorphism onto the connected manifold $R(\alpha)$ it is enough to notice that, if $z \in R(\alpha)$, $\|H_f(z) - z\| = \|\alpha(z + f(z)) - z\| \leq M\delta_3 < \delta_2$. Hence if $x, y \in R(\alpha)$ and $\|x - y\| \geq \delta_2$, $H_f(x) \neq H_f(y)$.

Lemma 6. *Let f be as in the previous lemma and set $K_f = H_f^{-1}$. Then for every $x, y \in R(\alpha)$, $\|x - y\| \leq \delta_2$, we have that $\|K_f(x) - K_f(y)\| \leq N\|x - y\|$, where $N = \max(1/a, 2\delta_2/\delta_3)$.*

Proof. Let $\|x - y\| \leq \delta_3$; we have that $\|x - y\| = \|H_f(K_f(x)) - H_f(K_f(y))\| \geq a\|K_f(x) - K_f(y)\|$ since $\|K_f(x) - K_f(y)\| \leq \|K_f(x) - x\| + \|x - y\| + \|K_f(y) - y\| \leq 2M\delta_3 + \delta_3 < \delta_2$. Therefore $\|K_f(x) - K_f(y)\| \leq (1/a)\|x - y\|$.

Assume now that $\|x - y\| \leq \delta_2$ and $\|x - y\| \geq \delta_3$. Then

$$\frac{\|K_f(x) - K_f(y)\|}{\|x - y\|} \leq \frac{\delta_2 + \frac{2}{3}\delta_2}{\delta_3} < 2\frac{\delta_2}{\delta_3},$$

which completes the proof.

Let $P = \max((N + 3), N + \varepsilon N + 1)$ and define the function p by $p(f) = \psi$, where $\psi(x) = K_f(x) + f(K_f(x)) - x$ for $x \in R(\alpha)$.

Lemma 7. *Let f be as in Lemma 5. Then $\|\psi\| < \delta_2$ and $\|\psi(x) - \psi(y)\| \leq P\|x - y\|$ for $x, y \in R(\alpha)$ and $\|x - y\| \leq \delta_2$.*

Proof. $\|\psi(x)\| \leq \|K_f(x) - x\| + \|f(K_f(x))\| \leq M\delta_3 + \delta_3 < \delta_2$. If $\|x - y\| \leq \delta_3$,

$$\|K_f(x) - K_f(y) + f(K_f(x)) - f(K_f(y)) + y - x\| \leq (N + \varepsilon N + 1)\|x - y\|.$$

If $\|x - y\| \leq \delta_2$, and $\|x - y\| \geq \delta_3$,

$$\|\psi(x) - \psi(y)\|/\|x - y\| \leq N + 2\delta_3/\delta_3 + 1 = N + 3.$$

Hence, for $\|x - y\| \leq \delta_2$, we have that $\|\psi(x) - \psi(y)\| \leq P\|x - y\|$.

Lemma 8. *Let f, g satisfy the assumptions of Lemma 5. Then*

$$\|p(f) - p(g)\| \leq (M/a + \varepsilon M/a + 1)\|x - y\|.$$

Proof. Put $\psi = p(f)$, $\phi = p(g)$. We have that

$$\begin{aligned}\psi(x) - \phi(x) &= K_f(x) + f(K_f(x)) - K_g(x) - g(K_g(x)) \\ &= K_f(x) - K_g(x) + f(K_f(x)) - f(K_g(x)) + (f - g)(K_g(x)),\end{aligned}$$

which implies

$$\|\psi(x) - \phi(x)\| \leq \|K_f(x) - K_g(x)\| + \varepsilon \|K_f(x) - K_g(x)\| + \|f - g\|.$$

On the other hand, from $H_f(K_f(x)) - H_g(K_f(x)) + H_g(K_f(x)) - H_g(K_g(x)) = 0$, we get that

$$\begin{aligned} a\|K_f(x) - K_g(x)\| &\leq \|H_g(K_f(x)) - H_g(K_g(x))\| \\ &= \|H_f(K_f(x)) - H_g(K_g(x))\| \leq M\|f(K_f(x)) - g(K_f(x))\| \leq M\|f - g\|. \end{aligned}$$

Consequently, $\|K_f(x) - K_g(x)\| \leq (M/a)\|f - g\|$, and we obtain finally that

$$\|\psi(x) - \phi(x)\| \leq (M/a + \varepsilon M/a + 1)\|f - g\|.$$

Let $\mathfrak{D}$ be the space of the continuous functions $\psi: R(\alpha) \rightarrow R^n$ such that $\alpha(x + \psi(x)) = x$, $x \in R(\alpha)$; $\sup_{x \in R(\alpha)} \|\psi(x)\| \leq \delta_0$, and $\|\psi(x) - \psi(y)\| \leq P\|x - y\|$ for $x, y \in R(\alpha)$, $\|x - y\| \leq \delta_2$. If we define, for $\psi, \phi \in \mathfrak{D}$, $d(\psi, \phi) = \sup_{x \in R(\alpha)} \|\psi(x) - \phi(x)\|$, $(\mathfrak{D}, d)$ becomes a compact metric space. Let $w: U \rightarrow U$ be a continuous function such that

$$(*) \quad \sup_{x \in U} \|(w - \alpha)(x)\| \leq \delta_3,$$

and

$$\|(w - \alpha)(x) - (w - \alpha)(y)\| \leq Q\|x - y\| \quad \text{for } x, y \in U$$

where $Q > 0$, and $Q(1 + P) < \varepsilon$, $Q(M/a + \varepsilon M/a + 1) < 1$.

For $\psi \in \mathfrak{D}$ define $\Omega(\psi) = f$, where $f(x) = w(x + \psi(x)) - x$, $x \in R(\alpha)$. Now we can state

Theorem 5. $p \circ \Omega$ is a contraction of $\mathfrak{D}$.

Proof. First we show that $P \circ \Omega$ is a mapping from $\mathfrak{D}$ to $\mathfrak{D}$. For $x \in R(\alpha)$ we have that $w(x + \psi(x)) - x = w(x + \psi(x)) - \alpha(x + \psi(x))$. Thus, $\|f\| = \|(w - \alpha) \cdot (x + \psi(x))\| \leq \delta_3$.

On the other hand, if $\|x - y\| \leq \delta_2$,

$$\begin{aligned} \|f(x) - f(y)\| &= \|(w - \alpha)(x + \psi(x)) - (w - \alpha)(y + \psi(y))\| \\ &\leq Q(\|x - y\| + \|\psi(x) - \psi(y)\|) \leq \varepsilon\|x - y\|. \end{aligned}$$

Hence f satisfies the assumptions of Lemma 5, and since $\alpha(x + p(f)(x)) = \alpha(K_f(x) + f(K_f(x))) = H_f(K_f(x)) = x$, it follows that $p(f) \in \mathfrak{D}$.

Next we prove that $p \circ \Omega$ is a contraction. Let ψ, ϕ belong to $\mathfrak{D}$; then we may apply Lemma 8 to $f = \Omega(\psi)$ and $g = \Omega(\phi)$ to get

$$d(p \circ \Omega(\psi), p \circ \Omega(\phi)) \leq (M/a + \varepsilon M/a + 1) \|f - g\|.$$

But

$$\begin{aligned} \|f - g\| &= \|w \circ (\alpha + \psi) - w \circ (\alpha + \phi)\| \\ &= \|(w - \alpha) \circ (\alpha + \psi) - (w - \alpha) \circ (\alpha + \phi)\| \leq Qd(\psi, \phi). \end{aligned}$$

Consequently if w satisfies the conditions (*), $p \circ \Omega$ is a contraction of $\mathfrak{O}$.

Corollary 3. *If w satisfies the conditions (*), there exists a manifold V homeomorphic to $R(\alpha)$ that is invariant under w .*

Proof. Let ψ be the fixed point of $p \circ \Omega$, and take $V = \{z \in U : z = x + \psi(x) \text{ for some } x \in R(\alpha)\}$. The function $x \rightarrow x + \psi(x)$ is a homeomorphism of $R(\alpha)$ onto V since $x + \psi(x) = y + \psi(y)$ implies $x = \alpha(x + \psi(x)) = \alpha(y + \psi(y)) = y$. Now, let us show that $w(x + \psi(x)) \in V$ for every $x \in R(\alpha)$. Let $w(x + \psi(x)) - x$ be denoted again by f ; since $\psi = p(f)$ we know that $\psi(H_f(x)) = K_f(H_f(x)) + f(K_f(H_f(x))) - H_f(x)$, i.e. $H_f(x) + \psi(H_f(x)) = x + f(x) = w(x + \psi(x))$. Since $H_f(x) + \psi(H_f(x)) \in V$, this completes the proof.

On account of the compactness of $\mathfrak{O}$ it is easy to conclude that for any $\delta > 0$ there is a positive integer n such that the n -iterate of $p \circ \Omega$ applied to any $\phi \in \mathfrak{O}$ is δ -close to the fixed point ψ of $p \circ \Omega$. Since, on the other hand, for each y near V there exists $\phi \in \mathfrak{O}$ such that $y = x + \phi(x)$ for some $x \in R(\alpha)$, we obtain that V is an attractor (as defined in the introduction) of the discrete flow $\Phi(x, n) = w^n(x)$, $x \in U$, $n = 1, 2, \dots$. Consequently if D is an invariant set of w lying near V and $w(D) = D$, then $D \subset V$. In particular, if w is a homeomorphism and D is a compact connected manifold of the dimension of V , then $D = V$. Hence, if $\pi: U \rightarrow U$ is a homeomorphism close to the identity that commutes with w , we obtain that V is also invariant under π since $\pi(V)$ is a compact connected manifold and $w(\pi(V)) = \pi(w(V)) = \pi(V)$. These remarks together with previous results imply readily the following corollary.

Corollary 4. *There is a $C^1(U)$ neighbourhood of X such that if Y belongs to it, the flow defined by Y has an attractor $V \subset U$ homeomorphic to A .*

Perhaps it is worthwhile to say that as a consequence of the implicit function theorem it may be shown that there is a compact neighbourhood of $\psi = 0$ in $\mathfrak{O}$ with a convex structure. The existence of invariant manifolds may also be obtained as an application of Schauder's fixed point theorem.

If instead of $\mathfrak{O}$ we consider a space $\mathfrak{O}_1$ of continuous functions $\psi: R(\alpha) \rightarrow R^n$, $\alpha(x + \psi(x)) = x$, that have continuous partial derivatives satisfying a uniform Lipschitz condition, we get with similar arguments the existence of an invariant C^1 manifold for any function $w: U \rightarrow U$, $C^1(U)$ close to α and such that $(w - \alpha)'$ has a small Lipschitz constant on U .

7. Consider again the space $\mathfrak{D}$. It is clear from its definition that there exists $\bar{P} \geq P$ such that $\|\psi(x) - \psi(y)\| \leq \bar{P}\|x - y\|$ for each $x, y \in R(\alpha)$ and each $\psi \in \mathfrak{D}$. Let $L > 0$ be such that

$$\|(\alpha + \psi \circ \alpha)(x) - (\alpha + \psi \circ \alpha)(y)\| \leq L\|x - y\|$$

for each $x, y \in U$ and each $\psi \in \mathfrak{D}$. We want to show that if $w: U \rightarrow U$ is a homeomorphism satisfying $(*)$ with a Q also less than $[2(\bar{P} + 1)(L + 1)]^{-1}$, then there is a retraction of U onto V (the invariant manifold of w) that commutes with w .

Let $\psi: R(\alpha) \rightarrow R^n$, $\alpha(x + \psi(x)) = x$, be such that $V = \{x + \psi(x): x \in R(\alpha)\}$, and define $\beta: U \rightarrow U$ by $\beta(x) = \alpha(x) + (\psi \circ \alpha)(x)$. Then β is a retraction, for $\beta \circ \beta = \alpha \circ \beta + \psi \circ \alpha \circ \beta = \beta$ since $\alpha \circ \beta = \alpha$. We also have from the invariance of $R(\beta) = V$ under w that $\beta \circ w \circ \beta = w \circ \beta$.

Let $\mathcal{L}$ be the space of continuous functions $\lambda: U \rightarrow R^n$ such that $\sup_{x \in U, x \neq \beta(x)} \|\lambda(x)\|/\|x - \beta(x)\| < \infty$, $\lambda \circ \beta = 0$, and $\beta \circ (\beta + \lambda) = \beta + \lambda$. For $\lambda, \mu \in \mathcal{L}$, set

$$d(\lambda, \mu) = \sup_{x \in U, x \neq \beta(x)} \|(\lambda - \mu)(x)\|/\|x - \beta(x)\|.$$

Then, with d as the distance, $\mathcal{L}$ becomes a complete metric space. Let w satisfy $(*)$ with such a Q that $2Q(\bar{P} + 1)(L + 1) < 1$ and for $\lambda \in \mathcal{L}$, set $O(\lambda) = w^{-1} \circ (\beta + \lambda) \circ w - \beta$. Since $(\beta + \lambda) \circ w \circ \beta = w \circ \beta$ we have that $O(\lambda) = 0$; we also have that $\beta \circ (\beta + O(\lambda)) = \beta + O(\lambda)$ as may be verified immediately.

Theorem 6. O is a contraction of $\mathcal{L}$.

Proof. If $x, y \in R(\beta)$ we have $x - y = \alpha(x) - \alpha(y) + \psi(\alpha(x)) - \psi(\alpha(y))$, which implies $\|x - y\| \leq \|\alpha(x) - \alpha(y)\| + \bar{P}\|\alpha(x) - \alpha(y)\|$ and $\|\alpha(x) - \alpha(y)\| \geq (1 + \bar{P})^{-1}\|x - y\|$. Then

$$\begin{aligned} \|w(x) - w(y)\| &\geq \|\alpha(x) - \alpha(y)\| - \|(w - \alpha)(x) - (w - \alpha)(y)\| \\ &\geq \left(\frac{1}{\bar{P} + 1} - \frac{1}{2(\bar{P} + 1)} \right) \|x - y\| = \frac{1}{2(\bar{P} + 1)} \|x - y\|. \end{aligned}$$

Therefore, since $x - y = w(w^{-1}(x)) - w(w^{-1}(y))$, we obtain that for $x, y \in R(\beta)$, $\|w^{-1}(x) - w^{-1}(y)\| \leq 2(\bar{P} + 1)\|x - y\|$.

Let $\lambda, \mu \in \mathcal{L}$. From $O(\lambda) - O(\mu) = w^{-1} \circ (\beta + \lambda) \circ w - w^{-1} \circ (\beta + \mu) \circ w$, we get that

$$\|O(\lambda)(x) - O(\mu)(x)\| \leq 2(\bar{P} + 1)\|\lambda(w(x)) - \mu(w(x))\|,$$

or

$$\|O(\lambda)(x) - O(\mu)(x)\| \leq 2(\bar{P} + 1)\|(\lambda - \mu)(w(x))\|.$$

Consequently

$$\begin{aligned}
 \|O(\lambda)(x) - O(\mu)(x)\| &\leq 2(\bar{P} + 1)d(\lambda, \mu)\|w(x) - \beta(w(x))\| \\
 &\leq 2(\bar{P} + 1)d(\lambda, \mu)(\|w(x) - w(\beta(x))\| + \|\beta(w(\beta(x))) - \beta(w(x))\|) \\
 &\leq 2(\bar{P} + 1)(L + 1)d(\lambda, \mu)\|w(x) - w(\beta(x))\| \\
 &= 2(\bar{P} + 1)(L + 1)d(\lambda, \mu)\|(w - \alpha)x - (w - \alpha)(\beta(x))\|.
 \end{aligned}$$

Thus, $d(O(\lambda), O(\mu)) \leq 2(\bar{P} + 1)(L + 1)Qd(\lambda, \mu)$, which implies the thesis of the theorem.

If λ is the fixed point of O , we have that $(\beta + \lambda)w = w(\beta + \lambda)$ and as $(\beta + \lambda) \circ (\beta + \lambda) = \beta \circ (\beta + \lambda) + \lambda \circ (\beta + \lambda) = (\beta + \lambda) + \lambda \circ \beta \circ (\beta + \lambda) = \beta + \lambda$, $\beta + \lambda$ is a retraction of U onto V .

Corollary 5. *If Y is a vector field close enough to X in the $C^1(U)$ topology, there is a retraction γ of U onto a manifold V that commutes with Y .*

Proof. Call Ψ the flow determined by Y , and let $T > 0$ be such that $w = \Psi_T$ satisfies the assumptions of the previous theorem. If γ is the retraction that commutes with w , then $\Psi_{-t}\gamma\Psi_t - \beta \in \mathcal{L}$ for small $|t|$, since, as we have already shown, $R(\gamma)$ is invariant under Ψ_t . But $w^{-1} \circ \Psi_{-t} \circ \gamma \circ \Psi_t \circ w = \Psi_{-t} \circ \gamma \circ \Psi_t$ and on account of the uniqueness of the fixed point of O , we must have $\Psi_{-t} \circ \gamma \circ \Psi_t = \gamma$ as had to be shown.

The uniqueness of γ as a retraction close to α that commutes with Y follows at once from the uniqueness of the manifold $R(\gamma)$.

As before, the differentiable case may be handled in a very similar way. However, the space $\mathcal{L}_1$ to be considered in that case should consist of the C^1 -functions $\lambda, \lambda \circ \beta = 0, \beta \circ (\beta + \lambda) = \beta + \lambda$, such that

$$\sup_{x \in U, x \neq \beta(x)} \frac{|\lambda'(x)|}{\|x - \beta(x)\|} < \infty,$$

where $|\lambda'(x)| = \sup_{v \in \beta'(x)v, \|v\|=1} \|\lambda'(x)v\|$ and the metric d for $\mathcal{L}_1$ should be $d = d_1 + d_2$, where $d_1(\lambda, \mu) = \sup_{x \in U} \|(\lambda - \mu)(x)\| + \sup_{x \in U} \|(\lambda - \mu)'(x)\|$, and

$$d_2(\lambda, \mu) = \sup_{x \in U, x \neq \beta(x)} \frac{|(\lambda - \mu)'(x)|}{\|x - \beta(x)\|}.$$

Remark. A generic periodic saddle point of an analytic vector field X need not have C^2 -retractions that commute with X . If such a retraction exists it is possible to define in a neighbourhood of the periodic motion a closed C^2 1-form w , such that $w(X) = 1$. Indeed, if r is the retraction onto the closed orbit p , consider on p the form dt , $dt(X) = 1$; then $w = r^*dt$ is obviously a closed form and satisfies $w(X) = dt(r'(X)) = 1$. Consider in $R^2 \times S^1$ the flow given by $\varphi(x, y, e^{i\theta}, t)$

$= (xe^{-t}, ye^{t/n}, e^{i\theta} e^{2\pi i t})$, where n is a positive integer. The retraction $r, r(x, y, e^{i\theta}) = (0, 0, e^{i\theta})$, commutes with the flow, and hence there exists a closed 1-form w , analytic in a neighbourhood of $p = \{(0, 0)\} \times S^1$ such that $w(X) = 1$, X being the vector field defined by the flow. Let $f(x, y, e^{i\theta}) = 1 + axy^n$, $a > 0$, and consider the flow associated to the vector field fX that has a generic periodic saddle point at $(0, 0, 1)$. If there were a closed 1-form w^* such that $w^*(fX) = 1$, then, since $\int_p w^* = \int_p w = 1$, we would have $w^* = w + dh$, h being a C^2 real function defined in a neighbourhood of p . From $1 = w^*(fX) = f(1 + X(h))$ we get that $X(h) = -axy^n(1 + axy^n)^{-1}$. Therefore

$$h(x/e_{n+1}, ye^{1/n}, 1) - h(x, y, 1) = -axy^n(1 + axy^n)^{-1};$$

however, if we take $\partial^{n+1}/\partial x \partial y^n$ on both sides and evaluate at $(0, 0)$, we get $0 = -a$. Thus, if we take $n = 1$, fX has no C^2 retraction commuting with it.

This fact may be used to show that there are C -generic attractors A of analytic vector fields X such that we can find a complex neighbourhood U of A and a sequence $\{X_n\}$ of analytic vector fields that converges uniformly to X on U , and so that there is no analytic retraction onto the invariant manifold A_n of X_n that commutes with X_n . (The periodic motion of the above example may lay on A_n .)

REFERENCES

1. J. L. Massera, *On Liapounoff's conditions of stability*, Ann. of Math. (2) **50** (1949), 705–721. MR **11**, 721.
2. J. W. Milnor, *Topology from the differentiable viewpoint*, University Press of Virginia, Charlottesville, Va., 1965. MR **37** #2239.
3. D. Montgomery and L. Zippin, *Topological transformation groups*, Interscience, New York, 1955. MR **17**, 383.
4. R. Sacker, *A new approach to the perturbation theory of invariant surfaces*, Comm. Pure Appl. Math. **18** (1965), 717–732. MR **32** #6003.
5. S. Smale, *Stable manifolds for differential equations and diffeomorphisms*, Ann. Scuola Norm. Sup. Pisa (3) **17** (1963), 97–116. MR **29** #2818b.

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