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LOCAL NORM CONVERGENCE OF STATES ON THE ZERO TIME BOSE FIELDS(1)

BY

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ABSTRACT. For a sequence of vector states on the Boson Fock space which are norm convergent on the Newton-Wigner local algebras, conditions are given which guarantee norm convergence on the relativistic local algebras also. These conditions are verified for the cutoff physical vacuum states of the $P(\phi)_2$ field theory, and yield a simplification of the proof of the locally normal property of the physical vacuum in that theory.

1. **Introduction.** In the C^*-algebraic approach to the $P(\phi)_2$ quantum field theory, the existence of a physical vacuum is established by first studying the ground states ω_g of the space cutoff Hamiltonian $H(g)$. The states ω_g are Fock space vector states, so that $\omega_g(A) = (\Omega_g, A\Omega_g)$, where Ω_g is a unit vector in Fock space. It is then shown that ω_g lie in a norm compact set of states on each of the local algebras generated by the Newton-Wigner fields [3], [6]. Thus a sequence of g's can be picked out converging toward 1 such that the corresponding states ω_g converge in the norm on the Newton-Wigner local algebras. In this paper we present a simplified version of an argument due to Glimm and Jaffe [3, Chapter 4], which shows that this sequence of vector states also converges in norm on the relativistic local algebras. This argument is based on the fact that operators in the relativistic local algebras can be approximated by operators in the Newton-Wigner local algebras in a topology which is stronger than the strong operator topology, but weaker than the norm topology. The topology is defined by a norm which is weaker than the usual operator norm, and the approximation is uniform in a way made precise in hypothesis (*) of Theorem 2.1.

The fact that the sequence of ω_g converges in norm on the local relativistic algebras implies that the limiting state ω is locally normal, and thus defines a locally normal representation. By using this fact and the fact that the cutoff

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Hamiltonian $H(g)$ locally defines a correct dynamics, it can be shown that the dynamics is implemented by a strongly continuous unitary group in the representation defined by ω, and the existence of the Hamiltonian for the limit theory follows. The details of this construction are in [3].

2. **The general theorem.** We will not consider the question of how states on the Newton-Wigner fields shall be identified with states on the relativistic fields, but simply assume that the states in question are vector states, given by the same vector on both types of fields.

The notation is as follows; we indicate in parenthesis what the concepts correspond to in the $P(\phi)_2$ theory.

$\mathfrak{F}$ is a separable Hilbert space ($\sim$ Fock space).

$\{\mathfrak{A}_n^0\}_n$ is an increasing sequence of type I factors on $\mathfrak{F}$ ($\sim$ local algebras of Newton-Wigner fields).

$\mathfrak{A}$ is a von Neumann algebra on $\mathfrak{F}$ ($\sim$ a relativistic zero time local algebra).

$\mathfrak{A}_d$ is a weakly dense sub*-algebra of $\mathfrak{A}$.

$\{N_n\}_n$ is a sequence of positive selfadjoint operators, such that N_n is affiliated with $\mathfrak{A}_n^0 \cap \mathfrak{A}_{n-1}^{0'}$ ($\mathfrak{A}_1^0$ for $n = 1$). This means that the spectral projections of N_n lie in $\mathfrak{A}_n^0 \cap \mathfrak{A}_{n-1}^{0'}$. ($\mathfrak{A}_{n-1}^{0'}$ = the commutant of $\mathfrak{A}_{n-1}^0$ in $\mathfrak{B}(\mathfrak{F})$.)

$\{d_n\}$ is a sequence of positive integers such that $\Sigma_n d_n^{1/2} = 1$.

Let $N = \Sigma_n d_n N_n$ be defined as in Lemma 2.2.

$E_n: \mathfrak{A}_d \to \mathfrak{A}_n^0$ is a mapping.

f is a continuous and strictly positive real function on R such that $\lim_{\lambda\to\infty} f(\lambda)$ exists in R.

Assume that f and $\{d_n\}$ can be chosen such that

$$(*)\qquad \begin{aligned}&\text{There exist constants } K \text{ and } K_m \text{ such that } \lim_{m\to\infty} K_m = 0\\ &\text{and } \|f(N)E_m(x)f(N)\| \le K\|x\| \quad \text{for } x \in \mathfrak{A}_d,\\ &\|f(N)(x - E_m(x))f(N)\| \le K_m\|x\| \text{ for } x \in \mathfrak{A}_d.\end{aligned}$$

Theorem 2.1. *If $\{\omega_n\}$ is a sequence of vector states in $\mathfrak{F}$ such that*

1. *$\omega_n(N_m) \le 1$ for all n and m,*
2. *$\{\omega_n\}$ converges in norm on each $\mathfrak{A}_m^0$,*

then $\{\omega_n\}$ converges in norm on $\mathfrak{A}$.

Proof. Let $\epsilon > 0$ be given. Let P_n be the spectral projection of N_n corresponding to the spectral interval $[0, 196\epsilon^{-2}d_n^{-1/2}]$, and define $P = \Pi_n P_n$. Since $I - P_n \le \epsilon^2 d_n^{1/2} N_n/196$ it follows from assumption 1 in the theorem that $\omega_m(I - P_n) \le \epsilon^2 d_n^{1/2}/196$. Since all the projections P_n commute we have that $I - P = I - \Pi_n P_n \le \Sigma_n (I - P_n)$, where the last sum is defined as in Lemma 2.2. Thus, by the monotone convergence theorem,

$$\omega_m(1-P) \leq \sum_n \omega_m(1-P_n) \leq \sum_n \frac{1}{196}\epsilon^2 d_n^{1/2} = \frac{1}{196}\epsilon^2.$$

Define positive linear functionals $\omega_m^{\epsilon}(\cdot) = \omega_m(P \cdot P)$. Then for $x \in \mathfrak{B}(\mathfrak{F})$:

$$|\omega_m(x) - \omega_m^{\epsilon}(x)| \leq |\omega_m(x(I-P))| + |\omega_m((I-P)xP)|$$
$$\leq (\omega_m(xx^*))^{1/2}(\omega_m(I-P))^{1/2} + (\omega_m(I-P))^{1/2}(\omega_m(Px^*xP))^{1/2}$$
$$\leq 2\|x\|(\omega_m(I-P))^{1/2} \leq \epsilon\|x\|/7.$$

Thus, as states on $\mathfrak{B}(\mathfrak{F})$:

(2.1) $$\|\omega_m - \omega_m^{\epsilon}\| \leq \epsilon/7.$$

Define $\overline{P}_n = \Pi_{k=1}^n P_n \in \mathfrak{A}_n^0$. We need other approximations to ω_m defined by $\omega_m^{\epsilon n}(\cdot) = \omega_m(\overline{P}_n \cdot \overline{P}_n)$. To estimate $\|\omega_m^{\epsilon n} - \omega_m^{\epsilon}\|$ we observe as before that for $x \in \mathfrak{B}(\mathfrak{F})$:

$$|\omega_m^{\epsilon n}(x) - \omega_m^{\epsilon}(x)| = |\omega_m((\overline{P}_n - P)x\overline{P}_n + Px(\overline{P}_n - P))| \leq 2\|x\|\omega_m(\overline{P}_n - P)^{1/2}.$$

Since

$$\overline{P}_n - P = \prod_{k=1}^{n} P_k - \prod_{k=1}^{\infty} P_k = \left(\prod_{k=1}^{n} P_k\right)\left(I - \prod_{k=n+1}^{\infty} P_k\right)$$

we have that:

$$\omega_m(\overline{P}_n - P) \leq \left(\omega_m\left(\prod_{k=1}^{n} P_k\right)\right)^{1/2}\left(\omega_m\left(I - \prod_{k=n+1}^{\infty} P_k\right)\right)^{1/2}$$
$$\leq \left(\omega_m\left(1 - \prod_{k=n+1}^{\infty} P_k\right)\right)^{1/2} \leq \left(\sum_{k=n+1}^{\infty} \omega_m(1-P_k)\right)^{1/2}$$
$$\leq \left(\sum_{k=n+1}^{\infty} \frac{1}{196}\epsilon^2 d_k^{1/2}\right)^{1/2},$$

which tends to zero as $n \to \infty$. Thus, as linear functionals on $\mathfrak{B}(\mathfrak{F})$:

(2.2) $$\lim_{n\to\infty} \|\omega_m^{\epsilon n} - \omega_m^{\epsilon}\| = 0, \quad \text{uniformly in } m.$$

$P = \Pi_n P_n$ is contained in a spectral projection of N corresponding to the interval $[0, \Sigma_n 196\epsilon^{-1} d_n^{-1/2} d_n] = [0, 196\epsilon^{-2}]$. This is because $\|PNP\| \leq \Sigma_n d_n \|PN_nP\| \leq \Sigma_n d_n \|P_nN_nP_n\| \leq \Sigma_n d_n 196\epsilon^{-2} d_n^{-1/2}$. Since f is continuous and strictly positive on $[0, 196\epsilon^{-2}]$ there exists a $\delta > 0$ such that $f(\lambda) \geq \delta$ for $\lambda \in [0, 196\epsilon^{-2}]$. Thus $P \leq \delta^{-1} f(N)$ and since P commutes with N it follows from Lemma 2.3 that $\|PxP\| \leq \delta^{-2}\|f(N)xf(N)\|$ for all $x \in \mathfrak{B}(\mathfrak{F})$, and then by assumption 3 of the theorem:

(2.3) $$\|PE_m(x)P\| \le \delta^{-2}K\|x\| \quad \text{for } x \in \mathfrak{A}_d,$$

(2.4) $$\|P(x - E_m(x))P\| \le \delta^{-2}K_m\|x\| \quad \text{for } x \in \mathfrak{A}_d.$$

We make a sharper version of (2.3). Assume $n \ge m$. Since $\mathfrak{A}_n^0$ is a type I factor, there exists a tensor product decomposition $\mathfrak{F} = \mathfrak{F}_1 \otimes \mathfrak{F}_2$ such that $\mathfrak{A}_n^0 = \mathfrak{B}(\mathfrak{F}_1) \otimes I_{\mathfrak{F}_2}$, see [2]. Since $\Pi_{k=1}^n P_k \in \mathfrak{A}_n^0$ and $\Pi_{k=n+1}^\infty P_k \in \mathfrak{A}_n^{0\prime}$, it follows that $\Pi_{k=1}^n P_k = Q_1 \otimes I$, $\Pi_{k=n+1}^\infty P_k = I \otimes Q_2$ for projections Q_1, Q_2 on $\mathfrak{F}_1, \mathfrak{F}_2$, and thus $P = Q_1 \otimes Q_2$. Since $E_m(x) \in \mathfrak{A}_n^0$, it is a $y \in \mathfrak{B}(\mathfrak{F}_1)$ such that $x = y \otimes I$. Thus

$$\|PE_m(x)P\| = \|(Q_1 \otimes Q_2)(y \otimes I)(Q_1 \otimes Q_2)\| = \|(Q_1 y Q_1) \otimes Q_2\|$$
$$= \|Q_1 y Q_1\|\,\|Q_2\| = \|Q_1 y Q_1\| = \|(Q_1 \otimes I)(y \otimes I)(Q_1 \otimes I)\| = \|\overline{P}_n E_m(x)\overline{P}_n\|.$$

From (2.3) it follows that:

(2.5) $$\|\overline{P}_n E_m(x)\overline{P}_n\| \le \delta^{-2}K\|x\| \quad \text{for } x \in \mathfrak{A}_d \text{ and } n \ge m.$$

We are now ready to prove the theorem. Let $x \in \mathfrak{A}_d$, and assume $m \le k$. We approximate $\omega_i(x)$ by $\omega_i^{\epsilon k}(E_m(x))$:

$$|\omega_i(x) - \omega_i^{\epsilon k}(E_m(x))| \le |\omega_i(x) - \omega_i^{\epsilon}(x)| + |\omega_i^{\epsilon}(x) - \omega_i^{\epsilon}(E_m(x))| + |\omega_i^{\epsilon}(E_m(x)) - \omega_i^{\epsilon k}(E_m(x))|$$
$$= |(\omega_i - \omega_i^{\epsilon})(x)| + |\omega_i(P(x - E_m(x))P)| + |(\omega_i^{\epsilon} - \omega_i^{\epsilon k})(\overline{P}_k E_m(x)\overline{P}_k)|$$
$$\le \epsilon\|x\|/7 + \delta^{-2}K_m\|x\| + \|\omega_i^{\epsilon} - \omega_i^{\epsilon k}\|\delta^{-2}K\|x\|.$$

The last estimate follows from (2.1), (2.4) and (2.5). By hypothesis (*) of the theorem, m can be chosen such that $\delta^{-2}K_m < \epsilon/7$. By (2.2), $k \ge m$ can then be chosen such that $\|\omega_i^{\epsilon} - \omega_i^{\epsilon k}\|\delta^{-2}K < \epsilon/7$ for all i. Thus:

(2.6) $$|\omega_i(x) - \omega_i^{\epsilon k}(E_m(x))| < 3\epsilon\|x\|/7 \quad \text{for } k \ge m \text{ and } m \text{ large}.$$

Thus, for large m:

$$|\omega_i(x) - \omega_j(x)| \le |\omega_i(x) - \omega_i^{\epsilon k}(E_m(x))| + |\omega_i^{\epsilon k}(E_m(x)) - \omega_j^{\epsilon k}(E_m(x))| + |\omega_j^{\epsilon k}(E_m(x)) - \omega_j(x)|$$
$$\le 3\epsilon\|x\|/7 + |(\omega_i - \omega_j)(\overline{P}_k E_m(x)\overline{P}_k)| + 3\epsilon\|x\|/7$$
$$\le 6\epsilon\|x\|/7 + \|(\omega_i - \omega_j)|\mathfrak{A}_k^0\|\delta^{-2}K\|x\|, \quad \text{by (2.5)}.$$

Thus it follows from hypothesis 2 that $|\omega_i(x) - \omega_j(x)| \le \epsilon\|x\|$ for all $x \in \mathfrak{A}_d$ provided i, j is large. By Kaplansky's density theorem [2] this is still true for all $x \in \mathfrak{A}$, and the theorem is proved.

We now prove the two lemmas needed in the proof of Theorem 2.1.

Lemma 2.2. *Let $\{A_n\}$ be a sequence of positive, mutually commuting, self-adjoint operators on a separable Hilbert space $\mathfrak{F}$. Then there exist a unique* *projection P on $\mathfrak{F}$ commuting with all A_n and a unique positive selfadjoint operator A on $P\mathfrak{F}$ such that*

$$A = \sup_n \left\{ \sum_{k=1}^{n} A_k \mid P\mathfrak{F} \right\} \quad \textit{and} \quad \sup_n \left(\psi, \sum_{k=1}^{n} A_k \psi \right) = \infty$$

for all $\psi \neq 0$ in $(I - P)\mathfrak{F}$.

Proof. The existence of P and A is easily established in a common spectral representation for $\{A_n\}$ in which each A_n is represented by multiplication by some measurable function. The uniqueness follows from the fact that we have

$$P\mathfrak{F} = \overline{\left\{ \psi \mid \sum_n (A_n^{1/2}\psi, A_n^{1/2}\psi) < \infty \right\}} \quad \text{and} \quad (A^{1/2}\psi, A^{1/2}\psi) = \sum_n (A_n^{1/2}\psi, A_n^{1/2}\psi),$$

where $\psi \in \mathfrak{D}(A^{1/2})$ if and only if the sum to the right converges.

We remark that if f is a Borel function such that $\lim_{\lambda \to \infty} f(\lambda) = a$ exists, we define $f(A) = f(AP) + a(I - P)$.

Lemma 2.3. *Let A and B be positive, bounded operators on $\mathfrak{F}$ such that $AB = BA$ and $B \leq A$. Then $\|B\,X\,B\| \leq \|A\,X\,A\|$ for all $X \in \mathfrak{B}(\mathfrak{F})$.*

Proof. Let E be the orthogonal projection onto the closure of the range of B. Then $EA = AE$, thus $\|AE\,X\,AE\| \leq \|A\,X\,A\|$, and it is enough to prove $\|BE\,X\,BE\| \leq \|AE\,X\,AE\|$, i.e., we can assume $E = I$. Then A^{-1} exists as a selfadjoint operator and $A^{-1}B \leq I$. Thus, for $X \in \mathfrak{B}(\mathfrak{F})$:

$$\|B\,X\,B\| = \|BA^{-1}A\,X\,AA^{-1}B\| \leq \|BA^{-1}\|\|A\,X\,A\|\|A^{-1}B\| \leq \|A\,X\,A\|.$$

3. **Application to $P(\phi)_2$.** Let $\mathfrak{F}$ be the Fock space over $L^2(\mathrm{R}^1)$ (see [4] for explanation of the terms in this chapter and the results mentioned in the introductory remarks); let $A(x)$ and $A^*(x)$ be the usual (configuration space) annihilation and creation bilinear forms, and let

$$\phi_0(x) = 2^{-1/2}(A^*(x) + A(x)), \qquad \pi_0(x) = i2^{-1/2}(A^*(x) - A(x)).$$

For $f_1, f_2 \in \mathcal{S}_{\mathrm{R}}(\mathrm{R})$, define the time zero Newton-Wigner fields by

$$\phi_0(f_1) = \int f_1(x)\phi_0(x)\,dx, \qquad \pi_0(f_2) = \int f_2(x)\pi_0(x)\,dx.$$

Then $\phi_0(f_1) + \pi_0(f_2)$ is a selfadjoint operator. Let μ be the positive, selfadjoint operator $(-d^2/dx^2 + m_0^2)^{1/2}$ on $L^2(R^1)$. For $f \in \mathcal{S}_{\mathrm{R}}(\mathrm{R})$, define the relativistic time zero fields by

$$\phi(f) = \phi_0(\mu^{-1/2}f), \qquad \pi(f) = \phi_0(\mu^{1/2}f).$$

Let B be an interval in R and define

$$\mathfrak{A}_d^0(B) = \left\{ \sum_{i=1}^{n} c_i \exp(i(\phi_0(f_i) + \pi_0(g_i))) \,\middle|\, c_i \in \mathbf{C},\ \operatorname{supp} f_i, g_i \subseteq B \right\},$$

$$\mathfrak{A}_d(B) = \left\{ \sum_{i=1}^{n} c_i \exp(i(\phi(f_i) + \pi(g_i))) \,\middle|\, c_i \in \mathbf{C},\ \operatorname{supp} f_i, g_i \subseteq B \right\}.$$

It follows from the commutation relations that $\mathfrak{A}_d^0(B)$ and $\mathfrak{A}_d(B)$ are *-algebras. Let $\mathfrak{A}^0(B)$ and $\mathfrak{A}(B)$ denote their weak closures. We will refer to the set of $\mathfrak{A}^0(B)$ as the Newton-Wigner local algebras and the $\mathfrak{A}(B)$ as the relativistic (time zero) local algebras.

$\mathfrak{A}^0(B)$ has a simple algebraic structure, it is the von Neumann algebra $R(L^2(B), L^2(B)/L^2(R))$ defined by Araki [1]. $\mathfrak{A}^0(B)$ is a type I factor. If $\mathfrak{F}(B)$ is the Fock space over $L^2(B)$ and $\Omega_{\sim B}$ is the vacuum of $\mathfrak{F}(\sim B)$, $\mathfrak{F}(B)$ may be identified with a subspace of $\mathfrak{F}$:

$$\mathfrak{F}(B) = \mathfrak{F}(B) \otimes \Omega_{\sim B} \subseteq S(\mathfrak{F}(B) \otimes \mathfrak{F}(\sim B)) = \mathfrak{F}$$

where S is the projection from the total, unsymmetrized "Fock space" $\mathfrak{F}_0 = \bigoplus_{n=0}^{\infty} L^2(R^n)$ into the Boson Fock space $\mathfrak{F} = \bigoplus_{n=0}^{\infty} L^2_{\mathrm{sym}}(R^n)$. It is easy to see that $\mathfrak{A}^0(B)$ leaves $\mathfrak{F}(B)$ (under this identification) invariant, and from the irreducibility of the Fock representation it follows that $\mathfrak{A}^0(B)E = \mathfrak{B}(\mathfrak{F}(B))$ where E is the projection from $\mathfrak{F}$ onto $\mathfrak{F}(B)$, and $\mathfrak{B}(\mathfrak{F}(B))$ is the algebra of all bounded operators on $\mathfrak{F}(B)$. Thus $\mathfrak{A}^0(B)$ is isomorphic to $\mathfrak{B}(\mathfrak{F}(B))$ in a canonical way.

Since $\mu^{\pm 1/2}$ transforms functions with compact support into functions with unbounded support, no such simple characterization is possible for $\mathfrak{A}(B)$.

$\mathfrak{A}(B)$ is the algebra $R(\mu^{-1/2}L^2(B), \mu^{1/2}L^2(B)/L^2(R))$ of Araki [1], and is a type III factor. The reason for considering the relativistic local algebras is that both the free and the $P(\phi)$ dynamics transform these algebras into themselves with propagation speed 1. They can thus be used to construct local algebras over space-time regions satisfying the Haag-Kastler axioms. The Newton-Wigner algebras are not even invariant under the free dynamics. The reason for considering these, is that they can be used to show convergence of the approximate vacuum. If ω_g is the ground state of a space cutoff Hamiltonian $H(g)$ in the $P(\phi)_2$ theory it is known that $\omega_g(N_{r,B}) \leq K$ where $N_{r,B} = d\Gamma(\chi_B \mu^r \chi_B)$, $0 \leq r < 1/4$ and K is a constant independent of g and of translation of B [6]. Relative to the decomposition $\mathfrak{F} = \mathfrak{F}(B) \otimes_s \mathfrak{F}(\sim B)$ we have $N_{r,B} = S(\hat{N}_{r,B} \otimes I)S$, and $\hat{N}_{r,B}$ is positive with compact resolvent [3]. Thus $\{\omega_g\}$ is contained in a norm compact subset of the dual of $\mathfrak{A}^0(B)$ [3] and we can extract from $\{\omega_g\}$ a sequence $\{\omega_n\}$ which converges in norm on each $\mathfrak{A}^0(B)$. We will show that the elements in $\mathfrak{A}_d(C)$ for a bounded region C can be approximated by elements in $\mathfrak{A}^0(B)$ in the sense of Theorem 2.1, thus $\{\omega_n\}$ converges in norm on each $\mathfrak{A}(C)$. The limiting state ω will therefore be locally normal, from which it follows that a Hamiltonian defining the $P(\phi)$ dynamics can be defined on the representation of the local algebras $\mathfrak{A}(C)$ defined by ω by the GNS-construction. We now turn to the proof of our main theorem.

Theorem 3.1. *Let $\{\omega_n\}$ be a sequence of vector states on $\mathcal{F}$ such that:*

1. $\omega_n(N_{0,[m,m+1)}) \leq K$ *for* $n = 1, 2, \cdots$; $m = 0, \pm 1, \pm 2, \cdots$, $K \in \mathbf{R}$ *is independent of* m *and* n.

2. *For each bounded interval* $B \subseteq \mathbf{R}$, ω_n *converges in norm on* $\mathfrak{A}^0(B)$. *Then, for each bounded interval* B, ω_n *converges in norm on* $\mathfrak{A}(B)$.

Proof. Let B be a fixed bounded interval. We will apply Theorem 2.1 with $N_m = N_{0,[m,m+1)}$, $m \in Z$ and

$$d_m = \begin{Bmatrix} 1/|m|^3 & \text{if } m \neq 0 \\ 1 & \text{if } m = 0 \end{Bmatrix} = \frac{1}{\{m\}^3},$$

where the definition of $\{m\}$ is obvious. Furthermore, we let $\mathfrak{A} = \mathfrak{A}(B)$, $\mathfrak{A}_d = \mathfrak{A}_d(B)$ and $\mathfrak{A}^0_m = \mathfrak{A}^0([-m, m))$. Define the mapping $E_m : \mathfrak{A}_d \to \mathfrak{A}^0_m$ by

$$E_m\left(\sum_{i=1}^{n} c_i \exp\left(i\phi_0(\mu^{-1/2}f_i) + i\pi_0(\mu^{1/2}f_i)\right)\right)$$
$$= \sum_{i=1}^{n} c_i \exp\left(i\phi_0(\chi_{[-m,m)}\mu^{-1/2}f_i) + i\pi_0(\chi_{[-m,m)}\mu^{1/2}f_i)\right).$$

Hypotheses 1 and 2 of Theorem 2.1 are fulfilled, so it remains to prove (*). To do so we expand an element $C \in \mathfrak{A}_d = \mathfrak{A}_d(B)$ in terms of creation and annihilation operators [3]:

$$C = \sum_{\alpha\beta} C_{\alpha\beta},$$

$$C_{\alpha\beta} = \int c_{\alpha\beta}(x_1 \cdots x_\alpha, x_{\alpha+1}, \cdots, x_{\alpha+\beta}) A^*(x_1) \cdots A^*(x_\alpha) A(x_{\alpha+1}) \cdots A(x_{\alpha+\beta})\, dx_1 \cdots dx_\alpha,$$

where the kernel $c_{\alpha\beta}(x)$ is symmetric in the creation and annihilation variables respectively. We will furthermore expand the (unbounded) operators $C_{\alpha\beta}$ in operators affiliated with the Newton-Wigner local algebras. By a slight modification of the terminology in [3], a localization index L is defined as an ordered pair $L =$

(L_α, L_β) where L_α and L_β are sequences of nonnegative integers:

$$L_\alpha = (\cdots k^\alpha_{-1}, k^\alpha_0, k^\alpha_1, \cdots), \quad L_\beta = (\cdots k^\beta_{-1}, k^\beta_0, k^\beta_1, \cdots),$$

where $\Sigma^\infty_{n=-\infty} k^\alpha_n = \alpha$, $\Sigma^\infty_{n=-\infty} k^\beta_n = \beta$. L corresponds to the annihilation of k^β_n particles and the creation of k^α_n particles in the region $[n, n+1\rangle$, $n = 0, \pm 1, \cdots$. $D(L)$ is defined as the number $D(L) = \Sigma^\infty_{n=-\infty} |n|(k^\alpha_n + k^\beta_n)$.

For fixed L, let f^α_L be a product of characterisitc functions of each of the creation variables $x_1 \cdots x_\alpha$, where the function $\chi_{[m,m+1\rangle}$ occurs k^α_m times. Let g^α be the symmetrization of f^α in the variables $x_1 \cdots x_\alpha$. g^α is then independent of the choice of f^α subject to the restriction above. Define g^β in the corresponding way for the annihilation variables, and define

$$\chi_L(x_1 \cdots x_\alpha, x_{\alpha+1}, \cdots, x_{\alpha+\beta}) = g^\alpha(x_1 \cdots x_\alpha) g^\beta(x_{\alpha+1}, \cdots, x_{\alpha+\beta}).$$

Then define, for $x = (x_1, \cdots, x_{\alpha+\beta})$:

$c^L_{\alpha\beta}(x) = \chi_L(x) c_{\alpha\beta}(x)$,

$c^L_{\alpha\beta} = \int c^L_{\alpha\beta}(x) A^*(x_1) \cdots A^*(x_\alpha) A(x_{\alpha+1}) \cdots A(x_{\alpha+\beta}) dx$,

$c^L_{\alpha\beta} =$ the operator $L^2_{\mathrm{sym}}(\mathbf{R}^\beta) \to L^2_{\mathrm{sym}}(\mathbf{R}^\alpha)$ with kernel $c^L_{\alpha\beta}(x)$.

Then $C_{\alpha\beta} = \Sigma_L\, L!\, C^L_{\alpha\beta}$ where $L! = \alpha!\beta!/(\Pi_m (k^\alpha_m!))(\Pi_m (k^\beta_m!))$, and the same relation holds for $c_{\alpha\beta}(x)$. The reason for the factor in front of $C^L_{\alpha\beta}$ is the following: In the decomposition:

$$1 = \sum_{m_1=-\infty}^{\infty} \cdots \sum_{m_\alpha=-\infty}^{\infty} \chi_{[m_1, m_1+1\rangle}(x_1) \cdots \chi_{[m_\alpha, m_\alpha+1\rangle}(x_\alpha)$$

a function of the form f^α occurs $\alpha!/\Pi_m (k^\alpha_m!)$ times.

Let Λ be the set of localization indices, and let Λ_m be the set of localization indices L such that $k^\alpha_n = 0 = k^\beta_n$ for $n < -m$ and $n \geq m$. Then it is clear that:

$$E_m(C) = \sum_{\alpha, \beta, L \in \Lambda_m} L!\, C^L_{\alpha\beta}, \qquad C - E_m(C) = \sum_{\alpha, \beta, L \notin \Lambda_m} L!\, C^L_{\alpha\beta}.$$

Thus, in order to prove hypothesis (*) of Theorem 2.1 it is enough to prove:

Proposition 3.2. *Let $f(\lambda) = e^{-M\lambda^2}$, $M \in R^+$. If M is sufficiently large there exist constants $R^L_{\alpha\beta}$ such that $\Sigma_{\alpha,\beta,L} R^L_{\alpha\beta} < \infty$, and $L!\|f(N) C^L_{\alpha\beta} f(N)\| \leq R^L_{\alpha\beta}\|C\|$, for all $C \in \mathfrak{A}_d(B)$.*

Proof. To estimate $\|f(N) C^L_{\alpha\beta} f(N)\|$ we use an estimate on $C^L_{\alpha\beta}$ derived by Glimm and Jaffe. It is based on the fact that the operators $\mu^{\pm 1/2}$ transform a function $f \in \mathcal{S}_{\mathbf{R}}(\mathbf{R})$ with compact support into a function $\mu^{\pm 1/2} f$ which is dominated by const $e^{-m_0|x|}$ for large values of the argument x. This leads to the estimate

(3.1) $$\|c^L_{\alpha\beta}\| \le \exp(K_1(\alpha+\beta) - m_0 D(L))\|C\|$$

where K_1 is a constant dependent on the region B [3, Lemma 4.3].

Next we decompose the Fock space into the tensor product of Fock spaces over the regions $[n, n+1\rangle$ (see [5] for the definition of infinite tensor products)

$$\mathcal{F} = S \bigotimes_{n=-\infty}^{\infty}{}^{\{\Omega_n\}} \mathcal{F}[n, n+1\rangle, \qquad \Omega = \bigotimes_n \Omega_n$$

and write $\mathcal{F}[n, n+1\rangle = \bigoplus_{m=0}^{\infty} \mathcal{F}_{n,m}$ where $\mathcal{F}_{nm}$ is the eigenspace of $N_{0,[n,n+1\rangle}$ corresponding to eigenvalue m, i.e. $\mathcal{F}_{n,m} = L^2_{\mathrm{sym}}([n, n+1\rangle^m)$. Let V be the set of sequences $\lambda = (\cdots \lambda_{-1}, \lambda_0, \lambda_1, \cdots)$ of nonnegative integers such that $\sum_{n=-\infty}^{\infty} \lambda_n < \infty$, and define

$$\mathcal{F}_\lambda = S \bigotimes_{n=-\infty}^{\infty}{}^{\{\Omega_n\}} \mathcal{F}_{n,\lambda_n} \quad \text{for } \lambda \in V.$$

This is the set of vectors in Fock space which has λ_n particles in the region $[n, n+1\rangle$, $n = 0, \pm 1, \cdots$. The spaces $\mathcal{F}_\lambda$ are mutually orthogonal and span $\mathcal{F}$: $\mathcal{F} = \bigoplus_{\lambda \in V} \mathcal{F}_\lambda$.

$\mathcal{F}_\lambda$ is an eigenspace of $N = \sum_n d_n N_{0,[n,n+1\rangle}$ of eigenvalues $\sum_n d_n \lambda_n$. $C^L_{\alpha\beta}$ transforms $\mathcal{F}_\lambda = S \otimes_n \mathcal{F}_{n,\lambda_n}$ into $S \otimes_n \mathcal{F}_{n,\lambda_n - k_n^\beta + k_n^\alpha}$ if $k_n^\beta \le \lambda_n$ for all n, and into $\{0\}$ otherwise. Furthermore

$$\|C^L_{\alpha\beta} | \mathcal{F}_\lambda\| = \begin{cases} \|c^L_{\alpha\beta}\| \prod_{n=-\infty}^{\infty} \left\{ \left(\dfrac{\lambda_n!}{(\lambda_n - k_n^\beta)!} \right)^{1/2} \left(\dfrac{(\lambda_n - k_n^\beta + k_n^\alpha)!}{(\lambda_n - k_n^\beta)!} \right)^{1/2} \right\} & \text{if } k_n \le \lambda_n \text{ for all } n \\ 0 & \text{otherwise.} \end{cases}$$

This follows from the definition of the creation and annihilation bilinear forms $A^*(x)$ and $A(x)$. Since $f(N)\, C^L_{\alpha\beta} f(N)$ transforms the mutually orthogonal spaces $\mathcal{F}_\lambda$ into each other, it follows that:

$$\begin{aligned} \|f(N) C^L_{\alpha\beta} f(N)\| &= \sup_{\lambda \in V} \|f(N) C^L_{\alpha\beta} f(N) | \mathcal{F}_\lambda\| \\ &= \sup_{\lambda \in V;\, k_n^\beta \le \lambda_n} f\left(\sum_n d_n (\lambda_n - k_n^\beta + k_n^\alpha) \right) \|c^L_{\alpha\beta}\| \\ &\quad \cdot \prod_n \left\{ \left(\frac{\lambda_n!}{(\lambda_n - k_n^\beta)!} \right)^{1/2} \left(\frac{(\lambda_n - k_n^\beta + k_n^\alpha)!}{(\lambda_n - k_n^\beta)!} \right)^{1/2} f\left(\sum_n d_n \lambda_n \right). \end{aligned}$$

Using estimate (3.1) and the explicit definitions of f and d_n it follows that:

$$L!\|f(N)C^L_{\alpha\beta}f(N)\|$$

$$\leq \sup_{\lambda\in V;\, k_n^\beta\leq\lambda_n} \alpha!\beta! \exp\left(-M\left(\sum_n \frac{1}{\{n\}^3}(\lambda_n - k_n^\beta + k_n^\alpha)^2\right)\right)$$

$$\cdot \exp\left(K_1(\alpha+\beta) - m_0 \sum_n |n|(k_n^\alpha + k_n^\beta)\right)\prod_n\left\{\lambda_n^{k_n^\beta/2}(\lambda_n - k_n^\beta + k_n^\alpha)^{k_n^\alpha/2}\right\}$$

$$\cdot \exp\left(-M\left(\sum_n \frac{1}{\{n\}^3}\lambda_n\right)^2\right)\|C\|$$

$$\leq \left\{\sup_{k\in V}\left(\sum_n k_n\right)! \exp\left(-\frac{M}{2}\left(\sum_n \frac{1}{\{n\}^3}k_n\right)^2 - \frac{m_0}{2}\left(\sum_n \{n\}k_n\right)\right)\right\}^2$$

$$\cdot\left\{\sup_{\lambda\in V;\, k_n^\beta\leq\lambda_n}\prod_n\left(\exp\left(-\frac{M}{2}\frac{1}{\{n\}^6}\lambda_n^2\right)\lambda_n^{k_n^\beta/2}\exp\left(\left(K_1+\frac{m_0}{2}\right)k_n^\beta - \frac{m_0}{2}|n|k_n^\beta\right)\right)\right\}$$

$$\cdot\left\{\sup_{\lambda\in V;\, k_n^\alpha\leq\lambda_n}\prod_n\left(\exp\left(-\frac{M}{2}\frac{1}{\{n\}^6}\lambda_n^2\right)\lambda_n^{k_n^\alpha/2}\exp\left(\left(K_1+\frac{m_0}{2}\right)k_n^\alpha - \frac{m_0}{2}|n|k_n^\alpha\right)\right)\right\}\|C\|$$

$$= S^2 \cdot T(k^\beta)\cdot T(k^\alpha)\|C\|.$$

Proposition 3.2 now follows from the following two lemmas, since we may choose $R^L_{\alpha\beta} = S^2T(k^\beta)T(k^\alpha)$, and thus $\Sigma_{\alpha\beta L} R^L_{\alpha\beta} = S^2(\Sigma_{k\in V} T(k))^2$.

Lemma 3.3. $S < \infty$.

Lemma 3.4. $\Sigma_{k\in V} T(k) < \infty$ *if* M *is sufficiently large.*

Proof of Lemma 3.3. When at least one $k_n \neq 0$ we have that:

$$\left(\sum_n k_n\right)! \exp\left(-\frac{M}{2}\left(\sum_n\left(\frac{1}{\{n\}^3}k_n\right)^2 - \frac{m_0}{2}\left(\sum_n \{n\}k_n\right)\right)\right)$$

$$\leq \exp\left(\log\left(\sum_n k_n\right)\left(\sum_n k_n\right) - \frac{M}{2}\left(\sum_n \frac{1}{\{n\}^3}k_n\right)^2 - \frac{m_0}{2}\left(\sum_n \{n\}k_n\right)\right)$$

$$\leq \exp\left(K_\epsilon\left(\sum_n k_n\right)^{1+\epsilon} - \frac{M}{2}\left(\sum_n \frac{1}{\{n\}^3}k_n\right)^2 - \frac{m_0}{2}\left(\sum_n \{n\}k_n\right)\right)$$

where K_ϵ is a constant. Define a function $g: V \longrightarrow R$ by

$$g(k) = K_\epsilon\left(\sum_n k_n\right)^{1+\epsilon} - \frac{M}{2}\left(\sum_n \frac{1}{\{n\}^3}k_n\right)^2 - \frac{m_0}{2}\left(\sum_n \{n\}k_n\right).$$

To prove Lemma 3.3 it is enough to show that g is bounded above. By Hölder's inequality:

$$\sum_n k_n = \sum_n (\{n\}^{-3/4} k_n^{1/4} \{n\}^{3/4} k_n^{3/4}) \leq \left(\sum_n \{n\}^{-3} k_n\right)^{1/4} \left(\sum_n \{n\} k_n\right)^{3/4}.$$

Defining $x = (\Sigma_n \{n\}^{-3} k_n)^{1/4}$, $y = (\Sigma_n \{n\} k_n)^{1/4}$ we thus obtain:

$$g(k) \leq K_\epsilon x^{1+\epsilon} y^{3+3\epsilon} - Mx^8/2 - m_0 y^4/2,$$

and the last function of x and y is easily shown to be bounded above when $\epsilon < 1/7$.

Proof of Lemma 3.4. Define, for $\lambda \geq k > 0$:

$$g(\lambda) = \exp\left(-\frac{M}{2}\frac{1}{\{n\}^6}\lambda^2 + \frac{k}{2}\log\lambda + \left(K_1 + \frac{m_0}{2}\right)k - \frac{m_0}{2}|n|k\right).$$

A calculation shows that

$$\sup_{\lambda \geq k} g(\lambda) \leq \exp\left(k\left(-\frac{1}{2} + 3\log\{n\} - \frac{1}{2}\log M + K_1 + \frac{m_0}{2} - \frac{m_0}{2}|n|\right)\right).$$

Thus, by choosing M large enough, we obtain

$$\sup_{\lambda \geq k} g(\lambda) \leq \exp(k(-m_0|n|/4 - 1)).$$

It follows that

$$\sum_{k \in V} T(k) \leq \sum_{k \in V} \prod_{n=-\infty}^{\infty} \exp\left(k_n\left(-\frac{m_0}{2}|n| - 1\right)\right)$$

$$= \prod_{n=-\infty}^{\infty}\left(\sum_{k_n=0}^{\infty} \exp\left(-k_n\left(-\frac{m_0}{4}|n| - 1\right)\right)\right) = \prod_{n=-\infty}^{\infty}\left(1 - \exp\left(\frac{m_0}{4}|n| - 1\right)\right)^{-1}.$$

Since $\Sigma_n \exp(-m_0|n|/4 - 1) < \infty$ this last infinite product converges, and the lemma follows.

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