EXTENDING CELL-LIKE MAPS ON MANIFOLDS

BY

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ABSTRACT. Let X be a closed subset of a manifold M and G_0 be a cell-like upper semicontinuous decomposition of X . We consider the problem of extending G_0 to a cell-like upper semicontinuous decomposition G of M such that $M/G \approx M$. Under fairly weak restrictions (which vanish if $M = E^n$ or S^n and $n \neq 4$) we show that such a G exists if and only if the trivial extension of G_0 , obtained by adjoining to G_0 the singletons of $M - X$, has the desired property. In particular, the nondegenerate elements of Bing's dogbone decomposition of E^3 are not elements of any cell-like upper semicontinuous decomposition G of E^3 such that $E^3/G \approx E^3$. Call a cell-like upper semicontinuous decomposition G of a metric space X *simple* if $X/G \approx X$ and say that the closed set Y is *simply embedded* in X if each simple decomposition of Y extends trivially to a simple decomposition of X . We show that tame manifolds in E^3 are simply embedded and, with some additional restrictions, obtain a similar result for a locally flat k -manifold in an m -manifold ($k, m \neq 4$). Examples are given of an everywhere wild simply embedded simple closed curve in E^3 and of a compact absolute retract which embeds in E^3 yet has no simple embedding in E^3 .

1. Introduction. We will be concerned with variations of the following general problem: If X is a closed subset of a manifold M and $f: X \rightarrow Y$ is a proper cell-like map of X onto a metric space Y , under what conditions is it possible to extend f to a proper cell-like map F defined on all of M such that $F(M)$ is homeomorphic to M ? (We do not assume that the boundary of a manifold is necessarily empty.) For 3-manifolds, analogous questions have been considered by R. H. Bing [9] and others (see, for example, [6], [12], [18], [19] and the discussion in §15 of [20]).

Since proper maps and upper semicontinuous decompositions of metric spaces correspond in a natural way, our results have equivalent formulations in terms of mappings and in terms of decompositions. In the latter terminology, the basic question considered becomes: Given a cell-like decomposition G_0 of X , when

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is there a cell-like decomposition G of M such that $G_0 \subset G$ and $M/G \approx M$? Under fairly weak restrictions (which vanish if M is S^n or E^n), we show that if M is an n -manifold, $n \neq 4$, then such a G exists only in case the trivial extension of G_0 , obtained by adjoining to G_0 all singletons in $M - X$, has the desired property. In particular, the set of nondegenerate elements of Bing's dogbone decomposition of E^3 cannot be embedded in any cellular decomposition of E^3 which yields E^3 as its decomposition space, which provides a negative answer to a question raised by Bing in [9].

We say that a cell-like upper semicontinuous decomposition G of a metric space X is *simple* if $X/G \approx X$ and that a closed subset Y of X is *simply embedded* in X if every simple decomposition of Y can be trivially extended to a simple decomposition of X (equivalently, if every proper cell-like map on Y can be extended to a proper cell-like map F on X such that $F(X) \approx X$ and $F|X - Y$ is a homeomorphism onto $F(X) - F(Y)$). We show that every tame manifold in E^3 is simply embedded in E^3 , and obtain similar results, with additional restrictions, for a locally flat k -manifold in an m -manifold ($k, m \neq 4$).

Local flatness is by no means a necessary condition for a manifold in E^n to be simply embedded; we show, in fact, that an arc in E^3 may be simply embedded even though it is wild at each of its points. With respect to more complicated sets, we observe that there exist 1-dimensional continua for which every embedding in E^3 is simple and, at the other extreme, show that a continuum may be embeddable in E^3 and yet have no simple embedding in E^3 . It will be shown in a later paper that every circularly chainable continuum can be simply embedded in E^3 , and the question of which continua have both simple and nonsimple embeddings in E^3 will be considered in greater detail.

2. Definitions and notation. Our usage of *upper semicontinuous (collection or decomposition)* and associated terms is standard (e.g., as in [30] and [2]).

If X and Y are metric spaces, a map $f: X \rightarrow Y$ is said to be *closed* if $f(C)$ is closed in Y for every closed subset C of X , and to be *compact* if $f^{-1}(K)$ is compact for every compact subset K of Y . If f is closed and $f^{-1}(y)$ is compact for each $y \in Y$, then f is said to be *proper*; it readily follows that f is proper if and only if it is compact.

We use the notation $f: X \twoheadrightarrow Y$ to indicate that f maps X onto Y (of course $f: X \rightarrow Y$ does not preclude $f: X \twoheadrightarrow Y$).

It is well known that if X and Y are metric spaces and $f: X \twoheadrightarrow Y$ is a proper map, then the set $G = \{f^{-1}(y) | y \in Y\}$ is an upper semicontinuous decomposition of X and $X/G \approx Y$; conversely, if G is any upper semicontinuous

decomposition of X , then X/G is a metric space and the projection map $P: X \rightarrow X/G$ is proper. Most of our results may be stated, equivalently, in terms of mappings and in terms of decompositions; generally, we give only one version formally, but may later use the alternative formulation without explicit comment.

By an n -manifold we mean a separable metric space each point of which has a neighborhood homeomorphic to a closed n -cell. As usual, if M is a manifold, the set of all points of M which have an open n -cell neighborhood in M is called the *interior* of M and is denoted by $\text{Int } M$; the set $M - \text{Int } M$ is called the *boundary* of M , and will be denoted by ∂M . We do not assume that manifolds are compact or connected, but whenever we speak of a manifold M as being "in" a manifold N , we always assume that M is embedded as a closed subset of N .

A subset X of an n -manifold M is said to be *cellular in M* if X is the intersection of a sequence $\{C_i\}$ of closed n -cells in M , with $C_{i+1} \subset \text{Int } C_i$ for each i . A continuum which is homeomorphic to a cellular subset of some Euclidean space E^n is said to be *cell-like*. A decomposition G is said to be *cell-like (cellular)* if each element of G is cell-like (cellular), and a mapping $f: X \rightarrow Y$ is *cell-like* if $f^{-1}(y)$ is cell-like for each $y \in Y$. (See [17] or [26].)

If U is a subset of a space X , then $\text{Cl } U$ will denote the closure of U in X .

If G is a collection of subsets of X , we use G^* to denote the union of the elements of G , and if $C \subset X$, we let G_C denote the set of all elements of G which intersect C . A subset C of X is said to be *saturated* with respect to G if $C = G_C^*$. If $f: X \rightarrow Y$, then $f(G)$ will denote $\{f(g) | g \in G\}$.

If Y is a closed subset of a metric space X and G is an upper semicontinuous decomposition of Y , then any upper semicontinuous decomposition G' of X such that $G \subset G'$ will be called an *extension* of G . The extension of G obtained by adding to G all singleton subsets of $X - Y$ will be called the *trivial extension* of G and will be said to be *generated by G* . This decomposition of X will be denoted by $\tilde{G}(X)$, or simply $\tilde{G}$ in case confusion is unlikely.

A cell-like upper semicontinuous decomposition G of a metric space X will be said to be *simple* if $X/G \approx X$ and to be *locally simple* if, for each $g \in G$, every neighborhood U of g in X contains a closed neighborhood C of g in X such that G_C generates a simple decomposition of X . (In general, a locally simple decomposition of a metric space X need not be simple, nor need a simple decomposition be locally simple. If X is a manifold, however, the situation is markedly different, as shown below.) A closed subset Y of X is said to be *simply embedded* in X if every simple decomposition of Y generates a simple decomposition of X .

3. Modifying cell-like maps on manifolds. The results of this section depend heavily on Siebenmann's recent theorem [26] on homeomorphic approximations of cell-like maps between m -manifolds (for $m = 3$, this result was proved earlier by Armentrout ([2], [3], [4], [5])). Siebenmann's theorem includes some restrictions which we find it convenient to abbreviate as follows.

Conditions $S(M, f)$ and $S(M, G)$. If M is an m -manifold and $f: M \rightarrow Y$ is a cell-like map of M onto a metric space Y , we will say that condition $S(M, f)$ holds provided that $m < 3$ or

- (1) if $m = 3$, then for each $y \in Y$, $f^{-1}(y)$ has a neighborhood in M which is embeddable in E^3 ,
- (2) if $m = 5$, then $f|_{\partial M}$ is bijective,
- (3) if $m = 4$ or $m > 5$, then for each $y \in Y$, $f^{-1}(y) \cap \partial M$ is either empty or cell-like.

If G is an arbitrary collection of cell-like subsets of M , then $S(M, G)$ is defined analogously (i.e., by replacing $f^{-1}(y)$ in (1) and (3) by an arbitrary $g \in G$ and requiring in (2) that no nondegenerate element of G intersect ∂M).

The following lemma provides a slight additional simplification of the statement of Siebenmann's theorem. That the first part of this lemma holds was indicated in §3 of [26], where it is also shown that the second part fails for $m \geq 4$.

3.1. Lemma. *If M and N are m -manifolds and $f: M \rightarrow N$ is a proper cell-like map, then $f(\partial M) = \partial N$ and, if $m \leq 3$, $f|_{\partial M}: \partial M \rightarrow \partial N$ is cell-like.*

Proof. Suppose $x \in \partial M$. If $f(x) \in \text{Int } N$, there is an open subset U of N such that $f(x) \in U$ and $U \approx E^m$. Then $f^{-1}(U)$ is an m -manifold with nonempty boundary, and since $f|_{f^{-1}(U)}: f^{-1}(U) \rightarrow U$ is a proper homotopy equivalence [17], it follows that $H_0(f^{-1}(U), \partial f^{-1}(U)) \approx H_C^m(f^{-1}(U)) \approx H_C^m(U) \approx H_C^m(E^m) = \mathbb{Z}$. (Here H_i denotes homology with integral coefficients and H_C^i cohomology with integral coefficients and compact supports.) But this is a contradiction since $f^{-1}(U)$ is path connected and $\partial f^{-1}(U) \neq \emptyset$. It follows that $f(\partial M) \subset \partial N$; a similar argument shows that if $y \in \partial N$, then $f^{-1}(y) \cap \partial M \neq \emptyset$, and hence $f(\partial M) = \partial N$.

If $m < 3$, it is easy to show directly that $f|_{\partial M}$ is monotone and therefore cell-like, so suppose $m = 3$. If $y \in \partial N$ and W is an open subset of ∂M containing $f^{-1}(y) \cap \partial M$, there is an open subset V of N containing y such that $V \approx E_+^3$ and $f^{-1}(V) \cap \partial M \subset W$. Then, as above, $H_2(f^{-1}(V), \partial f^{-1}(V)) \approx H_C^1(E_+^3) = 0$ and hence $H_1(\partial f^{-1}(V)) = 0$. It follows that $\partial f^{-1}(V)$ is an open 2-cell neighborhood of $f^{-1}(y) \cap \partial M$ lying in W ; since W was arbitrary, this implies that $f^{-1}(y) \cap \partial M$ is cellular in ∂M .

Theorem A (Siebenmann). *Suppose M and N are m -manifolds, $m \neq 4$, and $f: M \rightarrow N$ is a proper cell-like map such that $S(M, f)$ holds. If $\epsilon: M \rightarrow (0, \infty)$ is*

continuous, there is a homeomorphism $g: M \rightarrow N$ such that $d(f(x), g(x)) < \epsilon(x)$ for all $x \in M$.

3.2 Theorem. Suppose M and N are m -manifolds, $m \neq 4$, and $f: M \rightarrow N$ is a proper cell-like map such that $S(M, f)$ holds. If C is a closed subset of M such that $f^{-1}f(C) = C$, then $f|_C$ can be extended to a proper map $F: M \rightarrow N$ such that $F|_{M-C}$ is a homeomorphism onto $N - f(C)$.

Proof. Let ρ and d be metrics for M and N , respectively, and let $\epsilon(x) = \rho(x, C)$ for each $x \in M$. Applying Theorem A to $M - C$, $N - f(C)$, $f|_{M-C}$ and $\epsilon|_{M-C}$ yields a homeomorphism $g: M - C \rightarrow N - f(C)$ such that $d(f(x), g(x)) < \epsilon(x)$ for every $x \in M - C$. Define a function F on M by $F(x) = f(x)$ if $x \in C$, $F(x) = g(x)$ if $x \in M - C$. It is easily seen that F is continuous and, of course, $F(M) = N$ and $F|_{M-C}$ is a homeomorphism onto $N - f(C)$.

That F is proper follows directly from a result of Väisälä [28] in case $\partial M = \emptyset$; if $\partial M \neq \emptyset$, F can be shown to be proper by applying Väisälä's result to a manifold obtained by adjoining to M a homeomorphic copy of M which intersects M precisely in ∂M (and extending F to the resulting manifold in a natural way). Alternatively, it is easy to show directly that F is proper if the function $\epsilon(x)$ is modified so as to guarantee not only that $\epsilon(x_i) \rightarrow 0$ if $\{x_i\} \rightarrow x_0 \in C$, but also that $\epsilon(x_i) \rightarrow 0$ if $\{x_i\}$ has no accumulation point in M . (These properties for $\epsilon(x)$ may be realized easily by choosing ρ to be the metric obtained by restricting to M a metric ρ' for the one-point compactification $M + \infty$ of M , and defining $\epsilon(x)$ to be $\rho(x, C \cup \{\infty\})$. That $M + \infty$ is metrizable follows [14, p. 247] from the fact that M is locally compact and separable.)

3.3 Theorem. Suppose M is an m -manifold, $m \neq 4$, and $f: M \rightarrow Y$ is a proper cell-like map of M onto a metric space Y such that $S(M, f)$ holds. If for each $y \in Y$ there exist a closed neighborhood C_y of $f^{-1}(y)$ in M and a proper map F_y of M onto an m -manifold such that $\{f^{-1}f(x) | x \in C_y\} \subset \{F_y^{-1}F_y(x) | x \in M\}$, then Y is homeomorphic to M .

Proof. In view of Theorem A, it is only necessary to show that Y is an m -manifold.

Suppose $y_0 \in Y$. By hypothesis, there exist a closed neighborhood C of $f^{-1}(y_0)$ in M and a proper map F of M onto an m -manifold N such that $\{f^{-1}f(x) | x \in C\} \subset \{F^{-1}F(x) | x \in M\}$. Let $U = \{x \in C | f^{-1}f(x) \cap C \setminus (M - C) = \emptyset\}$. Since f is proper, it easily follows that U is open in M and that $f^{-1}f(U) = U$. Since C is a neighborhood of $f^{-1}(y)$ in M , $f^{-1}(y) \subset U$. Since f is proper, U is open in M , and $f^{-1}f(U) = U$, $f(U)$ is open in Y . The condition $\{f^{-1}f(x) | x \in C\} \subset \{F^{-1}F(x) | x \in M\}$ implies that also $F^{-1}F(U) = U$ and hence $F(U)$ is

open in N . Since $f|U$ and $F|U$ are both proper maps, it is easy to show that the function ϕ defined on $f(U)$ by $\phi(y) = F(f^{-1}(y))$ is a homeomorphism of $f(U)$ onto $F(U)$. Hence $f(U)$ is an open neighborhood of y_0 in Y which is homeomorphic to an open subset of N . It follows that Y is an m -manifold, and hence $Y \approx M$.

3.4 Corollary. *If M is an m -manifold, $m \neq 4$, and G is an upper semicontinuous cell-like decomposition of M such that $S(M, G)$ holds and such that for each $g \in G$ there is a closed neighborhood C of g in M such that G_C generates a simple decomposition of M , then G is a simple decomposition of M .*

Combining Theorem 3.2 and Corollary 3.4 yields the following result.

3.5 Corollary. *If M is an m -manifold, $m \neq 4$, and G is a cell-like upper semicontinuous decomposition of M such that $S(M, G)$ holds, then G is simple if and only if it is locally simple.*

Remark. It was shown by R. H. Bing [9] that if Y is a closed subset of E^3 (or S^3) such that each component of Y is compact and does not separate E^3 , and G is the collection of all components of Y , then there is a monotone upper semicontinuous decomposition G' of E^3 such that $G \subset G'$ and $E^3/G' \approx E^3$. Bing asked [9, p. 364] whether G' could be chosen to be a cellular decomposition in case each element of G is cellular in E^3 . Theorem 3.2 implies that a cell-like upper semicontinuous decomposition G of a closed subset of E^3 can be extended to a cell-like upper semicontinuous decomposition G' of E^3 such that $E^3/G' \approx E^3$ if and only if the trivial extension of G has this property. Hence, in particular, the set of nondegenerate elements of Bing's dogbone decomposition [8] cannot be extended to any cell-like decomposition of E^3 which yields E^3 as its decomposition space. Moreover, if G is any cell-like decomposition of E^3 such that $E^3/G \not\approx E^3$, then by Theorem 3.3 there is an element g_0 of G such that if C is any closed neighborhood of g_0 in E^3 , then the set of elements of G which intersect C is not a subset of any upper semicontinuous decomposition G' of E^3 —not necessarily cell-like or even monotone—such that E^3/G' is a 3-manifold.

We also note that if, in Theorem 3.2, $M = N$ and C does not intersect any nondegenerate $f^{-1}(y)$, $y \in M$, then $f|C: C \rightarrow M$ is topologically equivalent to the inclusion $i: C \rightarrow M$. Thus Theorem 3.2 provides a generalization of the results of §5 of [3]; in particular, the "repairing" of embeddings in the sense of [6] and [12] cannot be carried out within the category of cell-like maps.

3.6 Lemma. *If K is a k -manifold, $k \neq 4$, and $f: K \rightarrow K$ is a proper cell-like map such that $S(K, f)$ holds, then for each $y \in K$, there is a locally flat k -cell*

D in K such that D is a neighborhood of $f^{-1}(y)$ in K and (1) $D \cap \partial K = \emptyset$ if $f^{-1}(y) \cap \partial K = \emptyset$, (2) $D \cap \partial K$ is a locally flat $(k-1)$ -cell in ∂D if $f^{-1}(y) \cap \partial K \neq \emptyset$.

Proof. We first observe that if U is an open subset of K , then $f^{-1}(U)$, U and $f|f^{-1}(U)$ satisfy the hypothesis of Theorem A and hence $f^{-1}(U) \approx U$ (cf. [5, Theorem 4]). Hence if $y \in \text{Int } K$, $f^{-1}(y)$ has an open neighborhood in K which is homeomorphic to E^k , and the existence of a locally flat k -cell D in $\text{Int } K$ such that $f^{-1}(y) \subset \text{Int } D$ follows immediately. On the other hand, if $y \in \partial K$, then $f^{-1}(y)$ has an open neighborhood V in K such that $(V, \partial V) \approx (E_+^k, E^{k-1})$ and the existence of the desired k -cell D again follows easily.

Haver [16] has proved that if M is an m -cell, $m \neq 5$, then every cell-like map $f: \partial M \rightarrow \partial M$ can be extended to a map $F: M \rightarrow M$ such that $F| \text{Int } M$ is a homeomorphism onto $\text{Int } M$. In the next lemma we give a similar result in a more general setting.

3.7 Lemma. Suppose K is a locally flat k -cell in the interior of an m -cell M , and assume that $k \neq 4$ unless $k = m = 4$. Then every cell-like map $f: K \rightarrow K$ for which $S(K, f)$ holds can be extended to a map $F: M \rightarrow M$ such that $F| M - K$ is a homeomorphism onto $M - K$.

Proof. Suppose first that $k = m$, and let $S = \partial K$; by Lemma 3.1, $f(S) = S$. If $k = 5$, then condition $S(K, f)$ implies that $f|S$ is a homeomorphism, and the existence of F follows trivially. Hence assume $k \neq 5$. Then since $f|S: S \rightarrow S$ is cell-like and $\dim S = k - 1 \neq 4$, it follows from the complement to Theorem A, given in [26], that there is a map $\Phi: S \times I \rightarrow S \times I$ such that for each $t \in [0, 1]$, $\Phi|S \times \{t\}$ is a homeomorphism onto $S \times \{t\}$ and for each $x \in S$, $\Phi(x, 1) = (f(x), 1)$. Let $A = \text{Cl}(M - K)$; since K is locally flat in $\text{Int } M$, $A \approx S \times I$ and we may identify A and $S \times I$, with S corresponding to $S \times \{1\}$. Then $\Phi: A \rightarrow M$, $\Phi|S = f|S$ and $\Phi|A - S$ is a homeomorphism onto $A - S$. If $F: M \rightarrow M$ is defined by $F(x) = f(x)$ if $x \in K$, $F(x) = \Phi(x)$ if $x \in A$, then F has the desired properties.

Suppose now that $k < m$. Since K is locally flat in $\text{Int } M$, it may be assumed that M is the standard m -ball B^m in E^m and that $K \subset \text{Int } B^k$, where $B^k = \{(x_1, \dots, x_m) \in B^m | x_i = 0 \text{ for } i > k\}$.

Let $U = \text{Int } B^m$ and $V = \text{Int } B^k$. It follows from the first case considered that there is a map $F_1: V \rightarrow V$ such that $F_1|K = f$ and $F_1|V - K$ is a homeomorphism onto $V - K$. Since $(U, V) \approx (E^m, E^k)$, it follows from the Addendum to Corollary 4 of [25] that there is a map $F_2: U \rightarrow U$ such that $F_2(V) = V$, $\{F_2^{-1}(y) | y \in V\} = \{F_1^{-1}(x) | x \in V\}$ and $F_2|U - K$ is a homeomorphism onto $U - F_2(K)$.

Since F_1 and F_2 are proper maps [28] and $\{F_1^{-1}(x) | x \in V\} = \{F_2^{-1}(y) | y \in V\}$, the function H_1 defined on V by $H_1(y) = F_1(F_2^{-1}(y))$, for each $y \in V$, is a homeomorphism of V onto V and the diagram

is commutative. It is clear that H_1 may be extended to a homeomorphism $H_2: U \rightarrow U$. If $F^* = H_2 \circ F_2$, then $F^*: U \rightarrow U$, $F^*|K = f$ and $F^*|U - K$ is a homeomorphism onto $U - K$.

Let C_1 be a locally flat m -cell in U such that $K \subset \text{Int } C_1$. Let $S_1 = \partial C_1$ and $S_2 = F^*(S_1)$, and let C_2 denote the (locally flat) m -cell in U bounded by S_2 . It is easy to verify that $C_2 = F^*(C_1)$ and that $K \subset \text{Int } C_2$. Since $K \subset \text{Int } C_1 \cap \text{Int } C_2$, there exist homeomorphisms $b_1: M \rightarrow C_1$ and $b_2: M \rightarrow C_2$ such that $b_1|K = b_2|K = \text{id}_K$. If $F = b_2^{-1}F^*b_1$, then $F: M \rightarrow M$, $F|K = f$ and $F|M - K$ is a homeomorphism onto $M - K$, as required.

Remark. It is not difficult to show that if $k < m$, the extension F of Lemma 3.7 may be chosen so as to be the identity on ∂M ; if $k = m$, this is possible if and only if $f| \partial K$ is a map of degree 1. Since an orientation reversing homeomorphism of a closed interval K contained in a half-open interval M cannot be extended homeomorphically to all of M , it is clearly impossible to extend Lemma 3.7 directly to the case of an arbitrary locally flat k -manifold in the interior of an m -manifold. The next theorem shows that such problems can be overcome, in a sense, if one is allowed to change the embedding of K in M .

3.8 Theorem. *If K is a locally flat k -manifold in the interior of an m -manifold M , $k, m \neq 4$, and $f: K \rightarrow K$ is a proper cell-like map such that $S(K, f)$ holds, then there exist a proper map $F: M \rightarrow M$ and a locally flat embedding $b: K \rightarrow M$ such that the diagram*

is commutative, and such that $F|M - K$ is a homeomorphism onto $M - b(K)$.

Since the proof of this result seems more natural in terms of upper semi-continuous decompositions, we give an equivalent formulation of Theorem 3.8 in

these terms before proceeding with the argument.

3.8a Theorem. *If K is a locally flat k -manifold in the interior of an m -manifold M , $k, m \neq 4$, and G_0 is a simple decomposition of K such that $S(K, G_0)$ holds, then G_0 generates a simple decomposition G of M , and K/G_0 is locally flat in M/G .*

Proof. We will show that each $g \in G$ has a closed neighborhood C in M such that G_C generates a simple decomposition of M , and it will follow from Corollary 3.4 that G is a simple decomposition of M . Clearly we need consider only elements of G_0 .

Suppose $g_0 \in G_0$ and let D be a k -cell neighborhood of g_0 in K satisfying the conditions of Lemma 3.6. Since G_0 is upper semicontinuous, there exists an open neighborhood U of g_0 in K such that every element of G_0 which intersects $\text{Cl } U$ lies in $D - \text{Cl}(K - D)$. Let $G_1 = \{g \in G_0 \mid g \cap \text{Cl } U \neq \emptyset\}$ and let $A = G_1^*$. Then A is a closed neighborhood of g_0 in K , $A \subset D - \text{Cl}(K - D)$ and A is saturated with respect to G_0 . Since $g \cap \partial D = g \cap \partial K$ for every $g \in G_1$, condition $S(D, G_1)$ holds since $S(K, G_0)$ does.

Let G_2 denote the decomposition of D generated by G_1 and G_3 the decomposition of K generated by G_1 . Let $P_3: K \rightarrow K/G_3$ be the projection map for G_3 . Since $K/G_0 \approx K$, it follows from Theorem 3.2 that $K/G_3 \approx K$. We wish to show that $P_3(D) \approx D$, and for this it will be sufficient (applying Theorem A to $D, P_3(D), P_3|_D$) to show that $P_3(D)$ is a k -manifold.

If $x \in D - A$ and U is an open subset of K containing x and lying in $K - A$, then since $P_3|_{K-A}$ is a homeomorphism onto $K - A$, $P_3(U) \cap P_3(D) \approx U \cap D$ and hence $P_3(x)$ has a neighborhood in $P_3(D)$ which is homeomorphic to a k -manifold. If $x \in A$, there is an open neighborhood U of x in K such that $U \subset D - \text{Cl}(K - D)$ and U is saturated with respect to G_3 . Then $P_3(U)$ is open in $P_3(K)$, $P_3(U) \subset P_3(D)$ and, as in the proof of Lemma 3.6, $P_3(U) \approx U$. Hence $P_3(U)$ is an open subset of $P_3(D)$ which contains $P_3(x)$ and is homeomorphic to a k -manifold. It follows that $P_3(D)$ is itself a k -manifold, and hence $P_3(D) \approx D$. Since $P_3(D) = D/G_2$, G_2 is a simple decomposition of D . Since $S(D, G_1)$ holds, it readily follows that $S(D, G_2)$ also holds.

There is a locally flat m -cell B in M such that $D \subset \text{Int } B$, and there is a closed neighborhood C of g_0 in M such that $C \subset \text{Int } B$ and $C \cap K = A$. By Lemma 3.7, G_2 generates a simple decomposition $\tilde{G}_2$ of B , and D/G_2 is locally flat in $B/\tilde{G}_2$. Since every nondegenerate element of G which intersects C is an element of G_1 and therefore of G_2 , the decomposition of B generated by G_C is identical with that generated by G_2 ; hence G_C generates a simple decomposition of B , and it easily follows that G_C generates a simple decomposition $\tilde{G}_C$ of M , and that if $\tilde{P}: M \rightarrow M/\tilde{G}_C$ is the projection map, then $\tilde{P}(K)$ is locally

flat in $\tilde{P}(M)$ at the point $\tilde{P}(g_0)$. It follows from Corollary 3.4 that G is a simple decomposition of M , and from the observation above that if $P: M \rightarrow M/G$ is the projection map, then $P(K)$ is locally flat in $P(M)$.

Remark. Suppose K is a closed subset of a metric space M and f is a proper map of K onto K . If F_1, F_2 are proper maps of M onto M and b_1, b_2 are embeddings of K into M such that, for $i = 1, 2$, the diagram

$$\begin{array}{ccc} K & \xrightarrow{\text{incl}} & M \\ f \downarrow & & \downarrow F_i \\ K & \xrightarrow{h_i} & M \end{array}$$

is commutative and $F_i|_{M-K}$ is a homeomorphism onto $M - b_i(K)$, then (F_1, b_1) and (F_2, b_2) are *topologically equivalent*, in the sense that there is a homeomorphism $\psi: M \rightarrow M$ such that $\psi \circ F_1 = F_2$ and $\psi \circ b_1 = b_2$. To see this, let $G_0 = \{f^{-1}(x) | x \in K\}$ and let $G = \tilde{G}_0(M)$. Then $\{F_i^{-1}(x) | x \in M\} = G$, $i = 1, 2$. Let $P: M \rightarrow M/G$ be the projection map and define $\phi_i: M/G \rightarrow M$ by $\phi_i(y) = F_i(p^{-1}(y))$. Since F_i is a proper map, ϕ_i is a homeomorphism. It may be verified directly that the homeomorphism $\psi = \phi_2 \circ \phi_1^{-1}: M \rightarrow M$ has the desired properties. It follows, in particular, that if K, M and f satisfy the hypothesis of Theorem 3.8, then for every pair (F', b') satisfying the above conditions, b' is necessarily a locally flat embedding.

With a suitable definition of local flatness, Theorem 3.8 can be shown to hold without the requirement that K be a subset of $\text{Int } M$. We sketch below a proof for the case $K \cap \partial M = \partial K$.

3.9 Theorem. *In Theorems 3.8 and 3.8a, the hypothesis that $K \subset \text{Int } M$ may be replaced by the condition $K \cap \partial M = \partial K$.*

Outline of proof. The argument closely parallels that for Theorem 3.8a. We need to show that if $g_0 \in G_0$ and $g_0 \cap \partial M \neq \emptyset$, then there is a closed neighborhood C of g_0 in M such that G_C generates a simple decomposition $\tilde{G}_C$ of M , and if $\tilde{P}: M \rightarrow M/\tilde{G}_C$ is the projection map, then $\tilde{P}(K)$ is locally flat in $\tilde{P}(M)$ at the point $\tilde{P}(g_0)$.

As before, we use Lemma 3.6 to obtain a k -cell neighborhood D of g_0 in K such that D is locally flat in M and $D \cap \partial K$ is a locally flat $(k-1)$ -cell in ∂D . Since $D \cap \partial M = D \cap \partial K$ and K is locally flat in M , it follows that $D \cap \partial M$ is locally flat in ∂M . We again choose a closed neighborhood A of g_0 in K such that A is saturated with respect to G_0 and $A \subset D - \text{Cl}(K - D)$. There

is a locally flat m -cell B in M such that $D \subset B - \text{Cl}(M - B)$, $B \cap \partial M$ is an $(m - 1)$ -cell which is locally flat in M and in ∂B , and $D \cap \partial M \subset \text{Int}(B \cap \partial M)$. There is a closed neighborhood C of g_0 in M such that $C \cap K = A$ and $C \subset B$. We note that $\tilde{G}_C(B) = \{g \in G_0 \mid g \subset A\} \cup \{\{p\} \mid p \in B - A\}$, where $\tilde{G}_C(B)$ is defined to be the decomposition of B generated by G_C . In order to prove that $\tilde{G}_C(M)$ is a simple decomposition of M , it is sufficient to show that $\tilde{P}(B) \approx B$, where $\tilde{P}: M \rightarrow M/\tilde{G}_C(M)$ is the projection map. We must also, of course, show that $\tilde{P}(D)$ is locally flat in $\tilde{P}(B)$.

As usual, we let B^j denote the standard j -ball in E^j . We may identify B with $B^{k-1} \times I \times B^{m-k}$. We let $\mathcal{O}$ denote the center of B^{m-k} , and identify B^k with $B^{k-1} \times I \times \{\mathcal{O}\}$. In view of the local flatness conditions imposed on D and $D \cap \partial M$, we may assume that $D \subset B^k$ and $D \cap \partial M \subset \text{Int}(B^{k-1} \times \{0\} \times \{\mathcal{O}\})$.

The argument used in the proof of Theorem 3.8a to show that $P_3(D) \approx D$ may be used here to show that $\tilde{P}(B^k) \approx B^k$. Let $\phi: \tilde{P}(B^k) \rightarrow B^k$ be a homeomorphism and let $f = \phi \circ (\tilde{P}|B^k)$. Then f is a cell-like map of B^k onto B^k and $S(B^k, f)$ holds.

Let $L = \partial B^k - \text{Int}(B^{k-1} \times \{0\} \times \{\mathcal{O}\})$ and define $\epsilon: B^k \rightarrow [0, \infty)$ by $\epsilon(x) = d(x, L)$. We may apply the complement to Theorem A of [26] to $B^k - L$, $f|B^k - L$ and $\epsilon|B^k - L$ to obtain a homotopy $h: (B^k - L) \times I \rightarrow B^k - L$ such that $h_0 = f|B^k - L$, and for each $t \in (0, 1]$, h_t is a homeomorphism of $B^k - L$ onto itself such that $d(h_t(x), f(x)) < \epsilon(x)$ for all $x \in B^k - L$. Define a function H on $B^k \times I$ by $H(x, t) = h_t(x, t)$ if $x \in B^k - L$, $H(x, t) = f(x)$ if $x \in L$. It is easily shown that $H: B^k \times I \rightarrow B^k$, $H_0 = f$, and for each $t \in (0, 1]$, H_t is a homeomorphism of B^k onto B^k such that $d(H_t(x), f(x)) \leq \epsilon(x)$ for all $x \in B^k$.

Now define F on $B = B^{k-1} \times I \times B^{m-k}$ by $F(x, s, y) = (H_t(x, s), y)$, where t is the distance from y to $\mathcal{O}$ in B^{m-k} . Then $F: B^m \rightarrow B^m$ and $\{F^{-1}(p) \mid p \in B\} = \tilde{G}_C(B)$. It follows that $B/\tilde{G}_C(B) \approx F(B) = B$. Since $F(B^k) = B^k$ and $F|B - B^k$ is a homeomorphism onto $B - B^k$, $\tilde{P}(B^k)$ is locally flat in $\tilde{P}(B)$ and hence $\tilde{P}(K)$ is locally flat in $\tilde{P}(M)$ at the point $\tilde{P}(g_0)$.

As in the proof of Theorem 3.8a, it now follows that G_0 generates a simple decomposition G of M , and K/G_0 is locally flat in K/G .

4. Specializations to E^3 . If $M = E^3$ (or S^3), then for every cell-like map f of M onto a metric space Y and for every collection G of cell-like subsets of M , conditions $S(M, f)$ and $S(M, G)$ are satisfied. The theorems of §3 therefore have somewhat simpler hypotheses if $M = E^3$ and we give here a few additional results for this special case.

We will say that a closed subset Y of a metric space X is *strongly simply embedded* in X if every cell-like upper semicontinuous decomposition of Y , simple or not, generates a simple decomposition of X .

The following is an immediate consequence of the Moore theorem [22, Theorem 22] and Theorem 8 of [15]. As usual, we regard E^2 as the xy -plane in E^3 .

4.1 Lemma. *Every closed subset of E^2 is strongly simply embedded in E^3 .*

4.2 Theorem. *Suppose X is a closed subset of E^3 and G is a cell-like upper semicontinuous decomposition of X . If for each $g \in G$ there is a homeomorphism of E^3 onto itself which takes some closed neighborhood of g in X onto a subset of E^2 , then G generates a simple decomposition of E^3 .*

Proof. By hypothesis, for each $g \in G$ there is a tame 2-cell D in E^3 which contains a neighborhood of g in X . If A is a closed neighborhood of g in X such that $A \subset D$ and A is saturated with respect to G , then by Lemma 4.1, G_A generates a simple decomposition of E^3 . Hence, as in the proof of Theorem 3.8a, there is a closed neighborhood C of g in E^3 such that G_C generates a simple decomposition of E^3 , and it follows from Corollary 3.4 that G generates a simple decomposition of E^3 .

4.3 Corollary. *Every tame 2-manifold in E^3 is strongly simply embedded in E^3 .*

4.4 Corollary. *Every tame 1-dimensional polyhedron in E^3 is strongly simply embedded in E^3 .*

Specializing Theorem 3.8a to E^3 yields the following result (cf. [23, Theorem 4]).

4.5 Theorem. *Every tame 3-manifold in E^3 is simply embedded in E^3 .*

5. Examples. Since every proper subarc of an arc A generates a simple decomposition of A , it follows immediately that if A is simply embedded in E^3 , then every proper subarc of A must be cellular in E^3 . The arc A itself, however, need not be cellular. To see this, let A be an arc which is wild at each endpoint and locally tame elsewhere. Then A is noncellular [27, Theorem 10], but each proper subarc of A has at most one wild point and is therefore cellular [24, Theorem 3]. If G is a simple decomposition of A , then $\tilde{G}$ has only a countable number of nondegenerate elements, each is a cellular arc and all but at most two are tame. It follows from an easy modification of Theorem 3 of [7] that $E^3/\tilde{G} \approx E^3$, and hence A is simply embedded in E^3 .

Similarly, a simple closed curve in E^3 is simply embedded if it has at most one wild point, but if it has two *isolated* wild points, then it contains a non-cellular arc and therefore is not simply embedded. It is possible, however, for

an everywhere wild simple closed curve (or arc) in E^3 to be simply embedded.

5.1 Example. *There exists a (strongly) simply embedded simple closed curve in E^3 which is wild at each of its points.*

Proof. Let K denote a wild simple closed curve as described by Bothe [11] which has the property that each homeomorphism of K onto K can be extended to a homeomorphism of E^3 onto E^3 . The curve K is obtained as the intersection of a sequence $\{T_i\}$ of solid tori, with $T_{i+1} \subset \text{Int } T_i$ for every i ; each T_i is the union of a circular chain $\mathcal{C}_i = \{C_{i1}, C_{i2}, \dots, C_{in_i}\}$ of polyhedral 3-cells of diameter less than $1/i$ such that the intersection of each two adjacent elements of $\mathcal{C}_i$ is a disk which contains exactly one point of K , and for each i , $\mathcal{C}_{i+1}$ is a refinement of $\mathcal{C}_i$. For each i , let $F_i = K \cap \bigcup_{j=1}^{n_i} \partial C_{ij}$ and let $F = \bigcup_{i=1}^{\infty} F_i$. Then F is a countable dense subset of K , and for each i , F_i is finite and $F_{i+1} \subset F_i$.

Suppose G is a cell-like upper semicontinuous decomposition of K . Let H denote the set of nondegenerate elements of G and let D be a countable dense subset of $K - H^*$. Since F and $F \cup D$ are countable dense subsets of the simple closed curve K , there is a homeomorphism $\phi: K \rightarrow K$ such that $\phi(F \cup D) = F$. Let $F' = \phi(D)$ and let $H_1 = \phi(H)$.

Suppose U is an open subset of E^3 containing H_1^* and let A be an element of H_1 . Since F' is a dense subset of $K - H_1^*$, each endpoint of A is a limit point of F' and it follows that there is an arc $A' \subset K \cap U$ such that $A \subset \text{Int } A'$ and the endpoints of A' belong to F' . Since each element of $\mathcal{C}_i$ has diameter less than $1/i$, there is an integer n such that if $i > n$, every element of $\mathcal{C}_i$ which intersects A' lies in U . Since $F_i \supset F_{i+1}$ for every i , there is an integer $i_0 > n$ such that both endpoints of A' belong to F_{i_0} . If C denotes the union of all the elements of $\mathcal{C}_{i_0}$ whose interiors intersect A' , then C is a tame 3-cell lying in U , $A \subset \text{Int } C$, and $K \cap \partial C = \partial A' \subset F'$. Since $H_1^* \cap F' = \emptyset$, repetition of this process yields a null family of disjoint tame 3-cells in U such that each element of H_1 lies in the interior of some member of this family, and hence by Theorem 3 of [1], H_1 generates a simple decomposition of E^3 .

Since K is "homogeneously embedded," ϕ can be extended to a homeomorphism $\Phi: E^3 \rightarrow E^3$. Then $\Phi(\tilde{G})$ is the decomposition of E^3 generated by $\Phi(H)$. Since $\Phi(H) = \phi(H) = H_1$ and H_1 generates a simple decomposition of E^3 , so does H . Hence G generates a simple decomposition of E^3 , and it follows that K is strongly simply embedded in E^3 .

We note that there exist nondegenerate continua in E^3 for which every embedding into E^3 is a simple embedding. Such are, for example, the continua described by Whyburn [29] and Cook [13] which have no simple decomposition other than the trivial decomposition into singletons. The next example shows

that the other extreme is also possible—a continuum may be embeddable in E^3 , yet have no simple embedding at all.

We first collect some facts about upper semicontinuous decompositions of metric spaces. Let X be a metric space and G a collection of pairwise disjoint compact subsets of X such that $G^* = X$. Then it is known that G is an upper semicontinuous decomposition of X if and only if the following condition holds: for each sequence $\{p_i\}$, $p_i \in g_i \in G$, which converges to a point of $g \in G$ and any sequence $\{q_i\}$, $q_i \in g_i$, some subsequence of $\{q_i\}$ converges to a point of g .

The following observation, which is an easy consequence of the above characterization of upper semicontinuous decompositions, will be useful: If G is an upper semicontinuous decomposition of a metric space X and f is a proper map of X onto a metric space Y , then $f(G)$ is an upper semicontinuous decomposition of Y provided the elements of $f(G)$ are disjoint. (Note that it is not required that f map distinct elements of G onto distinct elements of $f(G)$, but only that if $g, g' \in G$ and $f(g) \cap f(g') \neq \emptyset$, then $f(g) = f(g')$.)

The next lemma may be compared with Theorem 20 of [19].

5.2 Lemma. *If X is a compact metric space, G and G_1 are upper semicontinuous decompositions of X such that G_1 is a refinement of G , and P_1 is the projection map of X onto X/G_1 , then $P_1(G)$ is an upper semicontinuous decomposition of X/G_1 and $X/G \approx (X/G_1)/P_1(G)$.*

Proof. That $P_1(G)$ is upper semicontinuous follows immediately from the observation made above.

To complete the proof, let $X_1 = X/G_1$, $G_2 = P_1(G)$ and $Y = X_1/G_2$. Let $P_2: X_1 \rightarrow Y$ be the projection map associated with G_2 and let $f = p_2 \circ p_1$. Then $f: X \rightarrow Y$, and in order to show that $Y \approx X/G$ it is sufficient to show that for each $g \in G$, $f^{-1}(f(g)) = g$. Since $f^{-1}(f(g)) = P_1^{-1}P_2^{-1}P_2P_1(g) = P_1^{-1}(P_2^{-1}P_2(g_2))$, where $g_2 = P_1(g)$, and $P_2^{-1}P_2(g_2) = g_2$ (because $g_2 \in G_2$ and P_2 is the projection map for G_2), it follows that $f^{-1}(f(g)) = P_1^{-1}(P_1(g))$.

If $x \in P_1^{-1}(P_1(g))$, then $P_1(x) \in P_1(g)$ and hence there is an $x' \in g$ such that $P_1(x) = P_1(x')$. Let g_1 be the element of G_1 containing x and g'_1 the element of G_1 containing x' . Then $g'_1 = P_1^{-1}(P_1(x')) = P_1^{-1}(P_1(x)) = g_1$. Since $x' \in g \cap g'_1$ and G_1 is a refinement of G , $g'_1 \subset g$. Since $x \in g_1 = g'_1 \subset g$, $x \in g$. Hence $P_1^{-1}(P_1(g)) \subset g$ and therefore $P_1^{-1}(P_1(g)) = g$.

5.3 Example. *There exists a compact absolute retract in E^3 which has no simple embedding in E^3 .*

Proof. Let B_0 denote the standard unit ball in E^3 and let $B_1, B_2, \dots$ be a sequence of spherical balls, each with center on the x -axis, such that, for $i = 1, 2, \dots$, B_{i-1} and B_i are externally tangent at a point p_i , and such that

the sequence $\{B_i\}_{i=0}^\infty$ converges to a point p . For each i let B_{-i} denote the reflection of B_i in the yz -plane, and let p_{-i} be the reflection of p_i . Let M denote the closure of the union of the balls B_i , $i = 0, \pm 1, \pm 2, \dots$.

Let A_0 be an arc from p_{-1} to p_1 which lies except for its endpoints in $\text{Int } B_0$ and which is wild at p_1 and locally tame at all other points. For $i = 1, 2, \dots$, let ϕ_i be a homeomorphism of B_0 onto B_i which takes p_{-1} onto p_i and p_1 onto p_{i+1} , and let $A_i = \phi_i(A_0)$. Let $A = \bigcup_{i=2}^\infty A_i \cup \{p\}$ and let $X = M/A$ (i.e., X is the decomposition space of M given by the decomposition whose only nondegenerate element is A).

Since A is cellular in E^3 [24, Theorem 3], $E^3/A \approx E^3$ and hence since $X = M/A \subset E^3/A$, X is embeddable in E^3 . Since M and A are absolute retracts, so is X [10, p. 131]. It remains to be shown that X has no simple embedding in E^3 ; this will be done by showing that X contains an arc K such that $X/K \approx X$ and such that if $f: X \rightarrow E^3$ is any embedding, then $f(K)$ is not cellular in E^3 . It will follow that $f(X)/f(K) \approx f(X)$ but $E^3/f(K) \not\approx E^3$, and therefore $f(X)$ is not simply embedded in E^3 .

Let $P: M \rightarrow M/A$ be the projection map. There is a homeomorphism of M onto itself which takes $A_0 \cup A_1 \cup A$ onto A , and hence $M/(A_0 \cup A_1 \cup A) \approx M/A = X$. By Lemma 5.2, $M/(A_0 \cup A_1 \cup A) \approx (M/A)/P(A_0 \cup A_1)$, and it follows that $X \approx X/P(A_0 \cup A_1)$.

Let A_0^- denote the reflection of A_0 in the yz -plane and let $K = P(A_0^- \cup A_1)$. Let $C = \text{Cl}(M - B_0)$ and let B'_0 and C' be disjoint copies of B_0 and C , respectively. Let A'_0 be the arc in B'_0 corresponding to A_0 , and let A'_1 and A' be the arcs in C' corresponding to A_1 and A , respectively. Let p'_{-1}, p'_1 be the points of B'_0 corresponding to p_{-1} and p_1 , and let p''_{-1}, p''_1 be the points of C' corresponding to p_{-1} and p_1 . Let $M' = B'_0 \cup C'$, and let f be a map of M' onto M which takes $\{p''_{-1}, p'_{-1}\}$ onto p_{-1} , and $\{p'_1, p''_1\}$ onto p_1 , such that f is 1-1 on $M' - \{p''_{-1}, p'_{-1}, p'_1, p''_1\}$ and takes $A'_0 \cup A'_1 \cup A'$ onto $A_0 \cup A_1 \cup A$. Let g be a map of M' onto M which takes $\{p''_{-1}, p'_1\}$ onto p_{-1} , and $\{p'_{-1}, p''_1\}$ onto p_1 , such that g is 1-1 on $M' - \{p''_{-1}, p'_{-1}, p'_1, p''_1\}$ and takes $A'_0 \cup A'_1 \cup A'$ onto $A_0^- \cup A_1 \cup A$. Then by Lemma 5.2, $M'/(A'_0 \cup A'_1 \cup A') \approx f(M')/f(A'_0 \cup A'_1 \cup A') = M/(A_0 \cup A_1 \cup A) \approx X$ and $M'/(A'_0 \cup A'_1 \cup A') \approx g(M')/g(A'_0 \cup A'_1 \cup A') = M/(A_0^- \cup A_1 \cup A)$. It follows that $X \approx M/(A_0^- \cup A_1 \cup A) \approx (M/A)/P(A_0^- \cup A_1) = X/K$.

If $f: X \rightarrow E^3$ is an embedding and $g = f|P(B_0 \cup B_1)$, then g is an embedding of the union of the 3-cells $P(B_0)$ and $P(B_1)$ and it follows [21, Lemma 4] that $f(K)$ is wild at each of its endpoints. Since $f(K)$ is locally tame at all other points except possibly $f(p_1)$, $f(K)$ is not cellular in E^3 [27, Theorem 10] and it follows, as indicated above, that $f(X)$ is not simply embedded in E^3 .

6. Questions and further comments. In order that a closed set $X \subset E^n$ should be simply embedded, it is necessary that each element of any simple decomposition of X be cellular in E^n ($n \neq 4$). That this condition is not sufficient in general may be seen from the following example: Let H denote the set of nondegenerate elements of Bing's dogbone decomposition [8], constructed so that each element of H has its upper endpoint on a given horizontal plane α and otherwise lies wholly below α , and let X denote the union of H^* and all vertical intervals of length 1 with lower endpoint in $\alpha \cap H^*$. Then if G is any monotone decomposition of X , each element of G is either a point or a tame arc; X is not simply embedded in E^3 , however, since the decomposition of X whose nondegenerate elements are the elements of H is a simple decomposition of X which does not generate a simple decomposition of E^3 . Whether the condition that each element of each simple decomposition of X be cellular in E^n is sufficient, in certain special cases, to insure that X be simply embedded is the gist of our first question.

6.1 Suppose X is a closed subset of E^n . Which, if any, of the following conditions implies that X is simply embedded in E^n ?

- (a) X is an arc and every proper subarc of X is cellular in E^n .
- (b) X is a simple closed curve and every arc in X is cellular in E^n .
- (c) X is a k -manifold and every k -cell in X is cellular in E^n , $k < n$.
- (d) X is an n -manifold and every $(n-1)$ -cell in ∂X is cellular in E^n .

It was remarked in §4 that every 1-dimensional polyhedron in E^3 is strongly simply embedded. Not every 2-dimensional polyhedron in E^3 is strongly simply embedded, however, since, for example, if X is a "book-with-one-page" (the product of a triod and an interval), then X contains a noncellular arc and hence has a cell-like decomposition which does not generate a simple decomposition of E^3 . In this case, at least, it appears that every simple decomposition of X generates a simple decomposition of E^3 , which suggests the following question.

6.2 Is every polyhedron in E^3 simply embedded in E^3 ?

Example 5.3 is a (3-dimensional) absolute retract in E^3 which has no simple embedding in E^3 , and it seems likely that a 2-dimensional AR with this property could be obtained similarly by replacing each B_i of Example 5.3 by a book-with-one-page. Every 1-dimensional AR, however, is embeddable in E^2 and hence, by Lemma 4.1, has a (strongly) simple embedding in E^3 ; dropping the "absolute retract" condition leads to the following question (which we conjecture has a negative answer).

6.3 Does every 1-dimensional continuum have a simple embedding in E^3 ?

REFERENCES

1. Steve Armentrout, *Upper semi-continuous decompositions of E^3 with at most countably many non-degenerate elements*, Ann. of Math (2) 78 (1963), 605-618. MR 27 #6255.

2. Steve Armentrout, *Concerning cellular decompositions of 3-manifolds that yield 3-manifolds*, Trans. Amer. Math. Soc. 133 (1968), 307–332. MR 37 #5859.
3. ———, *Concerning cellular decompositions of 3-manifolds with boundary*, Trans. Amer. Math. Soc. 137 (1969), 231–236. MR 38 #5224.
4. ———, *Cellular decompositions of 3-manifolds that yield 3-manifolds*, Bull. Amer. Math. Soc. 75 (1969), 453–456. MR 39 #935.
5. ———, *Cellular decompositions of 3-manifolds that yield 3-manifolds*, Mem. Amer. Math. Soc. No. 107 (1970).
6. Ralph J. Bean, *Repairing embeddings and decompositions in S^3* , Duke Math. J. 36 (1969), 379–385. MR 39 #4820.
7. R. H. Bing, *Upper semicontinuous decompositions of E^3* , Ann. of Math. (2) 65 (1957), 363–374. MR 19, 1187.
8. ———, *A decomposition of E^3 into points and tame arcs such that the decomposition space is topologically different from E^3* , Ann. of Math. (2) 65 (1957), 484–500. MR 19, 1187.
9. ———, *Extending monotone decompositions of 3-manifolds*, Trans. Amer. Math. Soc. 149 (1970), 351–369. MR 41 #7656.
10. Karol Borsuk, *Theory of retracts*, Monografie Mat., Tom 44, PWN, Warsaw, 1967. MR 35 #7306.
11. H. G. Bothe, *Ein homogen wilder Knoten*, Fund. Math. 60 (1967), 271–283. MR 35 #7327.
12. W. S. Boyd, *Repairing embeddings of 3-cells with monotone maps of E^3* , Trans. Amer. Math. Soc. 161 (1971), 123–144. MR 43 #8064.
13. H. Cook, *Continua which admit only the identity mapping onto non-degenerate subcontinua*, Fund. Math. 60 (1967), 241–249. MR 36 #3315.
14. James Dugundji, *Topology*, Allyn and Bacon, Boston, Mass., 1966. MR 33 #1824.
15. E. Dyer and M. E. Hamstrom, *Completely regular mappings*, Fund. Math. 45 (1958), 103–118. MR 19, 1187.
16. William Haver, *A characterization theorem for cellular maps*, Bull. Amer. Math. Soc. 76 (1970), 1277–1280. MR 42 #2440.
17. R. C. Lacher, *Cell-like mappings, I*, Pacific J. Math. 30 (1969), 717–731. MR 40 #4941.
18. H. W. Lambert, *Some comments on the structure of compact decompositions of S^3* , Proc. Amer. Math. Soc. 19 (1968), 180–184. MR 37 #901.
19. Jack W. Lamoreaux, *Decompositions of metric spaces with a 0-dimensional set of non-degenerate elements*, Canad. J. Math. 21 (1969), 202–216. MR 39 #2122.
20. Louis F. McAuley, *Some fundamental theorems and problems related to monotone mappings*, Proc. First Conf. on Monotone Mappings and Open Mappings (SUNY at Binghamton, Binghamton, N.Y., 1970), State Univ. of New York at Binghamton, Binghamton, N.Y., 1971, pp. 1–36. MR 44 #4722.
21. D. R. McMillan, *Some topological properties of piercing points*, Pacific J. Math. 22 (1967), 313–322. MR 35 #7319.
22. R. L. Moore, *Concerning upper semicontinuous collections of continua*, Trans. Amer. Math. Soc. 27 (1925), 416–428.
23. V. Nicholson, *Mapping cylinder neighborhoods*, Trans. Amer. Math. Soc. 143 (1969), 259–268. MR 40 #2038.

24. R. B. Sher, *Determining the cellularity of a 1-complex by properties of its arcs*, Proc. Amer. Math. Soc. 26 (1970), 491–498. MR 42 #5242.
25. ———, *Realizing cell-like maps in Euclidean space*, General Topology and Appl. 2 (1972), 75–89.
26. L. C. Siebenmann, *Approximating cellular maps by homeomorphisms*, Topology 11 (1972), 271–294. MR 45 #4431.
27. C. D. Sikkema, *A duality between certain spheres and arcs in S^3* , Trans. Amer. Math. Soc. 122 (1966), 399–415. MR 33 #7991.
28. J. Väisälä, *Invariance of domain under acyclic mappings*, Duke Math. J. 33 (1966), 679–681. MR 34 #815.
29. G. T. Whyburn, *A continuum every subcontinuum of which separates the plane*, Amer. J. Math. 52 (1930), 319–330.
30. ———, *Analytic topology*, Amer. Math. Soc. Colloq. Publ., vol. 28, Amer. Math. Soc., Providence, R.I., 1942. MR 4, 86.

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