WEIGHTED GROTHENDIECK SUBSPACES

BY

JOÃO B. PROLLA AND SILVIO MACHADO

ABSTRACT. Let V be a family of nonnegative upper semicontinuous functions on a completely regular Hausdorff space X . For a locally convex Hausdorff space E , let $CV_\infty(X; E)$ be the corresponding Nachbin space, that is, the vector space of all continuous functions f from X into E such that vf vanishes at infinity for all $v \in V$, endowed with the topology given by the seminorms of the type $f \mapsto \sup \{v(x)p(\tilde{f}(x)); x \in X\}$, where $v \in V$ and p is a continuous seminorm on E . Given a vector subspace L of $CV_\infty(X; E)$, the set of all pairs $x, y \in X$ such that either $0 = \delta_x|_L = \delta_y|_L$ or there is $t \in \mathbb{R}$, $t \neq 0$, such that $0 \neq \delta_x|_L = t\delta_y|_L$, is an equivalence relation, denoted by G_L , and we define for $(x, y) \in G_L$, $g(x, y) = 0$ or t , accordingly. The subsets KS_L , resp. WS_L , where $g(x, y) \geq 0$, resp. $g(x, y) \in \{0, 1\}$, are likewise equivalence relations. The G -hull (resp. KS -hull, WS -hull) of L is the vector subspace $\{f \in CV_\infty(X; E); f(x) = g(x, y)f(y) \text{ for all } (x, y) \in G_L \text{ (resp. } KS_L, WS_L)\}$, and L is a G -space (resp. KS -space, WS -space) if its G -hull (resp. KS -hull, WS -hull) is contained in its closure. This paper is devoted to the characterization, by invariance properties, of the G -spaces resp. KS -spaces and WS -spaces of a given Nachbin space $CV_\infty(X; E)$. As an application we derive an infinite-dimensional Weierstrass polynomial approximation theorem, and a Tietze extension theorem for Banach space valued compact mappings.

1. Introduction. Let X be a completely regular Hausdorff space, and let E be a locally convex Hausdorff space, not reduced to $\{0\}$. The vector space of all E -valued continuous functions on X will be denoted by $C(X; E)$, while $C_b(X; E)$ denotes the subspace of all $f \in C(X; E)$ which are bounded. A *weight* on X is a nonnegative upper semicontinuous function on X . A set V of weights on X is directed if, for all $u, v \in V$, there are $w \in V$ and $t > 0$ such that $u \leq tw$ and $v \leq tw$ pointwise. All sets of weights considered will be directed and $V > 0$, i.e., given $x \in X$ there is $v \in V$ such that $v(x) > 0$. Given such a V the *Nachbin space* $CV_\infty(X; E)$ is the vector subspace of all $f \in C(X; E)$ such that vf vanishes at infinity, for every $v \in V$, endowed with the topology determined by the seminorms $f \mapsto \sup \{v(x)p(f(x)); x \in X\}$, where $v \in V$, and $p \in s(E)$, a fixed

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determining set of continuous seminorms on E . The following definitions are from Blatter [2].

Let L be a vector subspace of $CV_\infty(X; E)$. The set G_L of all pairs $x, y \in X$ such that either $0 = \delta_x|L = \delta_y|L$ or there exists $t \in \mathbb{R}, t \neq 0$, such that $0 \neq \delta_x|L = t\delta_y|L$, is an equivalence relation for X , and we define a map $g: G_L \rightarrow \mathbb{R}$ as follows:

$$\begin{aligned} g(x, y) &= 0, & \text{if } 0 = \delta_x|L = \delta_y|L; \\ g(x, y) &= t, & \text{if } 0 \neq \delta_x|L = t\delta_y|L. \end{aligned}$$

The sets KS_L and WS_L of all pairs $(x, y) \in G_L$ such that $g(x, y) \geq 0$ and $g(x, y) \in \{0, 1\}$, respectively, are likewise equivalence relations for X . Let $\Delta \in \{G, KS, WS\}$. The vector subspace $\Delta(L) = \{f \in CV_\infty(X; E); f(x) = g(x, y)f(y) \text{ for all } (x, y) \in \Delta_L\}$ is called the Δ -bull of L in $CV_\infty(X; E)$, and L is called a Δ -space if $\Delta(L)$ is the closure of L in $CV_\infty(X; E)$. Notice that, by the hypothesis made on V , the closure of L is always contained in $\Delta(L)$. The letters G, KS , and WS stand for Grothendieck, Kakutani-Stone, and Weierstrass-Stone respectively.

The object of the present paper is to characterize, by invariance properties, those subspaces L which are Δ -spaces. Our results are as follows. Let $A(\Delta)$ be the algebra of all $b \in C_b(X; \mathbb{R})$ which are constant on the equivalence classes modulo Δ_L . Assume that $A(\Delta)$ separates the equivalence classes modulo Δ_L . (This is automatically satisfied in the case $\Delta = WS$.) Let $M = \{u^*(f); u^* \in E^*, f \in L\}$. Then:

- (1) L is a WS -space, if L is a polynomial algebra such that M is localizable under itself in $CV_\infty(X; \mathbb{K})$.
- (2) L is a KS -space, if L is a latticial subspace.
- (3) L is a G -space, if L is a Lindenstrauss-Wulbert subspace (see Definition 3.10).

In the spirit of Stone's paper [11], we apply some of these results to reobtain vector-valued versions of some classical theorems.

The following elementary result will be used many times.

1.1 Lemma. *Let L be a vector subspace of $CV_\infty(X; E)$. The following statements are equivalent.*

- (1) $M = \{u^*(f); u^* \in E^*, f \in L\}$ is a vector subspace of $CV_\infty(X; \mathbb{K})$ and $M \otimes E \subset L$. If L is closed, M is closed, too.
- (2) Given $u^* \in E^*, f \in L$, and $u \in E$, then $u^*(f)u \in L$.

Proof. It is obvious that (1) implies (2). Conversely, assume (2). M is clearly invariant under scalar multiplication. Let $u^*, v^* \in E^*$, and $f, h \in L$ be given. We may assume that u^* and v^* are linearly independent. Choose $u, v \in E$ such that $u^*(u) = v^*(v) = 1, u^*(v) = v^*(u) = 0$. By (2), $k = u^*(f)u + v^*(h)v$ belongs

to L . Let $w^* = u^* + v^*$. Then $u^*(f) + v^*(b) = w^*(k) \in M$. Obviously $M \otimes E \subset L$. Suppose now that L is closed. Let f be in the closure of M . Choose $u^* \in E^*$ and $u \in E$ such that $u^*(u) = 1$. Set $b = f \otimes u$. Given $v \in V$, $p \in s(E)$, and $\epsilon > 0$, there exists $w^*(k) \in M$ such that $v(x)|f(x) - w^*(k(x))| < \epsilon/(1 + p(u))$ for all $x \in X$. Then $w^*(k)u \in L$ and $v(x)p(b(x) - w^*(k(x))u) < \epsilon$ for all $x \in X$. Since L is closed, $b \in L$. Therefore, $f = u^*(b) \in M$, and M is closed.

2. Weierstrass-Stone spaces.

2.1 Definition. Let E and F be two nonzero topological vector spaces.

For each integer $n \geq 1$, $P_f(nE; F)$ denotes the vector subspace of $C(E; F)$ generated by the set of all maps of the form $x \mapsto u^*(x)^n u$, where $u^* \in E^*$, the topological dual of E , and $u \in F$. The elements of $P_f(nE; F)$ are called *n-homogeneous continuous polynomials of finite type* from E into F . The vector subspace generated by the union of all $P_f(nE; F)$, $n \geq 1$, and the constant maps, is denoted by $P_f(E; F)$. Similarly, $L_f(nE; F)$, for each integer $n \geq 1$, denotes the vector space generated by the set of all maps of the form $(x_1, \dots, x_n) \mapsto u_1^*(x_1) \dots u_n^*(x_n)u$ where $u_1^*, \dots, u_n^* \in E^*$ and $u \in F$. The elements of $L_f(nE; F)$ are called *n-linear continuous mappings of finite type* from E^n into F .

2.2 Lemma. Let L be a vector subspace of $CV_\infty(X; E)$. The following statements are equivalent.

- (1) Given $b_1, \dots, b_n \in L$ and $T \in L_f(nE; E)$, where $n \geq 1$, then $T(b_1, \dots, b_n) \in L$.
- (2) Given $b \in L$ and $p \in P_f(nE; E)$, where $n \geq 1$, then $p(b) \in L$.
- (3) $M = \{u^*(b); u^* \in E^*, b \in L\}$ is an algebra contained in $CV_\infty(X; K)$ such that $M \otimes E \subset L$.
- (4) (a) Given $b \in L$, $u^* \in E^*$, and $u \in E$, then $u^*(b)u \in L$.
 (b) There exists a continuous map $T: E \times E \rightarrow E$ such that $T(v, v) \neq 0$ for some $v \in E$ for which $T(av, bv) = abT(v, v)$ for all $a, b \in K$ and $T(b, k) \in L$ for all $b, k \in L$.

Proof. It is obvious that (1) implies (2). Assume (2). By the Lemma 1.1 the set M is a vector subspace of $CV_\infty(X; K)$ such that $M \otimes E \subset L$. Since $4u^*(b)v^*(k) = [u^*(b) + v^*(k)]^2 - [u^*(b) - v^*(k)]^2$, all that remains to prove is that $u^*(b)^2$ belongs to M for any $u^* \in E^*$ and $b \in L$. If $u^* = 0$, there is nothing to prove. Let $u \in E$ be such that $u^*(u) = 1$, and define $p(t) = u^*(t)^2 u$ for all $t \in E$. By (2), $k = p(b) \in L$. Since $u^*(k) = u^*(b)^2$, the set M is an algebra. Assume now (3). Part (a) follows immediately. To prove part (b) of (4), consider $u^* \in E^*$ and $v \in E$ with $u^*(v) = 1$. Define $T(s, t) = u^*(s)u^*(t)v$ for all $s, t \in E$. Then T satisfies all requirements. Finally, assume (4). Let $b_1, \dots, b_n \in L$, $u_1^*, \dots, u_n^* \in E^*$, and $u \in E$ be given. We claim that $u_1^*(b_1) \dots u_n^*(b_n)u \in L$.

The proof is by induction. If $n = 1$, the conclusion follows from part (a). Assume that the conclusion is true for m . Then, given $b_i, u_i^*, i = 1, \dots, m+1$, we have $u_1^*(b_1) \dots u_m^*(b_m)v \in L$. Call it b . Let $k = u_{m+1}^*(b_{m+1})v$. Then $k \in L$, by part (a), and $T(b, k) \in L$, by part (b). Choose $v^* \in E^*$ such that $v^*(T(v, v)) = 1$. By part (a), $v^*(T(b, k))u = u_1^*(b_1) \dots u_{m+1}^*(b_{m+1})u \in L$, and the proof is complete.

2.3 Definition. A vector subspace $L \subset CV_\infty(X; E)$ is called a *polynomial algebra* if any of the equivalent statements (1)–(4) is true.

2.4 Remark. The name "polynomial algebra" was introduced by Wulbert (unpublished) for vector subspaces satisfying (2), when one allows all continuous polynomials and not just those of finite type. Subspaces L of $C(X; E)$, for X compact and E a real Banach space, satisfying (1) were considered by Pełczyński [9], but he allowed T to be any multilinear continuous mapping and L contained the constants. Blatter, in [2], considered subspaces of $C_\infty(X; E)$, for X locally compact and E a real Banach space, satisfying (4).

2.5 Remark. When $E = K$, $L \subset CV_\infty(X; K)$ satisfies (1)–(4) if, and only if, L is an algebra under pointwise operations, and the equivalence relation WS_L is just the usual equivalence relation X/L defined by $x \sim y$ if, and only if, $b(x) = b(y)$ for all $b \in L$. We recall that, in this case, L is called *localizable* under itself in $CV_\infty(X; K)$ (see Nachbin [7]) when the following condition holds true: A function $f \in CV_\infty(X; K)$ is in the closure of L if (and always if), for any $v \in V$, any $\epsilon > 0$, and any equivalence class $Y \subset X$ modulo X/L , there exists $b \in L$ such that $v(x)|f(x) - b(x)| < \epsilon$ for all $x \in Y$. It is immediate that L is localizable if, and only if, L is a WS -space. Theorem 4 of Nachbin [7] gives a sufficient condition for L to be localizable in $CV_\infty(X; K)$.

2.6 Proposition. Suppose that $CV_\infty(X; E)$ is a polynomial algebra and let L be a vector subspace of $CV_\infty(X; E)$. The WS -hull of L is a polynomial algebra such that $M = \{u^*(b); u^* \in E^*, b \in WS(L)\}$ is localizable in $CV_\infty(X; K)$.

Proof. Let $b \in WS(L)$, $u^* \in E^*$, $u \in E$, and $n \geq 1$ be given. Let $k = u^*(b)^n u$. Since $CV_\infty(X; E)$ is a polynomial algebra, $k \in CV_\infty(X; E)$. Let $(x, y) \in WS_L$. Then $k(x) = u^*(b(x))^n u = u^*(g(x, y)b(y))^n u = g(x, y)k(y)$, since $g(x, y) \in \{0, 1\}$. Hence $k \in WS(L)$, and therefore $WS(L)$ is a polynomial algebra. On the other hand, let $f \in CV_\infty(X; K)$ belong to $WS(M)$. Choose $u \in E$ and $u^* \in E^*$ such that $u^*(u) = 1$. Then $b = f \otimes u$ belongs to $WS(L)$, because the formation of WS -hulls is idempotent. Therefore, $u^*(b) = f \in M$, and M is localizable in $CV_\infty(X; K)$.

2.7 Corollary. Suppose that $CV_\infty(X; E)$ is a polynomial algebra and let L be a Weierstrass-Stone space contained in $CV_\infty(X; E)$. Then its closure is a poly-

nomial algebra and $M = \{u^*(b); u^* \in E^*, b \in L\}$ is localizable in $CV_\infty(X; K)$.

2.8 Lemma. Let A be a selfadjoint subalgebra of $C(X; K)$ such that every $b \in A$ is bounded on the support of every $v \in V$, and let L be a vector subspace of $CV_\infty(X; E)$ which is an A -module. Then, $f \in CV_\infty(X; E)$ is in the closure of L if (and always only if), for every equivalence class $Y \subset X$ modulo X/A , every $v \in V$, every $p \in s(E)$, and every $\epsilon > 0$, there is $k \in L$ such that for all $x \in Y$, $v(x)p(f(x) - k(x)) < \epsilon$.

Proof. This is an immediate corollary to Theorem 2 of [8].

2.9 Theorem. Let L be a vector subspace of $CV_\infty(X; E)$ which is a polynomial algebra such that $M = \{u^*(b); u^* \in E^*, b \in L\}$ is localizable in $CV_\infty(X; K)$. Then L is a WS-space.

Proof. Let $A = A(WS)$ be the algebra of all $f \in C_b(X; R)$ which are constant on the equivalence classes modulo WS_L . Notice that $WS(M)$ is an A -module. Therefore, since M is localizable, $\bar{M}$ is an A -module. Let $b \in WS(L)$. Let $Y \subset X$ be an equivalence class modulo X/A , let $v \in V$, $p \in s(E)$, and $\epsilon > 0$ be given. Since M and A define the same equivalence relation on X , namely WS_L , b is constant on Y . Let $u \in E$ be this constant value. If $u = 0$, then $0 \in M \otimes E$ is such that $v(x)p(b(x) - 0) = 0 < \epsilon$ for all $x \in Y$. If $u \neq 0$, there exists $f \in L$ whose constant value on Y , say t , is not zero. Let $u^* \in E^*$ be such that $u^*(t) = 1$. Then $k = u^*(f)u$ belongs to $M \otimes E$ and $v(x)p(b(x) - k(x)) = 0 < \epsilon$ for all $x \in Y$. By Lemma 2.8, b belongs to the closure of $\bar{M} \otimes E$. Since $\bar{M} \otimes E$ is contained in the closure of L , L is a Weierstrass-Stone space.

2.10 Remark. A subset L of $CV_\infty(X; E)$ is called *separating* if for any $x, y \in X$ with $x \neq y$, there is $b \in L$ such that $b(x) \neq b(y)$, i.e., if WS_L reduces to the diagonal. In this case, $WS(L)$ is precisely the set of all $b \in CV_\infty(X; E)$ which vanish on the subset $Z(L) = \{x \in X; f(x) = 0 \text{ for all } f \in L\}$. When $Z(L) = \emptyset$, we say that L is *everywhere different from zero*.

2.11 Corollary. Let L be as in Theorem 2.9. Suppose that L is separating. Then, $f \in CV_\infty(X; E)$ belongs to the closure of L if, and only if, f vanishes on the set $Z(L)$.

2.12 Corollary. Let L be as in Theorem 2.9. Suppose that L is separating and everywhere different from zero. Then L is dense in $CV_\infty(X; E)$.

3. Kakutani-Stone and Grothendieck spaces. In this paragraph E is a real locally convex Hausdorff space.

3.1 Definition. A subset $M \subset C(X; R)$ is a lattice if $\sup(f, b) \in M$ and $\inf(f, b) \in M$ whenever $f, b \in M$.

3.2 Theorem (Nachbin [7]). *Let M be a sublattice of $CV_\infty(X; \mathbb{R})$. A function $f \in CV_\infty(X; \mathbb{R})$ is in the closure of M if, and only if, for every pair $x, y \in X$ and every $\epsilon > 0$, there exists $h \in M$ such that $|b(x) - f(x)| < \epsilon$ and $|b(y) - f(y)| < \epsilon$.*

3.3 Remark. It follows from Definition 3.1 that the KS-hull of any vector subspace M of $CV_\infty(X; \mathbb{R})$ is a vector lattice.

3.4 Theorem. *Any vector sublattice of $CV_\infty(X; \mathbb{R})$ is a Kakutani-Stone space.*

Proof. Let M be a vector sublattice of $CV_\infty(X; \mathbb{R})$ and let $b \in KS(M)$. Let $x, y \in X$ be given arbitrarily.

Case I. $0 = \delta_x|M$ or $0 = \delta_y|M$. Suppose $0 = \delta_x|M$. Then $g(x, x) = 0$ and $b(x) = 0$. If $(x, y) \in KS_M$, then $b(y) = 0$ too, and $0 \in M$ agrees with b at x and y . If $(x, y) \notin KS_M$, then there is some $f \in M$ such that $f(x) = 0 = b(x)$ and $f(y) = b(y)$. The case $0 = \delta_y|M$ is analogous.

Case II. $0 \neq \delta_x|M$ and $0 \neq \delta_y|M$. If $(x, y) \in KS_M$, then $0 \neq \delta_x|M = r\delta_y|M$ for some $r > 0$. Choose $f \in M$ such that $f(x) = b(x)$. Then $f(y) = b(y)$, because both f and b belong to $KS(M)$. If $(x, y) \notin KS_M$, $\delta_x|M \neq r\delta_y|M$ for all $r > 0$. Choose $f_x \in M$ such that $f_x(x) = 1$. If $f_x(y) \leq 0$, we can in fact suppose $f_x(y) = 0$, by taking $\sup(f_x, 0) \in M$, if necessary. Choose $f_y \in M$ with $f_y(y) = 1$. Then, the function $f = (b(x) - b(y)f_y(x))f_x + b(y)f_y$ belongs to M and is such that $f(x) = b(x)$ and $f(y) = b(y)$. If $f_x(y) > 0$, there is $b_y \in M$ such that $b_y(x) \neq (f_x(y))^{-1}b_y(y)$. Hence the system

$$af_x(x) + bb_y(x) = b(x), \quad af_x(y) + bb_y(y) = b(y)$$

can be solved for a and b . Then $f = af_x + bb_y \in M$ agrees with b at x and y . From Theorem 3.2, b belongs to the closure of M , and therefore, M is a Kakutani-Stone space.

3.5 Corollary. *Let M be a closed vector subspace of $CV_\infty(X; \mathbb{R})$. Then, M is a KS-space if, and only if, M is a vector sublattice.*

3.6 Proposition. *A vector subspace M of $CV_\infty(X; \mathbb{R})$ is a G-space if, and only if, $\bar{M}$ is determined by its restrictions to the two-point subsets of X .*

Proof. For any vector subspace M of $CV_\infty(X; \mathbb{R})$, it follows from the definition of $G(M)$ that $G(M)$ is determined by its restrictions to the two-point subsets of X . Conversely, if $\bar{M}$ is determined by its restrictions to the two-point subsets of X , then an obvious modification of the argument presented in the proof of Theorem 3.4 proves that M is a G-space.

3.7 Remark. For any vector subspace M of $CV_\infty(X; \mathbb{R})$, it follows from the definition of $G(M)$ that $G(M)$ contains the function $\sup(0, f, b) + \inf(0, f, b)$, whenever it contains f and b .

3.8 Theorem. *Let M be a vector subspace of $CV_\infty(X; \mathbb{R})$ such that M contains $\sup(0, f, b) + \inf(0, f, b)$ whenever it contains f and b . Then M is a Grothendieck space.*

Proof. This result can be proved by modifying conveniently the argument of Lindenstrauss and Wulbert presented in the proof of Theorem 2' of [5], making use of the fact that, for each $f \in CV_\infty(X; \mathbb{R})$, $v \in V$, and $\epsilon > 0$, the set $K = \{x \in X; v(x)|f(x)| \geq \epsilon\}$ is compact.

3.9 Remark. It follows from the above results that, if M is an algebra contained in $CV_\infty(X; \mathbb{R})$ such that its closure is a lattice, then M is localizable. More generally, if the closure of M is a G -space, then M is localizable.

3.10 Definition. A vector subspace $L \subset CV_\infty(X; E)$ is called a *lattice subspace* (respectively a *Lindenstrauss-Wulbert subspace*) if it satisfies the following conditions:

- (a) For each $u^* \in E^*$, $u \in E$, and $b \in L$, then $u^*(b)u \in L$.
- (b) There exists a continuous mapping $T: E \times E \rightarrow E$ such that for some $v \in E$, $T(v, v) \neq 0$ and, for all pairs $a, b \in \mathbb{R}$, $T(av, bv) = \max(a, b)T(v, v)$ (respectively $T(av, bv) = (\max(0, a, b) + \min(0, a, b))T(v, v)$) and $T(f, b) \in L$, for all $f, b \in L$.

3.11 Proposition. *Let L be a (closed) vector subspace of $CV_\infty(X; E)$ and let $M = \{u^*(f); u^* \in E^*, f \in L\}$. Consider the following statements.*

- (1) L is a lattice subspace.
- (2) M is a (closed) KS -space such that $M \otimes E \subset L$.
- (3) L is a Lindenstrauss-Wulbert subspace.
- (4) M is a (closed) G -space such that $M \otimes E \subset L$.

Then, (1) is equivalent to (2), and (3) is equivalent to (4).

Proof. The proofs are similar to the proof of Lemma 2.2, and make use of Theorems 3.4 and 3.8.

3.12 Proposition. *Let L be a vector subspace of $CV_\infty(X; E)$. Then $KS(L)$ is a lattice subspace, and $G(L)$ is a Lindenstrauss-Wulbert subspace.*

Proof. Let $u^* \in E^*$, $u \in E$, and $f \in KS(L)$ be given. Let $b = u^*(f)u$. For any pair $(x, y) \in KS_L$ we have: $b(x) = u^*(f(x))u = g(x, y)u^*(f(y))u = g(x, y)b(y)$. Hence $b \in KS(L)$. Now choose $v^* \in E^*$ and $v \in E$ such that $v^*(v) = 1$. Define $T: E \times E \rightarrow E$ by setting $T(s, t) = \max(v^*(s), v^*(t))v$ for all $s, t \in E$. Then T is continuous and $T(v, v) = v \neq 0$, and $T(av, bv) = \max(a, b)T(v, v)$. Let now $f, b \in KS(L)$. It follows that $T(f(x), b(x)) = \max(v^*(f(x)), v^*(b(x)))v = \max(g(x, y)v^*(f(y)), g(x, y)v^*(b(y)))v = g(x, y)T(f(y), b(y))$, for all $(x, y) \in KS_L$, so $T(f, b)$ belongs to $KS(L)$. Therefore $KS(L)$ satisfies (a) and (b) of Definition 3.10. The proof that $G(L)$ is a Lindenstrauss-Wulbert subspace is similar.

3.13 Corollary. *Let L be a vector subspace of $CV_\infty(X; E)$. If L is a KS -space (respectively a G -space), then its closure is a latticial subspace (respectively a Lindenstrauss-Wulbert subspace).*

3.14 Theorem. *Let L be a vector subspace of $CV_\infty(X; E)$, which is a latticial subspace (respectively a Lindenstrauss-Wulbert subspace). Assume that $A(KS)$ (respectively $A(G)$) separates the equivalence classes modulo KS_L (respectively G_L). Then L is a Kakutani-Stone space (respectively a Grothendieck space).*

Proof. Let $\Delta \in \{KS, G\}$. Let $M = \{u^*(f); u^* \in E^*, f \in L\}$. From Proposition 3.11, M is a Δ -space contained in $CV_\infty(X; \mathbb{R})$ such that $M \otimes E \subset L$. Since $\Delta(M) = \bar{M}$, $\bar{M} \otimes E$ is an $A(\Delta)$ -module. Let $f \in CV_\infty(X; E)$ belong to $\Delta(L)$. Let $Y \subset X$ be an equivalence class modulo $X/A(\Delta)$. Since $A(\Delta)$ separates the equivalence classes modulo Δ_L , the set Y is contained in some equivalence class modulo Δ_L . Fix $x_0 \in Y$. If $f(x_0) = 0$, then $f(x) = 0$ for all $x \in Y$, and the function $0 \in M \otimes E$ agrees with f throughout Y . If $f(x_0) \neq 0$, there is some $b \in L$ such that $b(x_0) \neq 0$. Choose $u^* \in E^*$ such that $u^*(b(x_0)) = 1$. Let $k = u^*(b)/f(x_0) \in M \otimes E$. Then k agrees with f throughout Y . By Lemma 2.8, f belongs to the closure of $\bar{M} \otimes E$. Since $\bar{M} \otimes E$ is contained in the closure of L , L is a Δ -space.

3.15 Remark. The hypothesis that $A(\Delta)$ separates the equivalence classes modulo Δ_L is equivalent to the hypothesis that $C_b(F; \mathbb{R})$ separates points, where F is the quotient space of X by the equivalence relation Δ_L . One instance in which this occurs is the following. Suppose that our completely regular Hausdorff space X is in fact a normal space and that Δ_L is an upper semicontinuous relation, i.e., Δ_L is closed (Willard [12, Theorem 9.9]). Then, F is a normal Hausdorff space too (Engelking [4, Theorem 5, p. 85]). Another instance occurs when Δ_L reduces to the diagonal. Indeed, in this case, $A(\Delta)$ is $C_b(X; \mathbb{R})$. Notice that, in this case, as was pointed out in 2.10, $\Delta(L)$ is the set of all $f \in CV_\infty(X; E)$ which vanish on the set $Z(L) = \{x \in X; b(x) = 0 \text{ for all } b \in L\}$.

3.16 Corollary. *Let L be a vector subspace of $CV_\infty(X; E)$, which is a latticial subspace (respectively a Lindenstrauss-Wulbert subspace). Suppose that KS_L (respectively G_L) reduces to the diagonal. Then, $f \in CV_\infty(X; E)$ belongs to the closure of L , if and only if, f vanishes on the set $Z(L) = \{x \in X; b(x) = 0 \text{ for all } b \in L\}$.*

3.17 Corollary. *Let L be as in Corollary 3.16, and suppose that KS_L (respectively G_L) reduces to the diagonal and that L is everywhere different from zero. Then L is dense in $CV_\infty(X; E)$.*

3.18 Remark. Our Theorem 3.14 does not contain Theorem 1.13 of Blatter

[2]. The argument presented there rests on the fact that $C_\infty(X; \mathbb{R})$ and the KS-hull (respectively the G -hull) of M have the approximation property. However, his arguments can be used to prove the following generalization of Theorem 1.13 of [2].

3.19 Theorem. *Let X be a completely regular Hausdorff k -space, and let E be a complete locally convex Hausdorff space. Suppose that for every compact subset $K \subset X$ and every $t > 0$, there exists $v \in V$ such that $v(x) \geq t$ for all $x \in K$. Let L be a vector subspace of $CV_\infty(X; E)$ which is a latticial subspace, respectively a Lindenstrauss-Wulbert subspace. Then L is a KS-space, respectively a G -space.*

Proof. The space $CV_\infty(X; \mathbb{R})$ has the approximation property, by Theorem 3, §5, Bierstedt [1].

4. Examples and applications. Let V be the set of characteristic functions of compact subsets of X . The Nachbin space $CV_\infty(X; E)$ is just $C(X; E)$ endowed with the compact-open topology. A subset $L \subset C(X; E)$ is called selfadjoint if $M = \{u^*(f); u^* \in E^*, f \in L\}$ is a selfadjoint subset of $C(X; \mathbb{K})$.

4.1 Theorem (Weierstrass-Stone). *Let L be a selfadjoint polynomial subalgebra of $C(X; E)$. A function $f \in C(X; E)$ belongs to the closure of L if, and only if, the following conditions hold:*

- (1) *For every $x, y \in X$ such that $f(x) \neq f(y)$, there is $b \in L$ such that $b(x) \neq b(y)$.*
- (2) *For every $x \in X$, such that $f(x) \neq 0$, there is $b \in L$ such that $b(x) \neq 0$.*

Proof. Since every selfadjoint subalgebra of $C(X; \mathbb{K})$ is localizable [6, Theorem 1, §30], Theorem 2.9 implies that L is a Weierstrass-Stone space, and therefore its closure is precisely $WS(L)$. It remains only to notice that $f \in WS(L)$ if, and only if, conditions (1) and (2) are satisfied.

4.2 Corollary. *Let L be a selfadjoint polynomial subalgebra of $C(X; E)$, separating and everywhere different from zero on X . Then L is dense in $C(X; E)$.*

4.3 Corollary. *Let A be a selfadjoint subalgebra of $C(X; \mathbb{K})$, separating and everywhere different from zero on X . Then $A \otimes E$ is dense in $C(X; E)$. In particular, $C(X; \mathbb{K}) \otimes E$ is dense in $C(X; E)$.*

Suppose that X is the Cartesian product of two completely regular Hausdorff spaces X_1 and X_2 . Let L be the set of all finite sums of functions of the form $(x_1, x_2) \mapsto f(x_1)b(x_2)$, where $f \in C(X_1; \mathbb{K})$ and $b \in C(X_2; E)$. Dieudonné's approximation theorem on Cartesian products, stating that $C(X_1; \mathbb{K}) \otimes C(X_2; E)$ is dense in $C(X_1 \times X_2; E)$, is now an easy consequence of Corollary 4.2.

4.4 Theorem. *Let E and F be two locally convex Hausdorff spaces over the reals. The closure in the compact-open topology of the set of all continuous polynomials p of finite type from E into F , with $p(0) = 0$, consists of all mappings $f \in C(E; F)$ such that $f(0) = 0$.*

Proof. This follows from a direct application of Theorem 4.1.

4.5 Theorem (Infinite-dimensional Weierstrass polynomial approximation theorem). *Let E and F be two locally convex Hausdorff spaces over the reals. Then $P_f(E; F)$ is dense in $C(E; F)$.*

Proof. Apply Corollary 4.2 to $L = P_f(E; F)$.

4.6 Remark. The space $P_f(E; F)$ is not selfadjoint in the complex case. Instead of $P_f(E; F)$ one has to consider the vector subspace $P_f^*(E; F)$ defined as follows. For each integer $n \geq 1$, $P_f^{*(n)}(E; F)$ denotes the vector subspace of $C(E; F)$ generated by the set of all maps of the form $x \mapsto u^*(x)^n u$, where $u \in F$ and u^* or its complex conjugate belongs to E^* . Then $P_f^*(E; F)$ is, by definition, the vector subspace generated by the union of all $P_f^{*(n)}(E; F)$ with $n \geq 1$ and the constant maps. Theorem 4.5 now holds with $P_f^*(E; F)$ substituted for $P_f(E; F)$.

Let X be a locally compact Hausdorff space and let $V = C_\infty^+(X)$, the set of all positive continuous functions vanishing at infinity on X . The Nachbin space $CV_\infty(X; E)$ is then $C_b(X; E)$ endowed with the strict topology β . Again we are in the bounded case of the approximation problem, and therefore every selfadjoint subalgebra of $C_b(X; \mathbb{K})$ is localizable under itself in the strict topology. Hence Theorem 4.1 and its corollaries hold when $C_b(X; E)$ with the strict topology is substituted for $C(X; E)$ with the compact-open topology.

5. Extension theorems. Let Y be a closed subset of X . Then, for every weight ν on X , its restriction to Y is a weight on Y . If V is a set of weights on X , we denote by $V|Y$ the set of weights on Y that are restriction to Y of some element of V . The corresponding Nachbin space $C(V|Y)_\infty(Y; E)$ will be denoted by $CV_\infty(Y; E)$. Let L be a vector subspace of $CV_\infty(X; E)$. A function $f \in CV_\infty(Y; E)$ is said to be *extensible in L* if there exists $h \in L$ such that $h|Y = f$, i.e. $h(x) = f(x)$ for all $x \in Y$. The set of all such extensible functions is denoted by $L|Y$. Notice that, if L is a polynomial algebra, or a latticial subspace, or a Linderstrauss-Wulbert subspace, the same is true of $L|Y$ as subspace of $CV_\infty(Y; E)$.

5.1 Theorem. *Let Y be a closed subset of a completely regular Hausdorff space X . Let L be a vector subspace of $CV_\infty(X; E)$ which is a polynomial algebra such that every $h \in L$ is bounded on the support of every $\nu \in V$. Assume that L is selfadjoint and that $L|Y$ is closed in $CV_\infty(Y; E)$. Then $f \in CV_\infty(Y; E)$ is extensible in L if, and only if, the following conditions are satisfied:*

(1) For any $x, y \in Y$ such that $f(x) \neq f(y)$, there is $b \in L$ such that $b(x) \neq b(y)$.

(2) For any $x \in Y$ such that $f(x) \neq 0$, there is $b \in L$ such that $b(x) \neq 0$.

Proof. Under the hypothesis made, $L|Y$ is a WS -space.

5.2 Corollary. Let Y and L be as in Theorem 5.1. If $L|Y$ is separating and everywhere different from zero on Y , then every $f \in CV_\infty(Y; E)$ is extensible in L .

5.3 Theorem. Let Y be a compact subset of a completely regular Hausdorff space X , let E be a Banach space, and let $f \in C(Y; E)$. There is $b \in C_b(X; E)$ such that $b|Y = f$ and $\|f\|_Y = \|b\|_X$.

Proof. Let $L = C_b(X; E) \subset C(X; E)$. An obvious modification of the argument given in Stone [11, p. 64] proves that $L|Y$ is closed in $C(Y; E)$. Since X is completely regular, $C_b(X; E)$ is separating and everywhere different from zero on X , in particular on Y . By Corollary 5.2, every $f \in C(Y; E)$ is extensible in $C_b(X; E)$. The final statement follows from the fact that for any function $f \in C(X; E)$, which is bounded on Y , there is $b \in C_b(X; E)$ such that $b|Y = f$ and $\|f\|_Y = \|b\|_X$.

5.4 Theorem. Let Y be a closed subset of a normal Hausdorff space X , let E be a Banach space, and let $f \in C(Y; E)$ be a compact mapping. There exists a compact mapping $b \in C(X; E)$ such that $b|Y = f$ and $b(X)$ is contained in the closed convex hull of $f(Y)$.

Proof. If βX and βY denote the Stone-Čech compactifications of X and Y respectively, it is known that βY identifies with the closure of Y in βX , (see [10]) and that a mapping $f \in C(Y; E)$ has an extension to βY if, and only if, $f(Y)$ is precompact. Hence we can extend f to βY though a function $w \in C(\beta Y; E)$. By Theorem 5.3 we can extend w to βX . If v denotes such an extension, let $u = v|X$. Then $u \in C(X; E)$ and $u|Y = f$. Since E is a Banach space, the closed convex hull K of $f(Y)$ is compact and, therefore, a retract of E (see [3]). Let r be a retraction of E onto K . Then $b = r(u)$ is a compact mapping that answers all the requirements.

5.5 Theorem. Let Y be a closed subset of a completely regular Hausdorff space X , let E be a Banach space. A function $f \in C_\infty(Y; E)$ is extensible in $C_\infty(X; E)$ if, and only if, the following conditions hold:

(1) For any $x, y \in Y$ such that $f(x) \neq f(y)$, there is $b \in C_\infty(X; E)$ such that $b(x) \neq b(y)$.

(2) For any $x \in Y$ such that $f(x) \neq 0$, there is $b \in C_\infty(X; E)$ such that $b(x) \neq 0$.

If $f \in C_\infty(Y; E)$ is extensible, we can choose an extension $h \in C_\infty(X; E)$ such that $\|f\|_Y = \|h\|_X$.

Proof. Take $L = C_\infty(X; E)$. As in Theorem 5.3, $L|Y$ is closed in $C_\infty(Y; E)$. Theorem 5.5 follows then from Theorem 5.1.

5.6 Corollary. Let Y be a closed subset of a locally compact Hausdorff space X , let E be a Banach space. Then, every $f \in C_\infty(X; E)$ is extensible in $C_\infty(X; E)$, and we can choose an extension with the same norm.

Proof. Since X is locally compact, the space $C_\infty(X; E)$ is separating and everywhere different from zero.

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INSTITUTO DE MATEMATICA, UNIVERSIDADE FEDERAL DO RIO DE JANEIRO, CAIXA POSTAL 1835, ZC-00, 20000 RIO DE JANEIRO, GB, BRASIL