TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 188, Issue 2, 1974

p-ABSOLUTELY SUMMING OPERATORS AND THE REPRESENTATION OF OPERATORS ON FUNCTION SPACES

BY

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ABSTRACT. We introduce a class of p-absolutely summing operators which we call p-extending. We show that for a logmodular function space $A(K)$, an operator $T: A(K) \longrightarrow X$ is p-extending if and only if there exists a probability measure μ on K such that T extends to an isometry $T: A^p(K, \mu) \longrightarrow X$. We use this result to give necessary and sufficient conditions under which a bounded linear operator is isometrically equivalent to multiplication by z on a space $L^p(K, \mu)$ and certain Hardy spaces $H^p(K, \mu)$.

1. Introduction. We wish to discuss the problem of representing an operator on a Banach space as multiplication by the independent variable z on the function spaces $L^p(K, \mu)$ and $H^p(K, \mu)$, where K is a compact subset of the complex plane and μ a positive measure on K such that $\mu(K) = 1$. Perhaps the best known instance of this arises via the spectral theorem for normal operators on a Hilbert space [7]. Given a normal operator N from a certain class called simple normal operators [4], the spectral theorem shows the existence of a positive measure μ on the spectrum $\sigma(N)$ of N, with $\mu(\sigma(N)) = 1$, such that N is unitarily equivalent to multiplication by z on $L^2(\sigma(N), \mu)$. Another important instance is provided by the subnormal operators on a Hilbert space. It is well known [1] that given any subnormal operator S having a cyclic vector, there exists a positive measure μ on the spectrum $\sigma(S)$ of S, with $\mu(\sigma(S)) = 1$ such that S is unitarily equivalent to multiplication by z on $H^2(\sigma(S), \mu)$. Such representations have proved important, for example, in the study of the invariant subspaces of these operators ([2], [18]).

On a general Banach space the analogue of a normal operator is a spectral operator of scalar type (Definition 2.1), which we shall, for the sake of brevity, call a scalar operator. The restriction of such an operator to an invariant subspace will be called a subscalar operator. In §2 we introduce these operators and discuss some of their properties. In particular we show that each type has a certain

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Presented to the Society, March 16, 1972; received by the editors April 30, 1972.

AMS (MOS) subject classifications (1970). Primary 47A60; Secondary 47B40, 47B20.

Key words and phrases. Banach algebra, functional calculus, p-absolutely summing operator, normal operator, subnormal operator, scalar-type spectral operator, topologically cyclic vector, Hardy space, logmodular, $\mathfrak{L}_{p,\lambda}$.

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kind of functional calculus which will be used to establish later results. In §3 we introduce p-absolutely summing operators and the particular properties of them that we shall require. In §4 we show how the theory of p-absolutely summing operators can be applied to the functional calculi obtained in §2, to derive several results concerning a weak form of similarity between operators. We close this section with an example to show that such results cannot in general be improved to similarity. In §5, we introduce p-extending operators, and use these to give necessary and sufficient conditions under which such operators are similar, by an isometry, to multiplication by z on $L^p(K, \mu)$ and $H^p(K, \mu)$ respectively. Finally, in §6, we apply our results to the special case of multiplication by z on $L^1(\sigma(T), \mu)$.

Notation. X and Y will always denote complex Banach spaces, and $L(X)$ the Banach algebra of bounded linear operators on X (with the usual norm). By "operator" we shall always mean an element of $L(X)$. The spectrum of an operator T will be denoted by $\sigma(T)$. The complex numbers will be denoted by C and the reals by R. K will denote a compact Hausdorff space and $C(K)$ (resp. $C_R(K)$) the Banach algebra of complex (resp. real) valued continuous functions on K. $A(K)$ will denote a closed subalgebra of $C(K)$, and if μ is a positive measure on K, $A^p(K, \mu)$ will denote the closed subspace of $L^p(K, \mu)$ generated by $A(K)$. When $K \subset C$, $P(K)$ will denote the closed subalgebra of $C(K)$, and $H^p(K, \mu)$ the closed subspace of $L^p(K, \mu)$ generated by the polynomials on K.

By a probability measure on K we mean a positive Borel measure on K of total mass one.

2. Scalar and subscalar operators. For a complete discussion of scalar operators we refer the reader to Dunford and Schwartz [8]. In particular we refer to them for a discussion of spectral measures. We shall consider the domain of any spectral measure to be the Borel sets of the complex plane and shall be concerned only with those spectral measures whose support (that is the intersection of all those closed sets on whose complement the measure vanishes) is compact.

Definition 2.1. An operator $T \in L(X)$ is called a *scalar operator* if there exists a spectral measure $E(\cdot)$ such that

$$T = \int_C z\,E(dz). \tag{1}$$

This spectral measure is unique [8]. It is called the resolution of the identity for T and its support is $\sigma(T)$. We can therefore replace C by $\sigma(T)$ in (1). Furthermore, we can integrate any Borel measurable function on C with respect to $E(\cdot)$. To lend some perspective to this observation we give the following definition.

Definition 2.2. Let $K \subset C$ and $A(K)$ be either a closed subalgebra of $C(K)$

containing the polynomials, or the Banach algebra of bounded Borel measurable functions on K with supremum norm. A *functional calculus* for an operator $T \in L(X)$, or an *$A(K)$-calculus* for T, is a homomorphism $h: A(K) \longrightarrow L(X)$, where h is bounded, $h(1) = I$ and $h(z) = T$.

Lemma 2.3 [8]. *Let $T \in L(X)$ and $B(\sigma(T))$ be the algebra of bounded measurable functions on $\sigma(T)$. T is a scalar operator if and only if it has a $B(\sigma(T))$-calculus $h: B(\sigma(T)) \longrightarrow L(X)$ such that for each $x \in X$ the map*

$$(**) \qquad h_x : B(\sigma(T)) \to X, \qquad h_x(f) = h(f)x,$$

is weakly compact.

Proof. If T is scalar we define, for each $f \in B(\sigma(T))$,

$$h(f) = \int_{\sigma(T)} f(z)E(dz),$$

where $E(\cdot)$ is the spectral measure for T. Obviously $h: B(\sigma T)) \longrightarrow L(X)$ is linear, and $h(1) = I$, $h(z) = T$. The boundedness and multiplicativity of h follow easily from the properties of spectral measures, while the weak compactness of the maps h_x follows from the weak countable additivity of $E(\cdot)$.

If $h: B(\sigma(T)) \longrightarrow L(X)$ is functional calculus for T we define for each Borel set σ of $\sigma(T)$, $E(\sigma) = h(\chi_\sigma)$, where χ_σ is the characteristic function of σ. Since h is a bounded algebra homomorphism it follows easily from the Boolean algebra properties of characteristic set functions that $E(\cdot)$ is a spectral measure, except for weak countable additivity. This latter property follows from the weak compactness of the maps h_x. Now, by using simple functions and passing to limits of simple functions one can prove that for each $f \in B(\sigma(T))$

$$h(f) = \int_{\sigma(T)} f(z)\,E(dz).$$

Applying this result to the function $f(z) = z$ shows that T is scalar.

As an improvement of this result we quote without proof the following theorem due to P. G. Spain [17].

Theorem 2.4. *An operator $T \in L(X)$ is scalar if and only if for some $K \subset C$ it has a $C(K)$-calculus $h: C(K) \longrightarrow L(X)$ satisfying*

$$(**) \qquad \text{for each } x \in X \text{ the map } h_x : C(K) \longrightarrow X, \quad h_x(f) = h(f)x, \text{ is weakly compact.}$$

Observe that if X is weakly complete, then by a theorem of Grothendieck [11], condition (**) is automatically satisfied. The connection between K and the spectrum of T is given by the following theorem which may be found in [3].

Theorem 2.5. *If $T \in L(X)$ has a $C(K)$-calculus h, then $\sigma(T) \subset K$ and the support of h is $\sigma(T)$.*

We shall now turn to the discussion of subscalar operators.

Definition 2.6. Let $T \in L(X)$. T is called a *subscalar* operator if there exists an operator $S \in L(Y)$ having a $C(K)$-calculus for some $K \subset C$, such that X is contained in Y as a closed invariant subspace of S, and the restriction of S to X is T. S is called a *quasi-scalar extension* of T. S is called a *minimal quasi-scalar extension* of T if for some $C(K)$-calculus h, for S, there exists no proper closed subspace of Y containing X which is invariant under $h(f)$ for every $f \in C(K)$. We call h a *minimal $C(K)$-calculus* of S relative to T.

Since normal operators on a Hilbert space are precisely those with a $C(K)$-calculus of norm 1 [14], subnormal operators form a special class of subscalar operators on a Hilbert space. The following theorem proves the existence of a minimal quasi-scalar extension for a subscalar operator.

Theorem 2.7. *Any subscalar operator has a minimal quasi-scalar extension.*

Proof. Let $T \in L(X)$ be a subscalar operator and $S \in L(Y)$ a quasi-scalar extension of T with a $C(K)$-calculus h. Let Z be the intersection of all closed subspaces of Y which contain X, and which are invariant under $h(f)$ for every $f \in C(K)$. Obviously Z itself is such a subspace. The restriction $\overline{S}$ of S to Z has the $C(K)$-calculus

$$\overline{h} : C(K) \to L(Z), \qquad \overline{h}(f) = h(f)_{|Z}.$$

Moreover, no proper closed subspace of Z containing X is invariant under $\overline{h}(f)$ for every $f \in C(K)$, since such a subspace would then be invariant under $h(f)$ for $f \in C(K)$ and so must contain Z. It follows that $\overline{S}$ is a minimal quasi-scalar extension of T with a minimal $C(K)$-calculus $\overline{h}$.

To obtain a satisfactory functional calculus for subscalar operators one requires the spectral inclusion theorem 2.9. The most important argument in this theorem is covered by the following lemma.

Lemma 2.8. *Let T be a subscalar operator, and S a minimal quasi-scalar extension of it. If T is invertible then so is S.*

Proof. Let h be a minimal $C(K)$-calculus for S relative to T. We assume $0 \in K$, otherwise, by Theorem 2.6, S is already invertible. Now, using Theorem 2.6 again, we must show that if T is invertible, 0 does not lie in the support of h.

Choose $\epsilon < \|T^{-1}\|^{-1}$ and let D_ϵ be the closed disc of radius ϵ and centre 0. Let $\phi \in C(K)$ have its support contained in $D_\epsilon \cap K$. For any $x \in X$ and positive integer n

$$\|h(\phi)x\| = \|h(\phi)S^n T^{-n}x\| \le \|h(z^n\phi)\| \|T^{-1}\|^n \|x\| \tag{2}$$

because h is a $C(K)$-calculus for S, and $Sx = Tx$ for any $x \in X$. From (2) it follows that for every $x \in X$ and integer $n > 0$

$$(3)\qquad \begin{aligned} \|b(\phi)x\| &\leq \|b\|\|x\| \sup_{z \in D_\epsilon \cap K} |z^n \phi(z)| \|T^{-1}\|^n \\ &\leq \|b\|\|x\| \sup_{z \in D_\epsilon \cap K} |\phi(z)| \cdot (\epsilon \|T^{-1}\|)^n. \end{aligned}$$

Since $\epsilon\|T^{-1}\| < 1$ it follows that $b(\phi)x = 0$ for every $x \in X$ and $\phi \in C(K)$ whose support lies in $D_\epsilon \cap K$. For any $f \in C(K)$ the support of $f\phi$ lies in $D_\epsilon \cap K$ so that

$$(4)\qquad b(\phi)b(f)x = b(\phi f)x = 0.$$

Now the subspace

$$(5)\qquad \{b(g)x | x \in X,\ g \in C(K)\}$$

is invariant under $b(f)$ for every $f \in C(K)$, and contains X since $b(1) = I$. Since b is a minimal $C(K)$-calculus of S relative to T, the closure of this subspace must be Y. From this and (4), it follows that $b(\phi) = 0$ for every $\phi \in C(K)$ whose support lies in $D_\epsilon \cap K$, so that 0 does not lie in the support of b.

Theorem 2.9. *Let $S \in L(X)$ be a subscalar operator and $S \in L(Y)$ a minimal quasi-scalar extension of T, then $\sigma(S) \subset \sigma(T)$.*

Proof. Let $b: C(K) \to L(Y)$ be a minimal $C(K)$-calculus for S relative to T. For any λ let $K_\lambda = \{\omega - \lambda | \omega \in K\}$ and define

$$(6)\qquad b_\lambda : C(K_\lambda) \to L(Y), \qquad b_\lambda(f) = b(f \circ (z - \lambda)).$$

Now b_λ is a $C(K_\lambda)$-calculus for $S - \lambda I$. Moreover it is obvious that any subspace of Y invariant under $b(f)$ for every $f \in C(K)$ is invariant under $b_\lambda(f)$ for every $f \in C(K_\lambda)$, and this for every $\lambda \in C$. It follows that $T - \lambda I$ is a subscalar operator with $S - \lambda I$ a minimal quasi-scalar extension. By Lemma 2.8, $S - \lambda I$ is invertible whenever $T - \lambda I$ is, so $\sigma(S) \subset \sigma(T)$.

Theorem 2.10. *Let $T \in L(X)$ be a subscalar operator. T has a functional calculus*

$$b : P(\sigma(T)) \to L(X).$$

Proof. Let S be a minimal quasi-scalar extension of T with a functional calculus

$$(7)\qquad j : C(K) \to L(Y).$$

By Theorem 2.6, if $\phi \in C(K)$ is one on a compact neighbourhood of $\sigma(S)$, then $j(f) = j(f\phi)$ for each $f \in C(K)$, so that for any such ϕ

$$(8)\qquad \|j(f)\| \leq \|j\| \sup_{t \in K} |f(t)\phi(t)|.$$

It follows that

$$\|j(f)\| \leq \|j\| \sup_{t\in\sigma(S)} |f(t)|. \tag{9}$$

Using (9), the spectral inclusion theorem, and the fact that for any polynomial $p(z)$, $p(T)$ is the restriction to X of $p(S)$, we obtain

$$\|p(T)\| \leq \|p(S)\| \leq \|j\| \sup_{t\in\sigma(S)} |p(t)| \leq \|j\| \sup_{t\in\sigma(T)} |p(t)|. \tag{10}$$

The algebra homomorphism which takes $p(z)$ to $p(T)$ may therefore be extended to an algebra homomorphism

$$b: P(\sigma(T)) \to L(X) \tag{11}$$

with $\|b\| \leq \|j\|$.

Corollary 2.11. *If $T \in L(X)$ is a subscalar operator, and if some quasi-scalar extension of T has a $C(K)$-calculus j with $\|j\| = 1$, then the functional calculus for T is an isometry. In particular, the functional calculus for a subnormal operator is an isometry.*

Proof. Using the formula for spectral radius of an operator one obtains

$$\sup_{t\in\sigma(T)} |p(t)| \leq \|p(T)\| \tag{12}$$

for every polynomial $p(z)$. If some quasi-scalar extension of T has a $C(K)$-calculus of norm less than one, then it is clear from the proof of Theorem 2.7 that T has a minimal quasi-scalar extension with such a $C(K)$-calculus. The corollary follows by putting $\|j\| = 1$ in Theorem 2.10 and recalling that a normal operator on a Hilbert space has a $C(K)$-calculus which is an isometric isomorphism.

3. **p-absolutely summing operators.** If we are to show that an operator $T \in L(X)$ is similar by an isometry to multiplication by z on some $L^p(K, \mu)$ we must construct an isometry U: $L^p(K, \mu) \to X$. If T is scalar, Theorem 2.4 provides a map b_x: $C(\sigma(T)) \to X$. Our line of approach will be to determine conditions under which there exists a probability measure μ on $\sigma(T)$ such that b_x extends to an isometry of $L^p(\sigma(T), \mu)$ onto X. This will not be possible unless the range of b_x is dense. Therefore before discussing the extension problem we introduce the following concepts.

Definition 3.1. Let $A(K)$ be a closed subalgebra of $C(K)$ and b: $A(K) \to L(X)$, a bounded algebra homomorphism. A vector $x \in X$ is called a *topologically cyclic vector* for b if the set of vectors $b(f)x$ is dense in X as f ranges through $A(K)$. Equivalently we could require that the range of b_x: $A(K) \to X$, where $b_x(f) = b(f)x$, be dense in X.

Lemma 3.2. *Let $h: A(K) \longrightarrow L(X)$ be a bounded algebra homomorphism with a topologically cyclic vector $x \in X$. h_x is a monomorphism if and only if h is.*

Proof. If $h(f) = 0$ for some $f \in A(K)$, $f \neq 0$, then $h(f)x = 0$. Therefore if h is not a monomorphism neither is h_x. On the other hand, if h is a monomorphism, assume $h_x(f) = 0$. For any $g \in A(K)$,

$$h(f) \cdot h(g)x = h(fg)x = h(gf)x = h(g)h(f)x = 0.$$

But the range of h_x is dense in X. Therefore $h(f) = 0$ and so $f = 0$ which shows that h_x is a monomorphism.

Definition 3.3. A scalar operator T is said to be simple if some $C(\sigma(T))$-calculus for T has a topologically cyclic vector.

We remark that if T is a normal operator on a Hilbert space our terminology is that of Dieudonné [5].

Definition 3.4. A vector $x \in X$ is said to be cyclic for $T \in L(X)$, if the subspace of X generated by $T^n x$, $n = 0, 1, 2, \cdots$, is dense in X. Equivalently we could require that the subcpace $p(T)x$ where $p(z)$ ranges though all polynomias, be dense in X.

Lemma 3.5. *Let $T \in L(X)$ have a functional calculus $h: P(K) \longrightarrow L(X)$ (where $K \subset C$). A vector $x \in X$ is topologically cyclic for h if and only if it is cyclic for T. In particular a functional calculus for a subscalar operator, such as derived in Theorem 2.10, has a topologically cyclic vector if and only if T has a cyclic vector.*

Proof. Let $x \in X$ be cyclic for T. For any polynomial $p(z)$, $h(p)x = p(T)x$. The set $h(f)x$ where f ranges through $P(K)$ contains the set $p(T)x$ where $p(z)$ ranges through all polynomials, and is therefore dense in X. Therefore x is topologically cyclic for h.

Suppose $x \in X$ is topologically cyclic for h. If $u \in X$, there exists a sequence of functions f_n in $P(K)$ such that $h_x(f_n)$ converges to u. Given some $\epsilon > 0$ we choose n such that $\|h_x(f_n) - u\| < \epsilon/2$. Since any $f \in P(K)$ is a uniform limit of polynomials, we choose a sequence of polynomials p_m such that p_m converges to f_n. Since h_x is continuous $h_x(p_m)$ converges to $h_x(f_n)$. We therefore choose m so that $\|h_x(p_m) - h_x(f_n)\| < \epsilon/2$. Now we have

$$\|h_x(p_m) - u\| \leq \|h_x(p_m) - h_x(f_n)\| + \|h_x(f_n) - u\| < \epsilon.$$

But $h_x(p_m) = p_n(T)x$. For any $\epsilon > 0$ we can find a polynomial $p(z)$ such that $\|p(T)x - u\| < \epsilon$. Since u was chosen arbitrarily, this shows that $p(T)x$ is dense in X as p ranges through all polynomials; that is x is cyclic for T.

Having given an indication as to which operators $T \in L(X)$ with an $A(K)$-calculus, give rise to a map $h_x: A(K) \longrightarrow X$ with dense range, we turn to the problem of extending h_x to a bounded linear operator on $A^p(K, \mu)$. The relevant concept here is the following.

Definition 3.6 (Pietsch [16], Lindenstrauss and Pełczyński [15]). A linear map $S: X \longrightarrow Y$ is called p-absolutely summing, $1 \leq p \leq \infty$, if there exists a $C > 0$ such that any $x_1, \cdots, x_n \in X$

$$\sum_{i=1}^{n} \|Sx_i\|^p \leq C \sup_{\alpha \in K^*} \sum_{i=1}^{n} |\alpha(x_i)|^p \tag{13}$$

where K^* is the weak star closure of the set of extreme points of the unit ball of X^*. The infimum of $C^{1/p}$ over all $C > 0$ for which (13) holds is denoted $a_p(S)$.

Using sets consisting of only one element in (13) shows that S is bounded, and $\|S\| \leq a_p(S)$.

Lemma 3.7. *Let $A(K)$ be a closed subspace of $C(K)$. A linear map $S: A(K) \longrightarrow X$ is p-absolutely summing if and only if there exists $C > 0$ such that for any $f_1, \cdots, f_n \in A(K)$*

$$\sum_{i=1}^{n} \|Sf_i\|^p \leq C \sup_{t \in K} \sum_{i=1}^{n} |f_i(t)|^p. \tag{14}$$

Proof. In the case of $A(K)$ every $\alpha \in K^*$ can be expressed $\alpha = a\delta_t$, where $a \in C$, $|a| = 1$ and δ_t is the evaluation functional at $t \in K$. For every $\alpha \in K^*$ we have $|\alpha(f)| = |f(t)|$ for some $t \in K$ and every $f \in A(K)$. The result follows trivially.

The relevance of p-absolutely summing operators is demonstrated by the following theorem (Pietsch [16], Lindenstrauss and Pełczyński [15]) which we quote without proof.

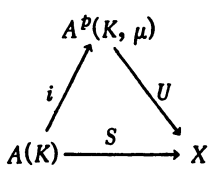
Theorem 3.8. *If $S: X \longrightarrow Y$ is a p-absolutely summing linear map, there exists a probability measure μ on K^* such that*

$$\|Sx\| \leq a_p(S) \left(\int_{K^*} |\alpha(x)|^p \, d\mu(\alpha) \right)^{1/p} \tag{15}$$

for all $x \in X$.

Conversely, if (15) holds with some β in place of $a_p(S)$ then S is p-absolutely summing and $a_p(S) \leq \beta$.

Corollary 3.9. *Let $A(K)$ be a closed subspace of $C(K)$. $S: A(K) \longrightarrow X$ is p-absolutely summing if and only if there exists a probability measure μ on K and a bounded linear map $U: A^p(K, \mu) \longrightarrow X$ such that the following diagram commutes:*



where $i: A(K) \longrightarrow A^p(K, \mu)$ is the canonical injection.

Proof. Such a U exists if and only if there exists a probability measure μ on K such that $S: A(K) \to X$ extends to a bounded linear map $U: A^p(K, \mu) \to X$. This will occur if and only if there exists $C > 0$ such that

$$\|Sf\| \leq C\left(\int |f(t)|^p \, d\mu\right)^{1/p} \quad \text{for all } f \in A(K),$$

and so the corollary follows from Theorem 2.8.

4. Application to weak similarity. In this section we apply the results of §3 to obtain some information about a kind of weak similarity between operators.

Definition 4.1. Given two operators $A \in L(X)$ and $B \in L(Y)$; if there exists a bounded linear map $S: X \to Y$ such that $SA = BS$ we say

(1) A *is weakly similar to* B if S has a densely defined unbounded inverse.

(2) A *is similar to* B if S has a bounded inverse.

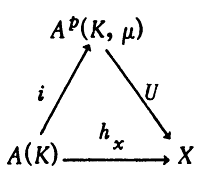
(3) A *is isometrically equivalent to* B if S is an invertible isometry.

We point out that while (2) and (3) are equivalence relations, (1) is only reflexive and transitive. Thus if A is weakly similar to B, it need not be true that B is weakly similar to A. If X and Y are the same Hilbert spaces, (3) becomes the usual unitary equivalence of operators.

Definition 4.2. Let $A(K)$ be a closed subalgebra of $C(K)$ and $h: A(K) \to L(X)$ a bounded algebra homomorphism. A vector $x \in X$ is said to be a p-summing cyclic vector for h if it is topologically cyclic for h and the map h_x is p-absolutely summing.

Theorem 4.3. *Let $T \in L(X)$ have a functional calculus $h: A(K) \to L(X)$ $(K \subset C)$ such that h is a monomorphism and has a p-summing cyclic vector. There exists a probability measure μ on K such that multiplication by z on $A^p(K, \mu)$ is weakly similar to T.*

Proof. By definition there exists a vector $x \in X$ such that h_x is p-absolutely summing. Consequently, by Corollary 3.9 there exists a probability measure μ on K and an operator $U: A^p(K, \mu) \to X$ such that the following diagram commutes:



By Lemma 3.2, h_x is a monomorphism, and, since x is topologically cyclic for h, h_x also has dense range. It therefore has a densely defined inverse which, composed with i, provides a densely defined inverse for U. Now let $w = h_x(g)$ for some $g \in A(K)$. We have

$$h_x(zg) = h(z)h(g)x = Th(g)x = T(h_x(g))$$

and, since the range of i is dense,

$$U(z \cdot f) = TU(f) \quad \text{for all } f \in A^p(K, \mu).$$

Denoting multiplication by z as M_z this gives $UM_z = TU$ which concludes the proof.

Corollary 4.4. *Let $T \in L(X)$ be a simple scalar operator. If a $C(\sigma(T))$-calculus for T has a p-summing cyclic vector, there exists a probability measure μ on $\sigma(T)$ such that multiplication by z on $L^p(\sigma(T), \mu)$ is weakly similar to T.*

Proof. Replace $A(K)$ by $C(\sigma(T))$ in the proof of Theorem 4.3.

Corollary 4.5. *Let $T \in L(X)$ be a subscalar operator with a cyclic vector. If some cyclic vector for T is a p-summing cyclic vector for a $p(\sigma(T))$-calculus for T, then there exists a probability measure μ on $\sigma(T)$ such that multiplication by z on $H^p(\sigma(T), \mu)$ is weakly similar to T.*

Proof. Replace $A(K)$ by $P(\sigma(T))$ in the proof of Theorem 4.3.

Corollary 4.6. *Let $T \in L(X)$ be a simple scalar operator. If a $C(\sigma(T))$-calculus for T has a p-summing cyclic vector for some $1 \le p \le 2$, there exists a simple normal operator N with $\sigma(N) \subset \sigma(T)$ such that N is weakly similar to T.*

Proof. If $S: X \longrightarrow Y$ is p-absolutely summing then it is q-absolutely summing for every $q \ge p$ (Pietsch [16]). If x is a p-summing cyclic vector for a $C(\sigma(T))$-calculus and $1 \le p \le 2$ then it is a 2-summing cyclic vector. By Corollary 4.3 there exists a probability measure μ on $\sigma(T)$ such that multiplication by z on $L^2(\sigma(T), \mu)$ is weakly similar to T. This is the normal operator N which we are required to find. The spectrum of N is the support of μ which is obviously contained in $\sigma(T)$. Further, every normal operator of this form is known to be simple [5].

Corollary 4.7. *Let $T \in L(X)$ be an operator with a $P(\sigma(T))$-calculus (in particular a subscalar operator). If this functional calculus has a p-summing cyclic vector for some $1 \le p \le 2$, there exists a subnormal operator S with a cyclic vector whose spectrum is contained in the polynomial convex hull $\hat{\sigma}(T)$ of the spectrum of T, such that S is weakly similar to T.*

Proof. We follow through the proof of Corollary 4.6 with $P(\sigma(T))$ and $H^2(\sigma(T), \mu)$ in place of $C(\sigma(T))$ and $L^2(\sigma(T), \mu)$ respectively. Multiplication by z on $H^2(\sigma(T), \mu)$ is known to be a subnormal operator with cyclic vector whose spectrum is contained in $\hat{\sigma}(T)$ [2].

We now demonstrate a large class of simple scalar operators to which Corollary 4.6 applies.

Definition 4.8 (Lindenstrauss and Pełczyński [15]). A Banach space X is called an $\mathfrak{L}_{p,\lambda}$ space, $1 \le p \le \infty$, $1 \le \lambda < \infty$, if for every finite dimensional subspace B, there is a finite dimensional subspace E containing it such that $d(E, l_p^n) \leq \lambda$ where l_p^n is the l_p space of dimension n, n is the dimension of E and for Banach spaces X and Y

$$d(X, Y) = \inf\{\|S\|\|S^{-1}\| | S : X \to Y,\ S \text{ invertible}\}.$$

If no such S exists put $d(X, Y) = \infty$.

A Banach space is called an $\mathfrak{L}_p$ space if it is an $\mathfrak{L}_{p,\lambda}$ space for some $\lambda \geq 1$.

Every space $L^p(K, \mu)$ is an $\mathfrak{L}_{p,\lambda}$ space for all $\lambda > 1$. For any K, $C(K)$ is an $\mathfrak{L}_{\infty,\lambda}$ space for all $\lambda > 1$. For $1 < p < \infty$ an $\mathfrak{L}_p$ space is isomorphic to a complemented subspace of some $L^p(K, \mu)$, and if X is an $\mathfrak{L}_{2,\lambda}$ space, there exists a Hilbert space Z such that $d(X, Z) \leq \lambda$. For these and many other results we refer the reader to the comprehensive study of $\mathfrak{L}_p$ spaces by Lindenstrauss and Pełczyński [15].

Theorem 4.9. *Let X be an $\mathfrak{L}_p$ space for some $1 \leq p \leq 2$ and T a simple scalar operator on X. There exists a simple normal operator N with $\sigma(N) \subset \sigma(T)$ such that N is weakly similar to T.*

Proof. Let h: $C(\sigma(T)) \to L(X)$ be a functional calculus for T with a topologically cyclic vector x. The map h_x: $C(\sigma(T)) \to X$ is a map from an $\mathfrak{L}_\infty$ space to an $\mathfrak{L}_p$ space, $1 \leq p \leq 2$. Theorem 2.3 of [15] states that any such map is 2-absolutely summing. The theorem follows from Corollary 4.6.

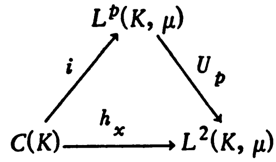
Theorem 4.10. *Let X be a Hilbert space and $T \in L(X)$ have a cyclic vector and a $P(\sigma(T))$-calculus. There exists a probability measure μ on the Šilov boundary $s(T)$ of $P(\sigma(T))$ such that multiplication by z on $H^2(s(T), \mu)$ is weakly similar to T.*

Proof. $P(s(T))$ is isometrically isomorphic to $P(\sigma(T))$, so we can regard T as having a functional calculus h: $P(s(T)) \to L(X)$, where h is a monomorphism and has a topologically cyclic vector x. Now $P(s(T))$ is a Dirichlet algebra, hence, by Lemma 1 of Foiaş, and Suciu [9], h_x is 2-absolutely summing. The result follows by Corollary 4.7.

Example 4.11. We conclude this section with an example to show that under our hypotheses, weak similarity cannot in general be strengthened to similarity. We choose for our Banach space X, the Hilbert space $L^2(K, \mu)$ where μ is a probability measure on K, and K is some infinite compact subset of C. Denote by M_f the operator of multiplication by the continuous function f. The operator M_z on $L^2(K, \mu)$ is a simple normal operator having as functional calculus

$$h : C(K) \to L(X), \qquad h(f) = M_f$$

and the function 1 is a topologically cyclic vector for h. This makes h_x the canonical injection i: $C(K) \to L^2(K, \mu)$. For every $p \geq 2$ the following diagram commutes:



where i, h_x, U_p are the appropriate canonical injections. This shows by Corollary 2.9 that h_x is p-absolutely summing for each $p \geq 2$. But for $p \neq 2$, $L^p(K, \mu)$ is not isomorphic to $L^2(K, \mu)$, so multiplication by z on $L^p(K, \mu)$ cannot be similar to M_z.

5. p-extending operators and applications. Example 4.11 indicates that if the map h_x: $C(K) \longrightarrow X$ is to extend to an isometry U: $L^p(K, \mu) \longrightarrow X$, more must be required of it than that it be p-absolutely summing. We set out to establish such requirements.

Definition 5.1. Let $A(K)$ be a closed subspace of $C(K)$. A bounded linear map S: $A(K) \longrightarrow X$ is called p-dominating if there exists $C > 0$ such that for every finite set $f_1, \cdots, f_n \in A(K)$

$$C \inf_{t \in K} \sum_{i=1}^{n} |f_i(t)|^p \leq \sum_{i=1}^{n} \|Sf_i\|^p. \tag{16}$$

We denote by $b_p(S)$, the supremum of $C^{1/p}$ over all those $C > 0$ for which (9) holds.

The importance of p-dominating maps will appear in the next two theorems. Compare Theorem 5.2 with Theorem 3.8.

Theorem 5.2. *If S: $A(K) \longrightarrow X$ is p-dominating there exists a probability measure ν on K such that*

$$b_p(S)\left(\int_K |f(t)|^p \, d\nu\right)^{1/p} \leq \|Sf\|. \tag{17}$$

Conversely, if (10 *holds with some $\beta > 0$ in place of $b_p(S)$, S is p-dominating and $b_p(S) \geq \beta$.*

Proof. We modify the proof of Theorem 2.8 given in [15].

To prove the second part, suppose (17) holds with some $\beta > 0$ in place of $b_p(S)$. In this case

$$\beta^p \int_K \sum_{i=1}^{n} |f_i(t)|^p d\nu \leq \sum_{i=1}^{n} \|Sf_i\|^p.$$

for any finite set $f_1, \cdots, f_n \in A(K)$. Since ν is a probability measure it follows that

$$\beta^p \inf_{t \in K} \left(\sum_{i=1}^{n} |f_i(t)|^p\right) \leq \sum_{i=1}^{n} \|Sf_i\|^p$$

so that S is p-dominating and $b_p(S) \geq \beta$.

Now assume that S is p-dominating and consider the set

$$W = \left\{ g \in C_R(K) \middle| g(t) = \sum_{i=1}^{n} |f_i(t)|^p \text{ and } \sum_{i=1}^{n} \|Sf_i\|^p = 1 \text{ for some } f_1, \cdots, f_n \in A(K) \right\}.$$

W is a convex set. Put $\gamma = (b_p(S))^p$ and define

$$V = \{w \in C_R(K) | w(t) = 2 - \gamma \cdot g(t) \text{ for some } g \in W\}.$$

V also is convex, and from (17) it follows that if $h \in V$, $\sup_{t \in K} h(t) \geq 1$. Hence V is disjoint from the set $N = \{g \in C_R(K) | \sup_{t \in K} f(t) < 1\}$.

Since N is an open convex set disjoint from V, there exists a linear functional α_0 on $C_R(K)$ such that $\alpha_0(g) \leq 1$ for all $g \in N$ and $\alpha_0(g) \geq 1$ for all $g \in V$. Since N contains the cone of negative functions, α_0 must be positive. Also, since the constant function 1 is in the closure of N, $\alpha_0(1) \leq 1$, so $\alpha_0 = a\alpha$ where $\alpha(1) = 1$ and $0 < a \leq 1$. By the Riesz representation theorem there exists a probability measure ν on K such that for each $f \in C_R(K)$, $\alpha(f) = \int_K f \, d\nu$. If $g \in W$, $2 - \gamma g \in V$, so

$$\int_K (2 - \gamma g) \, d\nu \geq \alpha_0(2 - \gamma g) \geq 1$$

and hence $\gamma \int_K g \, d\nu \leq 1$. For any $f \in A(K)$ the function $g(t) = |f(t)|^p / \|Sf\|^p$ is in W. Consequently

$$\gamma \int_K |f(t)|^p \, d\nu \leq \|Sf\|^p$$

and so

$$b_p(S) \left(\int_K |f(t)|^p \, d\nu \right)^{1/p} \leq \|Sf\|.$$

Definition 5.3. Let $A(K)$ be a closed subspace of $C(K)$. A linear map $S\colon A(K) \to X$ is said to be p-extending if for any $f_1, \cdots, f_n \in A(K)$

$$\inf_{t \in K} \left(\sum_{i=1}^{n} |f_i(t)|^p \right) \leq \sum_{i=1}^{n} \|Sf_i\|^p \leq \sup_{t \in K} \left(\sum_{i=1}^{n} |f_i(t)|^p \right). \tag{18}$$

The major result concerning these maps is Theorem 5.7. We observe that S is both p-absolutely summing and p-dominating.

Lemma 5.4. *If $S\colon A(K) \to X$ is p-extending, $\|S\| \leq 1$. If $A(K)$ contains the constant function* 1, *then $\|S(1)\| = 1$ and $\|S\| = 1$.*

Proof. Using a single $f \in A(K)$ in (18) shows

$$\|Sf\|^p \leq \sup_{t \in K} |f(t)|^p$$

or in other words $\|Sf\| \leq \|f\|$. Therefore $\|S\| \leq 1$. If $1 \in A(K)$ then (18) applied to 1 gives $\|S(1)\| = 1$. This combined with $\|S\| \leq 1$ shows that $\|S\| = 1$.

We recall the definition of a logmodular function space.

Definition 5.5. A closed subspace $A(K)$ of $C(K)$ is said to be logmodular if the set $\{\log |f| \mid f \in A(K), f(t) \neq 0 \text{ for any } t \in K\}$ is dense in $C_R(K)$.

We point out that our definition is slightly weaker than the usual one when $A(K)$ is a uniform algebra [10]. In that case $f \in A(K)$ is required to be invertible $A(K)$ before $\log |f|$ is considered admissible. Here we merely require it to be invertible in $C(K)$.

Observe that any Dirichlet algebra is logmodular. In particular for any K, $C(K)$ is logmodular. Moreover if $K \subset C$ and $s(K)$ is the Šilov boundary of $P(K)$, then $P(s(K))$ is Dirichlet and hence logmodular. For these and similar results we refer the reader to the books by Gamelin [10] and Hoffman [13].

Lemma 5.6. *A closed subspace $A(K)$ of $C(K)$ is logmodular if and only if for some $p > 0$, and so for all $p > 0$, the set $\{|f|^p \mid f \in A(K)\}$ is dense in the positive functions of $C_R(K)$.*

Proof. Assume that $A(K)$ is logmodular and let $u \in C_R(K)$ be strictly positive, so that $\log u$ is a well-defined function in $C_R(K)$. Let $p > 0$ be fixed. By hypothesis we can find a sequence $f_1, f_2, \cdots$ in $A(K)$ such that $\log |f_n|$ converges to $(1/p)\log u$ and so $\log |f_n|^p$ converges to $\log u$. Since K is compact the range of $\log u$ is compact and lies in some interval $[-A, A]$ where $A > 0$. Choose $\epsilon > 0$ and N such that for $n > N$ $\|\log |f_n|^p - \log u\| < \epsilon$. The range of $\log |f_n|^p$ for $n > N$ is contained in an interval $[-B, B]$ where $B > A + \epsilon$. The function e^x is uniformly continuous on $[-B, B]$, so if $\log |f_n|^p$ converges uniformly to $\log u$, $|f_n|^p$ converges uniformly to u. Since the strictly positive functions are dense in the positive functions we have that the set $\{|f|^p \mid f \in A(K)\}$ is dense in the positive functions of $C_R(K)$.

Now assume that this set is dense in the positive functions of $C_R(K)$, e^{pu} is a strictly positive function and since K is compact, its range is contained in an interval $[A, B]$, $B > A > 0$. Choose a sequence $f_1, f_2, \cdots$ in $A(K)$ such that $|f_n|^p$ converges uniformly to e^{pu}. Choose N such that for $n > N$, $\||f_n|^p - e^{pu}\| < A/2$, so that the range of $|f_n|^p$ is contained in the interval $[A/2, B + A/2]$ for $n > N$. Since $\log x$ is uniformly continuous on this interval $\log |f_n|^p$ converges uniformly to U.

Theorem 5.7. *Let $A(K)$ be a logmodular closed subspace of $C(K)$. A linear map $S: A(K) \longrightarrow X$ is p-extending if and only if there exists a probability measure μ on K such that S extends to an isometric isomorphism $U: A^p(K, \mu) \longrightarrow X$.*

Proof. Since S is both p-absolutely summing and p-dominating with $a_p(S) \leq 1$ and $b_p(S) \geq 1$, there exist, by Theorems 2.8 and 5.2, probability measures μ and ν on K such that for any $f \in A(K)$

(19) $$\int |f(t)|^p \, d\nu \leq \|Sf\|^p \leq \int |f(t)|^p \, d\mu.$$

Consider the real valued measure $\omega = \mu - \nu$. Since $\mu(K) = \nu(K) = 1$, $\omega(K) = 0$. As for any $f \in A(K)$, we have by (19)

$$\int_K |f(t)|^p \, d\omega \geq 0.$$

But functions of the form $|f|^p$ are dense in the positive functions on $C_R(K)$ by Lemma 5.6. Consequently ω is a positive measure with $\omega(K) = 0$ which means that $\omega = 0$. Hence $\mu = \nu$ and (19) becomes

(20) $$\|Sf\| = \left(\int_K |f(t)|^p \, d\mu\right)^{1/p}, \quad f \in A(K).$$

We can obviously extend S to an isometry $U: A^p(K, \mu) \longrightarrow X$.

For the converse, if S extends to an isometry $U: A^p(K, \mu) \longrightarrow X$ for some probability measure μ on K, then (20) holds for each $f \in A(K)$. From this we obtain for each finite set $f_1, \cdots, f_n \in A(K)$

$$\sum_{i=1}^n \|Sf_i\|^p = \int_K \sum_{i=1}^n |f_i(t)|^p \, d\mu.$$

Since μ is a probability measure it follows immediately that S is p-extending.

Definition 5.8. Let $T \in L(X)$ have a functional calculus $b: A(K) \longrightarrow L(X)$, where $K \subset C$ and $A(K)$ is a closed subalgebra of $C(K)$. A vector $x \in X$ is said to be a p-extending cyclic vector if the map $b_x: A(K) \longrightarrow X$ is p-extending and has dense range.

We observe that by Lemma 5.4 $\|b_x\| = 1$ and $\|x\| = \|b_x(1)\| = 1$.

Example 5.9. Let μ be a probability measure on $K \subset C$ and $A(K)$ a closed subalgebra of $C(K)$ containing the polynomials. For $f \in A(K)$ let $M_f: A^p(K, \mu) \longrightarrow A^p(K, \mu)$ denote multiplication by f. Since $A(K)$ is an algebra and is dense in $A^p(K, \mu)$ this operator is well defined. Indeed for any $g \in A^p(K, \mu)$

$$\|M_f(g)\| = \left(\int_K |fg|^p d\mu\right)^{1/p} \leq \sup_{t \in K} |f(t)| \cdot \left(\int_K |g|^p \, d\mu\right)^{1/p} = \|f\| \|g\|.$$

This further shows that the homomorphism

$$b: A(K) \longrightarrow L(A^p(K, \mu)), \quad b(f) = M_f$$

has $\|b\| \leq 1$ and constitutes a functional calculus for M_z.

We claim that the constant function $1 \in A^p(K, \mu)$ is a p-extending cyclic vector for b. In fact

$$\sum_{i=1}^n \|b(f_i)(1)\|^p = \sum_{i=1}^n \|Mf_i(1)\|^p = \int_K \sum_{i=1}^n |f_i|^p \, d\mu$$

from which it is obvious, since μ is a probability measure, that

$$\inf_{t \in K} \sum_{i=1}^{n} |f_i(t)|^p \leq \sum_{i=1}^{n} \|h(f_i)(1)\|^p \leq \sup_{t \in K} \sum_{i=1}^{n} |f_i(t)|^p,$$

Theorem 5.10. *Let $A(K)$ be a logmodular closed subalgebra of $C(K)$ (where $K \subset C$) containing the polynomials, and let $T \in L(X)$. There exists a probability measure μ on K such that T is isometrically equivalent to multiplication by z on $A^p(K, \mu)$ if and only if T has an $A(K)$-calculus which has a p-extending cyclic vector.*

Proof. Let μ be a probability measure on K such that T is isometrically equivalent, by an isometry U: $A^p(K, \mu) \longrightarrow X$, to multiplication by x on $A^p(K, \mu)$. Using the notation of Example 5.9 we have $T = UM_z U^{-1}$. Moreover, $h(f) = UM_f U^{-1}$ defines an $A(K)$-calculus for T. Put $z = U(1)$. Since U is an isomorphism and preserves norms, it follows easily, in view of Example 5.9, that x is a p-extending cyclic vector for h.

On the other hand, suppose T has an $A(K)$-calculus with a p-extending cyclic vector x. The map h_x: $A(K) \longrightarrow X$ is p-extending, so by Theorem 5.7 there exists a probability measure μ on K such that h_x extends to an isometric monomorphism U: $A^p(K, \mu) \longrightarrow X$. For each $f \in A(K)$, $h_x(zf) = Th_x(f)$. Since the range of h_x is dense, U is an epimorphism and $UM_z = TU$.

Corollary 5.11. *A scalar operator $T \in L(X)$ is isometrically equivalent to multiplication by z on $L^p(\sigma(T), \mu)$ for some probability measure μ on $\sigma(T)$, if and only if T has a $C(\sigma(T))$-calculus with a p-extending cyclic vector.*

Corollary 5.12. *Let $T \in L(X)$ and $s(T)$ be the Šilov boundary of $P(\sigma(T))$. T is a subscalar operator isometrically equivalent to multiplication by z on $H^p(s(T), \mu)$ if and only if T has a $P(\sigma(T))$-calculus and a cyclic vector $x \in X$ such that for any $f_1, \cdots, f_n \in P(\sigma(T))$*

$$\inf_{t \in s(T)} \sum_{i=1}^{n} |f_i(t)|^p \leq \sum_{i=1}^{n} \|h_x(f_i)\|^p \leq \sup_{t \in s(T)} \sum_{i=1}^{n} |f_i(t)|^p.$$

Proof. We apply Theorem 5.10 noting that $P(s(T))$ is logmodular and that, since $P(s(T))$ and $P(\sigma(T))$ are isometrically isomorphic, any $P(\sigma(T))$-calculus for T is a $P(s(T))$-calculus and vice versa.

Theorem 5.13. *Let T be an operator on a Banach space X, and $s(T)$ the Šilov boundary of $P(\sigma(T))$. T is isometrically equivalent to multiplication by z on $H^p(s(T), \mu)$ for some probability measure μ on $s(T)$ if and only if T has a cyclic vector x (of norm one) such that for any polynomials $p_1, \cdots, p_n$*

(21)
$$\sup_{t\in s(T)} \sum_{i=1}^{n} |p_i(t)|^p \le \sum_{i=1}^{n} \|p_i(T)x\|^p \le \sup_{t\in s(T)} \sum |p_i(t)|^p.$$

Proof. Necessity follows by Corollary 5.12 and the fact that in 5.12, $b(p_i) = p_i(T)$.

To prove sufficiency we first note that by the proof of Lemma 5.4, the map $b_x(p) = p(T)x$ satisfies $\|b_x(p)\| \le \|p\|$ where $\|p\|$ is the norm in $C(s(T))$. We extend b_x to a bounded map b_x: $P(s(T)) \to X$. Now $P(s(T))$ is a logmodular algebra. If, b_x is p-extending we will be able to extend it in the usual way to an isometric isomorphism U: $H^p(s(T), \mu) \to X$ by which T is isometrically equivalent to multiplication by z on $H^p(s(T), \mu)$. We must show that b_x is p-extending. Let $f_1, \cdots, f_n \in P(s(T))$. There exist sequences of polynomials $p_{i,k}$, $i = 1, \cdots, n$, $k = 1, 2, \cdots$, such that $p_{i,k}$ converges in $P(s(T))$ to f_i. Therefore $|p_{i,k}|^p$ converges to $|f_i|^p$ in $C(s(T))$ and $\|b_x(p_{i,k})\|^p$ converges to $\|b_x(f_i)\|^p$. Given $\epsilon > 0$ choose M such that for $m > M$:

$$|\|b_x(p_{i,m})\|^p - \|b_x(f_i)\|^p| < \epsilon/2n \quad \text{and} \quad \||p_{i,m}|^p - |f_i|^p\| < \epsilon/2n.$$

The two inequalities together show that

$$\inf_{t\in s(T)} \sum_{i=1}^{n} |f_i(t)|^p - \epsilon/2 \le \inf_{t\in s(T)} \sum_{i=1}^{n} |p_{i,m}(t)|^p,$$

$$\sup_{t\in s(T)} \sum_{i=1}^{n} |p_{i,m}(t)|^p \le \sup_{t\in s(T)} \sum_{i=1}^{n} |f_i(t)|^p + \epsilon/2$$

and

$$\sum_{i=1}^{n} \|b_x(p_{i,m})\|^p - \epsilon/2 \le \sum_{i=1}^{n} \|b_x(f_i)\|^p \le \sum_{i=1}^{n} \|b_x(p_{i,m})\|^p + \epsilon/2$$

for $m > M$. These together with (14) show that for any $\epsilon > 0$

$$\inf_{t\in s(T)} \sum_{i=1}^{n} |f_i(t)|^p - \epsilon \le \sum_{i=1}^{n} \|b_x(f_i)\|^p \le \sup_{t\in s(T)} \sum_{i=1}^{n} |f_i(t)|^p + \epsilon$$

which is enough to prove that b_x is p-extending.

Corollary 5.14. *Let T be an operator on a Hilbert space X, with a cyclic vector x. T is a subnormal operator whose minimal normal extension has its spectrum contained in $s(T)$ if and only if for any polynomials $p_1, \cdots, p_n$*

(22)
$$\inf_{t\in s(T)} \sum_{i=1}^{n} |p_i(t)|^2 \le \sum_{i=1}^{n} \|p_i(T)x\|^2$$

and T has an isometric $P(\sigma(T))$-calculus.

Proof. From our hypotheses T has a functional calculus $h: P(s(T)) \longrightarrow L(X)$ which is an isometry. By Lemma 1 in [9] we have for any cyclic vector with $\|x\| = 1$ and polynomials $p_1, \cdots, p_n$

$$\sum_{i=1}^{n} \|h_x(p_i)\|^2 \leq \sup_{t \in s(T)} \sum_{i=1}^{n} |p_i(t)|^2.$$

The result follows from this and (22) by Theorem 5.13 and the fact that the minimal normal extension of multiplication by z on $H^2(s(T), \mu)$ has its spectrum contained in $s(T)$.

It would be interesting to know if $s(T)$ can be replaced by $\sigma(T)$ in Theorems 5.13 and 5.14, thereby giving a more satisfactory discussion of the representation of subscalar operators on $H^p(K, \mu)$ spaces. We must point out here that our methods fail to deal with this problem because the convex cone generated by the functions $|f|^p$, where $f \in P(K)$ is not dense in $C_R^+(K)$ if K is not the Šilov boundary of $P(K)$. The denseness of this cone is essential to our proof of Theorem 5.7, and all later results depend on this. In fact we can prove the following general theorem.

Theorem 5.15. *Let $A(K)$ be a uniform algebra on a compact Hausdorff space, whose Šilov boundary is T. If $T \neq K$ then the convex cone generated by the functions $|f|^p$ where $f \in A(K)$ (p is fixed) is not dense in $C_R^+(K)$.*

Proof. Let K^* be the weak star closure of the set of extreme points of the unit sphere of the dual $A(K)^*$ of $A(K)$. According to [15, p. 285], for any $f_1, \cdots, f_n \in A(K)$

$$\sup_{\alpha \in A(K)^*, |\alpha| = 1} \sum_{i=1}^{n} |\alpha(f)|^p = \sup_{\alpha \in K^*} \sum_{i=1}^{n} |\alpha(f_i)|^p.$$

Since any $\alpha \in K^*$ can be written $a\delta_t$, where $a \in C$, $|a| = 1$, $t \in T$ and δ_t is the evaluation functional at t, we have

$$\sup_{t \in K} \sum_{i=1}^{n} |f_i(t)|^p = \sup_{t \in T} \sum_{i=1}^{n} |f_i(t)|^p.$$

This means that any function $g \in C_R^+(K)$ of the form $g = \Sigma_{i=1}^n |f_i(t)|^p$ takes its supremum on T. Since the set of all such functions forms the cone under consideration we can complete the proof by showing that if $T \neq K$, the set of functions in $C_R^+(K)$ which take their supremum on T form a proper closed subset of $C_R^+(K)$. We prove that the complement of this set is open. If $f \in C_R^+(K)$ does not take its supremum on T, put $\epsilon = (M - m)/2$ where M and m are the suprema of f on K and T respectively. If $g \in C_R^+(K)$ and $\|f - g\| < \epsilon$, then on T we have $g(t) < (M + m)/2$, while if f takes its supremum at $t \notin T$, $g(t) > (M + m)/2$. Hence no function in the neighbourhood $\|f - g\| < \epsilon$ takes its supremum on T. The set of functions in $C_R^+(K)$ taking their supremum on T is therefore closed. A trivial application of Urysohn's lemma using T and some $t \notin T$ shows that this subset is proper.

6. **Multiplication by z on $L^1(K, \mu)$.** In this section we use the results of §5 to characterise those simple scalar operators T which are isometrically equivalent to multiplication by z on $L^1(\sigma(T), \mu)$, purely in terms of their associated spectral measures. We begin by showing that the relevant operators are 1-dominating (Theorem 6.2).

Lemma 6.1. *Let $S: A(K) \to X$ be a bounded linear map with dense range, where $A(K)$ is a closed subalgebra of $C(K)$. There exists a unique bounded algebra homomorphism $h: A(K) \to L(X)$ with $h(1) = I$ such that $S = h_x$ with $x = S(1)$, if and only if there exists $C > 0$ such that for any $f, g \in A(K)$*

$$\|S(fg)\| \leq C\|f\|\,\|S(g)\|. \tag{23}$$

In this case $\|h\|$ is the infimum of all $C > 0$ for which (23) is true.

Proof. Let $h: A(K) \to L(X)$ be such a bounded algebra homomorphism with $x \in X$ such that h_x has dense range. Put $S = h_x$. We have $x = T(1)$ and

$$\|S(fg)\| = \|h(f)h(g)x\| \leq \|h\|\,\|f\|\,\|h_x(g)\| = \|h\|\,\|f\|\,\|S(g)\|$$

so that (23) holds.

For sufficiency, observe that by (23) the map $h(f)$ defined on the range of S by $h(f)(S(g)) = S(fg)$ is bounded and $\|h(f)\| \leq C\|f\|$. Hence it can be extended to a bounded linear map $h(f): X \to X$ since the range of S is dense. It is trivial to check that h is an algebra homomorphism with $h(1) = I$ (first do it on the range of S, then extend to X). Using (23) again shows $\|h\| \leq C$.

Theorem 6.2. *Let $S: C(K) \to X$ be weakly compact and satisfy $\|S(fg)\| \leq$*

$$\|S(fg)\| \leq \|f\|\ \|Sg\| \qquad \text{for any } f, g \in C(K) \tag{24}$$

and $\|S(1)\| = 1$. S is 1-dominating with $b_p(S) \geq 1$.

Proof. The functional $\rho: C(K) \to R$ defined by $\rho(f) = \|Sf\|$ is obviously a seminorm on $C(K)$. On the one dimensional subspace of constant functions we define $\theta(c) = c$. Since $\|S(1)\| = 1$ we have $|\theta(c)| = \|S(c)\|$. By the Hahn-Banach theorem θ extends to a bounded linear functional on $C(K)$ such that for all $f \in C(K)$

$$|\theta(f)| \leq \|S(f)\| \leq \|f\|$$

where the right-hand inequality is just (24) with $g = 1$. By the Riesz representation theorem there exists a measure ν on K such that

$$\theta(f) = \int_K f\, d\nu \quad \text{for each } f \in C(K).$$

Since $\|\theta\| \leq 1$ and $\theta(1) = 1$, ν is a probability measure and

$$\left|\int_K f\, d\nu\right| \leq \|S(f)\|. \tag{25}$$

Let $B(K)$ denote the bounded Borel measurable functions on K. For any $f \in B(K)$ there exists $\alpha \in B(K)$ such that $\alpha \cdot f = |f|$. Indeed we define

$$\alpha(t) = e^{i\theta} \quad \text{if } f(t) = re^{-i\theta}, \quad r \neq 0,$$
$$= 0 \quad \text{if } f(t) = 0.$$

Since S is weakly compact it extends without change of norm to a bounded linear map $U: B(K) \longrightarrow X$. Using pointwise bounded limits shows that U satisfies (24) for every $f, g \in B(K)$, and in particular

$$\|U(|f|)\| = \|U(\alpha \cdot f)\| \leq \|\alpha\| \|U(f)\| = \|U(f)\|.$$

So for $f \in C(K)$ we have $\|S(|f|)\| \leq \|S(f)\|$. Using this and (25) gives

$$\int_K |f| \, d\nu \leq \|S(|f|)\| \leq \|S(f)\|.$$

By Theorem 5.2, S is 1-dominating and $b_1(S) \geq 1$.

Lemma 6.3. *Let $T \in L(X)$ be a simple scalar operator and $h: C(\sigma(T)) \longrightarrow L(X)$ a $C(\sigma(T))$-calculus with $\|h\| = 1$. h has a* 1*-extending cyclic vector if and only if it has a* 1*-summing cyclic vector $x \in X$ with $\|x\| = 1$ and $a_1(h_x) = 1$.*

Proof. Necessity is obvious. On the other hand let x be a 1-summing cyclic vector with $a_1(h_x) = 1$ and $\|x\| = 1$. It follows that $\|h_x(1)\| = 1$ and $\|h_x(fg)\| \leq \|f\| \, \|h_x(g)\|$ since $\|h\| = 1$. Therefore, by Theorem 6.2, h_x is 1-dominating with $b_1(h_x) \geq 1$. Since it is also 1-summing, with a $a_1(h_x) \leq 1$ it is 1-extending.

We now turn to the problem of finding out when a linear map $S: C(K) \longrightarrow X$ is absolutely summing. If S is weakly compact, there exists a Borel measure $F(\cdot)$ on K with values in X, such that $S(f) = \int_K f \, dF$ for each $f \in C(K)$. (See [6].)

Definition 6.4. Let $F(\cdot)$ be a Borel measure on K with values in X. For each Borel set σ of K we define

$$\phi(\sigma) = \sup_{P(\sigma)} \sum_{i=1}^{n} \|F(\alpha_i)\|$$

where $P(\sigma)$ is the family of all partitions of σ in a finite union of pairwise disjoint subsets $\{\alpha_k\}^n$ of σ. If $\phi(K)$ is finite we put $\|F\|_1 = \phi(K)$ and call this the *total variation of* $F(\cdot)$ and say that $f(\cdot)$ is of *bounded variation*. In this case ϕ is a finite positive measure on k called the *variation measure* of $F(\cdot)$. We define the *semivariation* of $F(\cdot)$ to be

$$\sup \left\{ \left\| \sum_{i=1}^{n} a_i F(\alpha_i) \right\| \, \middle| \, \{\alpha_i\}_1^n \in P(\sigma), \ |a_i| \leq 1, \ i = 1, \cdots, n \right\}.$$

The following theorem is a special case of a theorem due to Diestel [4]. We offer a simpler proof in this case.

Theorem 6.5. *Let* $S: C(K) \longrightarrow X$. *S is absolutely summing if and only if there exists an X-valued measure* $F(\cdot)$ *on K of bounded variation, such that* $S(f) = \int f\, dF$ *for every* $f \in C(K)$. *In this case* $a_1(S) = \|F\|_1$.

Proof. Assume that such an F exists. Let $S = \Sigma_{i=1}^{n} a_i \chi_{\alpha_i}$ be a simple function on K. We have

$$\left\| \int_K S\, dF \right\| \le \sum_{i=1}^{n} |a_i| \|F(\alpha_i)\| \le \sum_{i=1}^{n} |a_i| \phi(\alpha_i) \le \int_K |S|\, d\phi.$$

Using uniform limits of simple functions, for any $f \in C(K)$, we have

$$|S(f)| = \left\| \int_K f\, dF \right\| \le \int_K |f|\, d\phi = \|F\|_1 \int_K |f|\, d\mu$$

where μ is a probability measure such that $\phi = \|F\|_1$. This shows that S is absolutely summing and that $a_1(S) \le \|F\|_1$ (Theorem 3.8).

If S is absolutely summing then it is weakly compact [16] and can therefore be written $S(f) = \int_K f\, dF$ where F is an X-valued measure on K. By Theorem 3.8 there exists a probability measure μ on K such that

$$\left\| \int_K f\, dF \right\| \le a_1(S) \int_K |f|\, d\mu$$

for every $f \in C(K)$, and hence using weak compactness for every $f \in B(K)$. Let $\{\alpha_K\}_{k=1}^{n}$ be any partition of K into pairwise disjoint Borel sets. If χ_{α_k} is the characteristic function of α_k

$$\sum_{k=1}^{n} \|F(\alpha_k)\| = \sum_{k=1}^{n} \left\| \int \chi_{\alpha_k} dF \right\| \le a_1(S) \int_K \sum_{k=1}^{n} \chi_{\alpha_k} d\mu = a_1(S).$$

Consequently F is of bounded variation and $\|F\|_1 \le a_1(S)$.

We conclude with the following characterisation.

Theorem 6.6. *Let* $T \in L(X)$ *be a simple scalar operator and* $E(\cdot)$ *be the resolution of the identity. There exists a probability measure* μ *on* $\sigma(T)$ *such that T is isometrically equivalent to multiplication by z on* $L^1(\sigma(T), \mu)$ *if and only if the following conditions are satisfied:*

(i) $E(\cdot)$ *has semivariation one.*

(ii) *There exists an* $x \in X$ *such that* $\|x\| = 1$ *and the total variation of* $E(\sigma)x$ *is one.*

(iii) *The subspace of X spanned by the set* $E(\sigma)x$ *as* σ *varies over the Borel sets of* $\sigma(T)$ *is dense in X.*

If these conditions hold then $\mu(\sigma) = \|E(\sigma)x\|$.

Proof. First we observe that (i) is equivalent to the condition that the map

$$h : C(\sigma(T)) \to L(X), \qquad h(f) = \int_{\sigma(T)} f(x)\, E(dz)$$

have $\|h\| = 1$. (See [6].) By Theorem 6.5, (ii) and (iii) are equivalent to the condition that h have a 1-summing cyclic vector $x \in X$ with $\|x\| = 1$ and $a_1(h_x) = 1$. Hence (ii) and (iii) together are equivalent to the condition that h have a 1-extending cyclic vector. Since h is the $C(\sigma(T))$-calculus for T, the theorem follows from Corollary 5.11 for the case $p = 1$.

If conditions (i)–(iii) hold then there exists an isometry U: $L^1(\sigma(T), \mu) \to X$ for some probability measure μ on $\sigma(T)$, such that

$$h(f)U = UM_f, \qquad U(1) = x.$$

Therefore for any Borel set σ of $\sigma(T)$,

$$\|E(\sigma)x\| = \|UM_{\chi_\sigma} U^{-1}x\| = \|U(\chi_\sigma)\|.$$

Since U is an isometry it follows that $u(\sigma) = \|E(\sigma)x\|$.

Remarks in conclusion. We consider that the approach we have used to obtain our results should be capable of producing many more. By perhaps introducing different techniques of proof one could deal with operators whose functional calculi are based on algebras such as $P(K)$ for a general K, or the uniform closure of rational functions on some $K \subset C$. By considering the algebra of k times continuously differentiable functions one could also give conditions under which an operator can be represented as multiplication by z on certain types of Sobolev spaces. The material concerning scalar operators can all be generalised to the case of unbounded scalar operators, but we shall defer a discussion of this to a later publication.

I should like to thank my supervisor Dr. J. D. Gray for his constant encouragement and many helpful discussions of this work.

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