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# $K_1$ OF A CURVE OF GENUS ZERO(1)

BY

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ABSTRACT. We determine the structure of the vector bundles on a curve of genus zero and calculate the "universal determinant" $K_1$ of such a curve.

**1. Introduction.** Let $F$ be a field. Then there is a bijection between (nonsingular projective irreducible) curves of genus zero over $F$ and central simple algebras of rank 4 over $F$. If $X$ is such a curve then $X$ is isomorphic to a plane curve of degree 2, and there is a separable extension $[K: F]$ of degree 2 such that $X \times_F K \cong P^1_K$, the projective line over $K$.

Let $\mathcal{V}$ be the category of vector bundles (= locally free sheaves of finite type) on $X$ and $\mathfrak{M}$ the (abelian) category of coherent sheaves on $X$. Let $K_1$ be the "universal determinant" $K_1$ as defined in [3, Chapter VIII]. The groups $K_1(\mathcal{V})$ and $K_1(\mathfrak{M})$ are both defined. Set $K_1(X) = K_1(\mathcal{V})$. In this paper we prove that if $X$ is the curve of genus zero over $F$ corresponding to the central simple algebra $A$ then $K_1(X) \cong K_1(F) \oplus K_1(A)$. At the end of the paper it is proved that the inclusion of categories $\mathcal{V} \to \mathfrak{M}$ induces an isomorphism $K_1(\mathcal{V}) \to K_1(\mathfrak{M})$ so $K_1(X)$ could have been defined with coherent sheaves instead of vector bundles. If $A$ is the ring of $2 \times 2$ matrices over $F$, then $X = P^1_F$ and the formula reads $K_1(X) \cong K_1(F) \oplus K_1(A) = F^* \oplus F^*$ (where $F^*$ denotes the nonzero elements of $F$). This has already been proved in [8] or [9] (working with coherent sheaves in the first case and vector bundles in the second) so we can confine ourselves to the case where $A$ does not split, i.e. is a division ring of rank 4.

Recently [7] Quillen has developed a theory of higher $K$'s for schemes, and in [7] he calculates the $K$-theory for Severi-Brauer schemes, the simplest example of which are the curves of genus zero. The result that I have obtained agrees with his, although Gersten in [6] has proved that if $X$ is a nonsingular elliptic curve over $\mathbf{C}$ Quillen's $K_1$ is not the same as the "universal determinant" $K_1$.

This paper is based on the second half of [11]. Throughout $\mathbf{Z}$ denotes the integers, $\mathbf{R}$ the real numbers, and $\mathbf{C}$ the complex numbers.

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**2. The structure of vector bundles on $X$.** The results in this section seem to be more or less "known" (see the discussion in [7, §8]). I include them here because I need the detailed description of the vector bundles for my calculation of $K_1$ and do not know a suitable reference.

Let $X$ be a curve of genus 0 over the field $F$, $X \not\cong P^1_F$, and let $K$ be a separable extension of degree 2 such that $X_K = X \times_F K \cong P^1_K$. If $f: X_K \to X$ is the morphism obtained from the change of field, then $f^*$ is an injection on isomorphism classes of vector bundles or coherent sheaves [12, remark at the end of §1]. We use this, together with the known structure of vector bundles on $P^1_K$ to determine the structure of vector bundles on $X$.

The Krull-Schmidt theorem holds for vector bundles on $X$ and $X_K$ because all Hom's are finite dimensional vector spaces over the ground field [2]. That is, every vector bundle can be written in a unique manner as the direct sum of indecomposable vector bundles. If we write $P^1_K = \mathrm{Proj}\ K[T_0, T_1]$ and let $\mathcal{O}(1)$ be the canonical line bundle determined by this projective structure, then the indecomposable vector bundles on $X_K$ are just the line bundles $\mathcal{O}(n)$. Some discussion of this can be found in [10]. Furthermore $\Gamma(\mathcal{O}(n))$ is a vector space of dimension $n+1$ over $K$ if $n \geq 0$ and is zero otherwise. Also $\mathrm{Hom}\,(\mathcal{O}(m), \mathcal{O}(n)) = \Gamma(\mathcal{O}(-m) \otimes \mathcal{O}(n)) = \Gamma(\mathcal{O}(n-m))$. Therefore there are no nonzero morphisms from $\mathcal{O}(m)$ to $\mathcal{O}(n)$ unless $n \geq m$.

The Picard group of $X_K$ is $\mathbf{Z}$, generated by $\mathcal{O}(1)$. The Picard group of $X$ is $\mathbf{Z}$ also, generated by $\mathcal{O}(1)$, where $\mathcal{O}(1)$ is defined by the projective structure of $X$ as a second degree curve. Also $f^*\mathcal{O}(1) = \mathcal{O}(2)$. (I will write simply $\mathcal{O}(n)$, it being clear from the context whether this is a bundle on $X$ or $X_K$.)

The following lemma will be used throughout this discussion:

**Lemma.** *Let $Y$ be a scheme of finite type over a noetherian ring $A$. Let $B$ be a flat $A$-algebra, and let $U$ be an open subset of $Y$. Write $Y' = Y \otimes_A B$ and let $f: Y' \to Y$ be the morphism induced by change of base. Let $N$ be a quasicoherent sheaf on $Y$ and $N' = f^*(N)$. Then $\Gamma(f^{-1}(U), N') = B \otimes_A \Gamma(U, N)$.*

**Proof.** First of all the lemma is true if $U$ is affine, by the construction of product in the category of schemes and the behavior of $f^*$ in the affine case. If $U$ is not affine then $U$ can be covered by a finite number of affine schemes $U_i$ $(1 \leq i \leq n)$. If we write $U_{ij} = U_i \cap U_j$ then $U_{ij}$ will be affine also. We have an exact sequence

$$0 \to \Gamma(U, N) \to \Pi_i \Gamma(U_i, N) \rightrightarrows \Pi_{i,j} \Gamma(U_{ij}, N)$$

since $N$ is a sheaf. Tensoring with $B$ and using the lemma for affine sets we get an exact sequence

$$0 \to B \otimes_A \Gamma(U, N) \to \Pi_i \Gamma(f^{-1}(U_i), N') \rightrightarrows \Pi_{i,j} \Gamma(f^{-1}(U_{ij}), N').$$

But $f^{-1}(U)$ is covered by the $f^{-1}(U_i)$ and $f^{-1}(U_{ij}) = f^{-1}(U_i) \cap f^{-1}(U_j)$. Therefore we have an exact sequence

$$0 \to \Gamma(f^{-1}(U), N') \to \Pi_i \Gamma(f^{-1}(U_i), N') \rightrightarrows \Pi_{i,j} \Gamma(f^{-1}(U_{ij}), N').$$

Therefore $B \otimes_A \Gamma(U, N) = \Gamma(f^{-1}(U), N')$ as required.

In the application $A = F$, $B = K$ and $Y = U = X$. The lemma is false if $B$ is not a flat $A$-module. For example, let $A = K[T_1, T_2]$, $Y = \operatorname{Spec} A$, $U = \operatorname{Spec} A -$ (origin) and $B = K$, the homorphism $A \to B$ given by sending $T_1$ and $T_2$ to 0. If $N$ is the structure sheaf of $Y$, then $\Gamma(U, N) = A$, and $B \otimes_A \Gamma(U, N) = B$. But $f^{-1}(U) = \emptyset$, so $\Gamma(f^{-1}(U), N') = 0$. (This example was pointed out to me by Paul-Jean Cahen.)

The structure of the vector bundles on $X$ is given by the following theorem:

**Theorem 1.** *Let $X$ be a curve of genus 0 over the field $F$, $X \not\cong P^1_F$, and let $K$ be a separable extension of degree 2 such that $X_K = X \times_F K \cong P^1_K$. Let $f: X_K \to X$ be the morphism obtained from change of base. Then the vector bundle $E(n) = f_*\mathcal{O}(n)$ ($n$ odd) on $X$ is indecomposable of rank 2. Every vector bundle on $X$ can be written uniquely (up to order of summands) as the direct sum of line bundles $\mathcal{O}(n)$, and the bundles $E(n)$.*

**Proof.** First we show that $E(n)$ is indecomposable. If $E(n)$ is decomposable, then $E(n) = \mathcal{O}(n_1) \oplus \mathcal{O}(n_2)$ and $f^*E(n) = \mathcal{O}(2n_1) \oplus \mathcal{O}(2n_2)$. The Galois group $\mathbf{Z}/2\mathbf{Z}$ of $K$ over $F$ acts on the vector bundles on $X_K$ in an obvious way (denoted by a $\bar{\ }$). By (2′) of [12] we have $f^*f_*\mathcal{O}(n) = \mathcal{O}(n) \oplus \overline{\mathcal{O}(n)}$. The equation $f^*\mathcal{O}(1) = \mathcal{O}(2)$ proves that the Galois group acts trivially. Therefore we must have $f^*(E(n)) = \mathcal{O}(n) \oplus \mathcal{O}(n)$. Hence $E(n)$ must be indecomposible, otherwise the Krull-Schmidt theorem would be violated.

If $n$ is even, $f_*\mathcal{O}(n) = \mathcal{O}(n/2) \oplus \mathcal{O}(n/2)$. For we get $\mathcal{O}(n) \oplus \mathcal{O}(n)$ on both sides if we apply $f^*$, and $f^*$ is an injection on isomorphism classes. Now suppose that $V$ is a vector bundle on $X$. Then $f^*(V) = \bigoplus_i \mathcal{O}(n_i)$, so $f_*f^*(V) = V \oplus V$ is the direct sum of the $\mathcal{O}(n_i)$ and $E(n_i)$. By the Krull-Schmidt theorem so also is $V$. This completes the proof of Theorem 1.

I will conclude this section by making some general remarks about the vector bundles on $X$. Let $\operatorname{Hom}_F$ denote morphisms of vector bundles on $X$, and $\operatorname{Hom}_K$ denote morphisms of vector bundles on $X_K$. First of all, $\operatorname{Hom}_F(\mathcal{O}(n), \mathcal{O}(m)) = 0$ if $n > m$ and is nonzero if $n \leq m$. Also $\operatorname{Hom}_F(\mathcal{O}(n), E(m)) \otimes_F K =$

$\mathrm{Hom}_K (f^*\mathcal{O}(n), f^*E(m)) = \mathrm{Hom}_K (\mathcal{O}(2n), \mathcal{O}(m) \oplus \mathcal{O}(m))$ so $\mathrm{Hom}_F (\mathcal{O}(n), E(m)) = 0$ if $2n > m$ and is nonzero if $2n < m$. By applying $f^*$ and using the fact that $f^*$ is an injection on isomorphism classes we can prove that $E(n) \otimes \mathcal{O}(m) \cong E(n+2m)$, $E(n)^* = E(-n)$ (*denotes dual) and $E(n) \otimes E(m) \cong 4\,\mathcal{O}((m+n)/2)$. From the last isomorphism it follows that $\mathrm{Hom}\,(E(n), E(m)) \cong E(-n) \otimes E(m) \cong 4\,\mathcal{O}((m-n)/2)$. Thus $\mathrm{Hom}_F (E(n), E(m)) = 0$ if $n > m$ and is nonzero if $n \leq m$. Hence we may linearly order the vector bundles $\cdots\ E(-3), \mathcal{O}(-1), E(-1), \mathcal{O}, E(1), \mathcal{O}(1), E(3), \mathcal{O}(2), \cdots$ with nonzero morphisms going only to the right. One can also show that $\Lambda^2 E(1) = \mathcal{O}(1)$ by applying $f^*$ to both sides.

Now we consider $\mathrm{Hom}_F (E(n), E(n))$. From the above it is a 4 dimensional vector space over $F$, and since $E(n)$ is indecomposable, there are no nontrivial idempotents. Finally $\mathrm{Hom}_F (E(n), E(n)) \otimes_F K$ is the ring of $2 \times 2$ matrices over $K$. Thus $\mathrm{Hom}_F (E(n), E(n))$ is semisimple and therefore a division ring over $F$. The $\mathrm{Hom}_F (E(n), E(n))$ are all isomorphic, since $E(n) \otimes \mathcal{O}(m) \cong E(n+2m)$.

More precisely, if $F$ is a field of characteristic $\neq 2$ then the equation for for the plane curve $X$ (in homogeneous co-ordinates) is (for suitable choice of variables) $T_0^2 - aT_1^2 - bT_2^2 = 0$, $a, b \in F$ and $\mathrm{Hom}_F (E(-1), E(-1))$ is isomorphic to the quaternion algebra $(a, b)$ (as defined on p. 96 of [13]). If the characteristic $F$ is 2, then $X$ is given by the equation $aT_1^2 + T_1T_2 + bT_2^2 + cT_0^2 = 0$ with $a, b, c \in F$, and $\mathrm{Hom}_F (E(-1), E(-1))$ is isomorphic to the Clifford algebra of the quadratic form $acv^2 + cuv + bcv^2$ as defined in [1, p. 150]. The characteristic $\neq 2$ case was proved by a straightforward but tedious calculation in [11] and the characteristic 2 case can be proved in a similar manner. I will omit these proofs because all we need to know for the calculation in §3 is that $\mathrm{Hom}_F (E(-1), E(-1))$ is a division ring of dimension 4 over its centre $F$. In fact, $\mathrm{Hom}_F (E(-1), E(-1))$ is just the central simple algebra corresponding to $X$ in the bijection mentioned at the beginning of the paper.

3. **Calculation of $K_1$.** We now calculate the group $K_1(X)$, where $X$ is as in §2. Let $V$ be a vector bundle on $X$, with automorphism $\alpha$. Let $V = n_1V_1 \oplus n_2V_2 \oplus \cdots \oplus n_rV_r$ be an expression for $V$ as the direct sum of indecomposable vector bundles $V_i$ which are ordered so that there exist nonzero morphisms $V_i \to V_j$ if and only if $i \leq j$. Using this direct sum decomposition $\alpha$ can be represented by a lower triangular matrix, with $r$ $n_i \times n_i$ blocks $\alpha_i$ down the diagonal having entries in either $F$ or $A$ depending on whether $V_i$ is of rank 1 or 2. Here $A = \mathrm{Hom}_F (E(-1), E(-1))$, which is the quaternion algebra $(a, b)$ that determines the curve if the characteristic $\neq 2$, or a certain Clifford algebra is characteristic $= 2$. In both cases $A$ is a division ring. The $\alpha_i$ are invertible. One of the defining relations of $K_1$ is that if we have a short exact sequence $0 \to V_1 \to V_2 \to V_3 \to 0$ in $\mathcal{V}$ and $\beta_1, \beta_2, \beta_3$ are automorphisms such that

$$\begin{array}{ccccccccc} 0 & \to & V_1 & \to & V_2 & \to & V_3 & \to & 0 \\ & & \downarrow \beta_1 & & \downarrow \beta_2 & & \downarrow \beta_3 & & \\ 0 & \to & V_1 & \to & V_2 & \to & V_3 & \to & 0 \end{array}$$

is commutative, then $\kappa_1(V_2, \beta_2) = \kappa_1(V_1, \beta_1) + \kappa_1(V_3, \beta_3)$, where $\kappa_1$ denotes the canonical image in $K_1$. Repeated application of this proves that $\kappa_1(V, \alpha) = \Sigma_{i=1}^{r} \kappa_1(n_i V_i, \alpha_i)$. Write $A^*$ and $F^*$ for the nonzero elements of $A$ and $F$ respectively. One sees easily that $\kappa_1(n_i V_i, \alpha_i) = \kappa_1(V_i, a_i)$ for some $a_i \in A^*$ or $F^*$ (depending on whether rank $V_i = 2$ or 1). Thus we have found generators for $K_1(X)$. We now try to reduce the number of generators by using exact sequences.

First of all, there is an exact sequence $0 \to \mathcal{O}(-1) \to \mathcal{O} \oplus \mathcal{O} \to \mathcal{O}(1) \to 0$ on $X$ since $\mathcal{O}(1)$ is generated by two global sections. (The kernel of the resulting map $\mathcal{O} \oplus \mathcal{O} \to \mathcal{O}(1) \to 0$ is a line bundle and must be $\mathcal{O}(-1)$ because the degree is additive.) Tensoring with $\mathcal{O}(n)$ we get exact sequences of the form $0 \to \mathcal{O}(n-1) \to \mathcal{O}(n) \oplus \mathcal{O}(n) \to \mathcal{O}(n+1) \to 0$ and these enable us to replace all the generators of the form $\kappa_1(\mathcal{O}(n), \lambda)$ by those of the form $\kappa_1(\mathcal{O}, \lambda)$ and $\kappa_1(\mathcal{O}(1), \lambda)$, $\lambda \in F^*$. Tensoring with $E(n)$ yields exact sequences of the form $0 \to E(n-2) \to E(n) \oplus E(n) \to E(n+2) \to 0$ and these enable us to replace the generators $\kappa_1(E(n), \mu)$, $\mu \in A^*$, by those of the form $\kappa_1(E(-1), \mu)$ and $\kappa_1(E(1), \mu)$. There is a nonzero morphism $E(1) \to \mathcal{O}(1)$ which must be onto, otherwise the image would be isomorphic to $\mathcal{O}(n)$ for some $n \leq 0$ and there are no nonzero maps $E(1) \to \mathcal{O}(n)$, $n \leq 0$. The kernel is a line bundle, which must be isomorphic to $\mathcal{O}$ since we have seen that $\Lambda^2 E(1) \cong \mathcal{O}(1)$. Therefore we have an exact sequence $0 \to \mathcal{O} \to E(1) \to \mathcal{O}(1) \to 0$. This enables us to get rid of the generators of the form $\kappa_1(\mathcal{O}(1), \lambda)$, $\lambda \in F^*$. Finally, on $X_K$ we have an exact sequence $0 \to \mathcal{O}(-1) \to \mathcal{O} \oplus \mathcal{O} \to \mathcal{O}(1) \to 0$. If we apply $f_*$ we get an exact sequence $0 \to E(-1) \to \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O} \to E(1) \to 0$. The map $\mathcal{O} \oplus \mathcal{O} \to \mathcal{O}(1) \to 0$ on $X_K$ is onto on global sections. Therefore so also is the map $4\mathcal{O} \to E(1)$ on $X_F$ (since the global sections remain the same). If $\mu \in \operatorname{Aut} E(1) = A^*$ then $\mu$ induces an automorphism of $\Gamma E(1)$ which is a 4 dimensional vector space over $F$ (as can be seen by applying $f^*$). Therefore $\mu$ can be lifted to an automorphism of $4\mathcal{O}$, and hence to an automorphism of the whole exact sequence $0 \to E(-1) \to 4\mathcal{O} \to E(1) \to 0$. By taking duals any automorphism of $E(-1)$ also extends to an automorphism of the sequence. The generators of $K_1(X)$ are finally reduced to elements of the form $\kappa_1(\mathcal{O}, \lambda)$ and $\kappa_1(E(1), \mu)$ for $\lambda \in F^*$, $\mu \in A^*$. This can be rephrased by saying there is a surjection $\phi$: $F^* \oplus H^* \to K_1(X)$ defined by $\phi(\lambda) = \kappa_1(\mathcal{O}, \lambda)$ $(\lambda \in F^*)$ and $\phi(\mu) = \kappa_1(E(1), \mu)$ $(\mu \in H^*)$.

Now consider the reduced norm $N: A^* \to F^*$. It is proved in [14, Corollary p. 334], that the kernel of $N$ is the commutator subgroup of $A^*$ (the index being 2 which is square free). Let $NA^*$ denote the image of $N$. The abelianized group of $A^*$ is therefore $NA^*$. Therefore $\phi$ induces a surjection (also denoted $\phi$) $\phi: F^* \oplus NA^* \to K_1(X)$.

We now have homomorphisms det: $K_1(X) \to F^*$ and $\chi: K_1(X) \to F^*$. The map det is defined by taking exterior powers. If $V$ is a vector bundle of rank $r$, then $\alpha \in \operatorname{Aut} V$ induces an automorphism $\det \alpha$ of $\Lambda^r V$. But $\Lambda^r V$ is a line bundle so $\operatorname{Aut} \Lambda^r V \cong F^*$ (canonically). Then $\det \kappa_1(V, \alpha) = \det \alpha$. The vector spaces $H^0(X, V)$ and $H^1(X, V)$ are finite dimensional, and $\alpha \in \operatorname{Aut} V$ induces automorphisms of these vector spaces. These automorphisms will be denoted $\alpha_0$ and $\alpha_1$ respectively. Then $\chi(V, \alpha) = (\det \alpha_0)(\det \alpha_1)^{-1}$. If $0 \to (V_1, \alpha_1) \to (V, \alpha) \to (V_2, \alpha_2) \to 0$ is exact, then $\chi(V, \alpha) = \chi(V_1, \alpha_1)\chi(V_2, \alpha_2)$ by the exact sequence of cohomology. That $\chi(V, \alpha\beta) = \chi(V, \alpha)\chi(V, \beta)$ is obvious, so $\chi$ defines a homomorphism $\chi: K_1(X) \to F^*$.

We now examine what these homomorphisms do to the generators of $K_1(X)$. It is clear that $\det(\mathcal{O}, \lambda) = \lambda$, $\lambda \in F^*$. End $E(1) = A$ is a subalgebra of $\operatorname{End} E(1) \otimes_F K = \operatorname{End}_K (\mathcal{O}(1) \oplus \mathcal{O}(1))$ which is the ring of $2 \times 2$ matrices over $K$. Det commutes with base change. Therefore by the definition of reduced norm [5, p. 142], we have that $\det(E(1), \mu) = N\mu (\mu \in A^*)$. The Riemann-Roch theorem says that $\dim_F H^0(X, V) - \dim_F H^1(X, V) = \text{degree}(\Lambda^r V) + r$, where rank $V = r$. If we take $V = \mathcal{O}$, then degree $\mathcal{O} = 0$, $r = 1$, so we get $\dim H^1(X, \mathcal{O}) = 0$. Therefore $\chi(\mathcal{O}, \lambda) = \lambda$ also. If we take $V = E(1)$, then $\dim H^0(X, E(1)) = 4$, degree $(\Lambda^2 E(1)) =$ degree $\mathcal{O}(1) = 2$, and $r = 2$. Therefore $\dim H^1(X, E(1)) = 0$. Thus $\chi(E(1), \mu) = \det \mu_0$. But $H^0(X, E(1)) = \Gamma(E(1))$ is a one dimensional vector space over $A$, so $\det \mu_0$ is the usual norm, which is the square of the reduced norm. That is, $\det \mu_0 = (N\mu)^2$.

Now define a homomorphism $\psi: K_1(X) \to F^* \oplus F^*$ by $\psi = ((\det)^2\chi^{-1}, \chi(\det)^{-1})$. Then $\psi\kappa_1(\mathcal{O}, \lambda) = (\lambda, 1)$ and $\psi\kappa_1(E(1), \mu) = (1, N\mu)$. Therefore the image of $\psi$ is $F^* \oplus NA^*$ and $\psi\phi: F^* \oplus NA^* \to F^* \oplus NA^*$ is the identity. We have already seen that $\phi$ is onto. Therefore $\phi$ is an isomorphism. This proves

**Theorem 2.** *Let $F$ be a field and let $X$ be a nonsingular curve of genus 0 which is not isomorphic to $P^1_F$. Let the division algebra $A$ be the endomorphism ring of the indecomposable vector bundle $E(1)$ of rank 2 on $X$. Let $NA^*$ denote the image of the reduced norm $N: A^* \to F^*$. Then there is an isomorphism $\phi: F^* \oplus NA^* \to K_1(X)$. (Note that $F^* = K_1(F)$ and $NA^* = K_1(A)$.)*

4. **Further remarks.** We first consider the homomorphism $\Phi: K_0(X) \otimes_Z F^* \to K_1(X)$ defined by $\Phi\kappa_0(V) \otimes \lambda = \kappa_1(V, \lambda)$. Here $K_0$ denotes the Grothendieck

group of vector bundles with relations coming from short exact sequences, and $\kappa_0(V)$ denotes the image of $V$ in $K_0$. Then $K_0(X) = \mathbf{Z} \oplus \mathbf{Z}$ with the second copy of $\mathbf{Z}$ being the Picard group. If we use this to identify $K_0(X) \otimes_{\mathbf{Z}} F^*$ with $F^* \oplus F^*$, then $\psi\Phi(\lambda, \mu) = \psi\kappa_1(\mathcal{O}, \lambda) + \psi\kappa_1(\mathcal{O}(1), \mu) = (\lambda, 1) + (\mu^{-1}, \mu^2) = (\lambda\mu^{-1}, \mu^2)$. If $F = \mathbf{R}$, since $\psi$ is an isomorphism, we see that $\Phi$ is onto but has nontrivial kernel $(-1, -1)$. We note that in general $NA^*$ will be bigger than $(F^*)^2$. If this is the case $\psi\Phi$ will not be onto, so neither is $\Phi$.

In [8] it was proved that $\Phi$ is an isomorphism for $X$ a projective nonsingular variety over an algebraically closed field.

We now consider the homomorphism $f^*: K_1(X) \to K_1(X_K)$ induced by the change of base. $K_1(X_K) = K^* \oplus K^*$ generated by $\kappa_1(\mathcal{O}, \lambda)$ and $\kappa_1(\mathcal{O}(1), \lambda)$, $\lambda \in K^*$. The action of the Galois group $G = \mathbf{Z}/2\mathbf{Z}$ of $K$ over $F$ on $K_1(X_K)$ is just the obvious action on each copy of $K^*$. The image of $f^*$ is contained in $K_1(X_K)^G$ (the fixed subgroup under the action of $G$) and by §2 of [12], the kernel and cokernel of the map $f^*: K_1(X) \to K_1(X_K)^G = F^* \oplus F^*$ are both killed by 2. One can check (using the above identifications of $K_1(X_K)$ with $K^* \oplus K^*$ and $K_1(X)$ with $F^* \oplus NA^*$) that $f^*(\lambda, \mu) = (\lambda, \mu)$, $\lambda \in F^*$, $\mu \in NA^*$. Therefore $f^*$ is injective, and the cokernel of $f^*: K_1(X) \to K_1(X_K)^G$ is $F^*/NA^*$ which is indeed killed by 2 because every square is a reduced norm.

We conclude by proving that coherent sheaves and vector bundles give the same $K_1$.

**Theorem 3.** *Let $Y$ be a regular projective scheme of finite type over a field $F$. Let $\mathcal{V}$ be the category of vector bundles and $\mathcal{M}$ the category of coherent sheaves on $Y$. Then the homomorphism $K_1(\mathcal{V}) \to K_1(\mathcal{M})$ induced by the inclusion of categories is an isomorphism.*

**Proof.** This follows from Theorem 5, p. 72 of [4]. The hypotheses are all immediate except (c). Let $N$ be an object in $\mathcal{M}$. If $n$ is sufficiently large then $N \otimes \mathcal{O}(n)$ will be generated by its global sections (which form a finite dimensional vector space over $F$ since $Y$ is projective). If $\alpha$ is an endomorphism of $N$, then $\alpha \otimes 1$ induces an endomorphism of the global sections of $N \otimes \mathcal{O}(n)$. Suppose $\dim \Gamma(N \otimes \mathcal{O}(n)) = m$. Choose a basis for $\Gamma(N \otimes \mathcal{O}(n))$. We have a surjection $m\mathcal{O}_Y \to N \otimes \mathcal{O}(n) \to 0$ by mapping the unit sections of the copies of $\mathcal{O}_Y$ to the corresponding basis vectors for $\Gamma(N \otimes \mathcal{O}(n))$. If $\Gamma(\alpha \otimes 1)$ has matrix $A$, then the endomorphism of $m\mathcal{O}$ given by the same matrix lifts $\alpha \otimes 1$. Tensoring with $\mathcal{O}(-n)$ we see that $\alpha$ can be lifted to an endomorphism of $m\mathcal{O}(-n)$, which proves (c). Theorem 3 now follows.

If $X$ were affine a similar result holds by [4, Theorem 3], but if $Y$ is neither affine nor projective then I do not know if the corresponding result holds. If

$A_F^2 = \text{Spec } F[T_0, T_1]$ (the affine plane) and $Y = A_F^2 -$ (origin) then I suspect that $\kappa_1(H, \lambda)$ where $H$ the structure sheaf of Spec $F[T_0]$ restricted to $Y$ and $\lambda$ is multiplication by $T_0$ does not lie in the image of the homomorphism $K_1(\mathcal{C}) \to K_1(\mathfrak{M})$ but I do not know how to prove it.

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