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EQUIVARIANT METHOD FOR PERIODIC MAPS

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ABSTRACT. The notion of coherency with submanifolds for a Morse function on a manifold is introduced and discussed in a general way. A Morse inequality for a given periodic transformation which compares the invariants called qth Euler numbers on fixed point set and the invariants called qth Lefschetz numbers of the transformations is thus obtained. This gives a fixed point theorem in terms of qth Lefschetz number for arbitrary q.

Let f be a periodic transformation of a closed m-dimensional manifold M with fixed point set N. We develop in this note an equivariant approach using Morse theory. We introduce in §2 the notion of coherency with a submanifold S of M for a Morse function and show that such S-coherent Morse functions are dense in $C^\infty(M)$. Furthermore, in this approximation f-invariance will be preserved (§3). The coherency with the fixed point set N of f makes it possible to compare the difference of qth Euler number of N and qth Lefschetz number of f. More precisely, let $\beta_q(N)$ and $\lambda_q(f)$ be respectively the qth Betti numbers of N and the trace of f^* on the qth homology group $H_q(M)$ with real coefficients. Let $B_q(N)$ and $\Lambda_q(f)$ be their alternative sums respectively, i.e.,

$$\beta_q(N) = \beta_q(N) - \beta_{q-1}(N) + \cdots + (-1)^q\beta_0(N),$$

$$\Lambda_q(f) = \lambda_q(f) - \lambda_{q-1}(f) + \cdots + (-1)^q\lambda_0(f),$$

where $0 \leq q \leq m$. We establish in §5 an inequality for arbitrary q that $|\beta_q(N) - \Lambda_q(f)|$ is no greater than the qth Morse difference of an arbitrary f-invariant N-coherent Morse function. We obtain as corollaries a fixed point theorem in terms of arbitrary Λ_q (when $q = m$, this is the Lefschetz fixed point theorem) and a more geometric proof of the fact that $\beta_n(N) = \Lambda_n(f)$, i.e., the Euler number of N is equal to the Lefschetz number of f.

The Lemma 1 (§1) which states that a smooth function can be approximated by a Morse function with prescribed "boundary value" is essential to the construction of the approximations.

1. **A Morse extension.** For a real-valued smooth function F on M, let $C(F)$ denote the set of all critical points of F. F is called a *Morse function* if for any $p \in C(F)$, the determinant of the Hessian at p does not vanish.

We assume without loss of generality that M is a riemannian manifold with a metric g. Let g_{ij} be the metric tensor of g with respect to a local coordinate (x^i)

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and let g^{ij} be the inverse of g_{ij} as matrices. Using the metric g, the differential $dF(x)$ of F at x has a natural way to be identified with a tangent vector at x which is called the gradient $\nabla F(x)$ at x. Locally we have $\nabla F(x) = g^{ij}(\partial F/\partial x^i)(\partial/\partial x^j)$.

We define $\|dF(x)\|$ by

$$\|dF(x)\|^2 = g(\nabla F, \nabla F) \quad \text{at } x$$

and define $\|F\|_{0,\Omega}$ and $\|F\|_{1,\Omega}$ of F on an open set Ω in M by

$$\|F\|_{0,\Omega} = \sup\{|F(x)|; x \in \Omega\},$$

$$\|F\|_{1,\Omega} = \sup\{|F(x)| + \|dF(x)\|; x \in \Omega\}.$$

Let ϕ: $\mathbf{R} \to \mathbf{R}$ be a C^∞-function with $0 \le |\phi(r)| \le 1$, $\phi(0) = 1$, $\phi''(0) < 0$ and $\phi(r) = 0$ for $|r| \ge 1$. We denote throughout the induced function of mollifier by ϕ_ε for each positive number ε, i.e. $\phi_\varepsilon(r) = \phi(r/\varepsilon)$.

There exists a constant $a > 1$ such that

$$(1) \qquad |\phi_\varepsilon'(r)| < a/\varepsilon.$$

It is well known ([4] or [3]) that any given real-valued smooth function on a compact manifold M can be approximated by a Morse function in the norm $\| \ \|_{1,M}$. The following lemma establishes this approximation theorem even when the "boundary value" of the desired Morse function has been given.

Lemma 1. *Let Ω and D be open sets of a smooth manifold M such that Ω has a compact closure Ω with smooth boundary $\partial\Omega$ and $\bar{D} \subset \Omega$. Let F be a Morse function defined on $M - \bar{D}$. Then $F \mid M - \Omega$ can be extended to a Morse function $\tilde{F}$: $M \to$* **R**. *Moreover if a smooth function G on M with $\|F - G\|_{0,M-\bar{D}} < \varepsilon$, is given, then the above Morse extension can be made so that $\|\tilde{F} - G\|_{0,M} < 2\varepsilon$.*

Proof. Choose a metric g for M. For a point x inside Ω, we denote by $r(x)$ the distance with respect to g from x to $\partial\Omega$. Let Ω_r be the set $\{x \in \Omega \mid r(x) > r\}$. Since $C(F)$ is discrete and Ω is compact, there exist positive numbers η, R and δ such that

$$(2) \qquad \delta < \min\{1/2(1 + a), \sqrt{\varepsilon/\eta}\} \quad \text{and} \quad \|dF(x)\| > \eta$$

for all x in the strip $\overline{\Omega}_{R-\delta} - \Omega_{R+2\delta}$ contained in $\Omega - D$.

Define $H: M \to \mathbf{R}$ by patching together F and G in $\Omega_{R+\delta} - \Omega_{R+2\delta}$ as follows:

$$\begin{aligned} H(x) &= F(x), & x &\in M - \Omega_{R+\delta}, \\ (3) \qquad &= G(x) + \varphi_\delta(R + \delta - r(x))(F(x) - G(x)), & x &\in \Omega_{R+\delta} - \Omega_{R+2\delta}, \\ &= G(x), & x &\in \Omega_{R+2\delta}. \end{aligned}$$

It follows that $\|H - G\|_{0,M} < \varepsilon$.

Let E be a Morse function on $\Omega_{R-\delta}$ approximating $H \mid \Omega_{R-\delta}$ such that

$$\|E - H\|_{1,\Omega_{R-\delta}} < \delta^2\eta < \varepsilon. \tag{4}$$

Finally we define $\tilde{F}$ on M by patching together E and F in the strip $\Omega_{R-\delta} - \Omega_R$ as above. In order to see that $\tilde{F}$ is a Morse function on M, it suffices to show that $\tilde{F}$ has no critical point in $\bar{\Omega}_{R-\delta} - \Omega_R$. In fact, for x in $\bar{\Omega}_{R-\delta} - \Omega_R$, we have $H(x) = F(x)$ and

$$\begin{aligned} \|d\tilde{F}(x)\| &\geq \|dF(x)\| - |\varphi_\delta(R - r(x))| \cdot \|dE(x) - dH(x)\| \\ &\quad - \|d\varphi_\delta(R - r(x))\| \cdot |E(x) - H(x)| \\ &> \eta - \delta^2\eta - (a/\delta)\delta^2\eta > \eta(1 - \delta(1 + a)) > \eta/2 > 0, \end{aligned}$$

since we have the estimates (1), (2) and (4). The approximation of $\tilde{F}$ to G follows evidently from the construction.

2. **Coherency with submanifold.** Let S be a closed embedding submanifold of M. In this section we define S-coherent Morse functions and show an approximation theorem of smooth functions by S-coherent Morse functions.

Definition 1. A Morse function F on M is called *S-coherent* if for each p in $C(F \mid S)$, there is a coordinate neighborhood $(U, (x_i))$ with origin at p, $U \cap S = \{x_{s+1} = \cdots = x_m = 0\}$, and

$$F(x_1 \cdots x_m) = F(0) - x_1^2 - \cdots - x_\lambda^2 + \cdots + x_m^2$$

where s is the dimension of S at p with $s \geq \lambda$.

Such a $(U, (x_i))$ is called an S-coherent coordinate neighborhood of p for F. Evidently, if F is an S-coherent Morse function on M, then $F \mid S$ is a Morse function on S with $C(F \mid S) \subset C(F)$ and at each p of $C(F \mid S)$, the index of $F \mid S$ is equal to the index of F.

For the convenience of later use, we fix the following notation:

Definition 2. Given a smooth function ψ defined on a closed embedding submanifold S of M, we denote by ψ^* an extension of ψ on a tubular neighborhood T_ρ of S with radius ρ defined as follows. Let ρ be so small that for any x in T_ρ, there is a unique geodesic joining x to a point x' of S and having the length equal to the distance $r(x)$ from x to S. Let

$$\psi^*(x) = \psi(x') \cdot (2 - \varphi_\rho(r(x)))$$

where φ_ρ is the mollifier relative to ρ (see §1).

If ψ is a Morse function, so is ψ^*. In fact,

$$C(\psi) = C(\psi^*) \quad \text{and} \quad \varphi''(0) < 0.$$

Note that at $p \in C(\psi)$, the index of ψ equals the index of ψ^*.

Theorem 1. *Given a closed submanifold S of M, any smooth function G on M can be approximated uniformly by an S-coherent Morse function F.*

Proof. Let g be a Morse function on S approximating $G \mid S$. By Lemma 1, the g^* on a tubular neighborhood of S can be extended to a Morse function F on M. F is evidently S-coherent. If the tubular neighborhood of S is sufficiently small, F can be made to approximate G. Q.E.D.

3. **Review of isometric actions.** In general, for a compact riemannian manifold (M, g), let $\mathrm{ISO}(M,g)$ denote the full isometry group. Let G be a closed subgroup of $\mathrm{ISO}(M,g)$ and p a point in M. By the *isotropy group* G^p, we mean the subgroup of isometries which leave p fixed. The *orbit* $G(p)$ of G at p is the set $\{\gamma(p); \gamma \in G\}$.

Each orbit is a closed submanifold embedded in M. An orbit $G(p)$ is called *principal* if

(1) for any $q \in M$, $\dim G^p \leq \dim G^q$, and

(2) the number of components of G^p is no greater than the number of components of G^q whenever $\dim G^p = \dim G^q$.

We quote the following well-known result.

Lemma 2 [5]. *Let G be a closed subgroup of* $\mathrm{ISO}(M,g)$ *of a complete riemannian manifold* (M, g). *Then the union of all the principal orbits of G is open and dense in M.*

We return to our given periodic map f of M with order ν. Without loss of generality, we may assume that f is an isometry of (M, g) with some metric. In fact we can modify an arbitrarily given metric $\bar{g}$ by taking the mean of the induced metrics $(f^k)^*g$ for $k = 1, 2, \ldots, \nu$.

Let Γ be the subgroup generated by f in $\mathrm{ISO}(M,g)$. Γ is finite and cyclic with order ν. By the order of an orbit of Γ, we mean the cardinal number of the orbit. For the integer k such that there exists an orbit Γ with order k, let M_k be the union of the orbits of order l where l is a divisor of k. Thus we have a lattice consisting of these M_k's with inclusion as the partial ordering. The lower bound of the lattice is evidently the fixed point set $N = M_1$.

We now consider some geometries about N and more generally about M_k's.

Lemma 3. *The fixed point set N of an isometry f is a closed totally geodesic submanifold embedded in M* [2]. *If the isometry f is periodic, then each* M_k, *defined in the above, is a closed totally geodesic submanifold embedded in M as well as in each* M_j *with j being a multiple of k.*

Proof. For the first statement, one can refer to [2]. An elementary proof with clearer geometric insight can be obtained by using the following two facts as the basis of induction to construct, in an obvious way, local coordinates of N for proving that N is a submanifold of M.

(1) For two points p and q of N which are sufficiently close to each other, the unique geodesic connecting p and q is contained in N.

(2) Let γ_1 and γ_2 be two geodesics of M which are contained in N and intersect with each other at a point p of N. Then the parallel transportation of γ_1 along γ_2 generates a 1-parametered family of geodesics whose union is entirely contained in N.

For the second statement of the lemma, we need only to notice that M_k is exactly the fixed point set of f^k acting on M as well as on M_j with j being a multiple of k. This completes the proof.

For any two M_k and M_l, the intersection $M_k \cap M_l$ is evidently the $M_{(k,l)}$ where (k, l) is the greatest common divisor of k and l. On the other hand, $M = M_\nu$. In fact, for each M_l and each x in M_l, choose a convex neighborhood U of x such that for any y in U, the geodesic joining y to x in U is the only curve joining y to $\Gamma(x)$ and having the length equal to the distance from y to $\Gamma(x)$. It follows that $\Gamma^y \subset \Gamma^x$ and therefore the order of $\Gamma(x)$ is a divisor of that of $\Gamma(y)$. By Lemma 2, we see that the order of $\Gamma(x)$ is a divisor of ν.

4. The approximation.

Theorem 2. *Given a periodic transformation f of M with fixed point set N, an f-invariant smooth function $G: M \to \mathbf{R}$ can be uniformly approximated by an f-invariant N-coherent Morse function F.*

Proof. We construct F inductively in the following steps.

Step 1. Let h_1 be a Morse function on N approximating $G \mid N$ uniformly. Recalling the Definition 2, we extend h_1 to h_1^* on a tubular neighborhood $T_{2\rho}$ of N.

Step 2. For each prime number p which is a divisor of ν, we shall extend $h_1^* \mid T_\rho \cap M_p$ to an f-invariant Morse function $h_p: M_p \to \mathbf{R}$ which approximates $G \mid M_p$.

For a general k with $1 \leq k \leq \nu$, let U_k denote the union of all orbits of order k. By Lemma 2, U_k is open and dense in M_k. Now $h_1^* \mid T_\rho \cap U_p$ induces a Morse function

$$\tilde{h}_1^*: (T_\rho \cap U_p)/\Gamma \to \mathbf{R}$$

where the quotient by Γ means the orbit space of $T_\rho \cap U_p$ under Γ. By Lemma 1, $\tilde{h}_1^*$ can be extended to a Morse function

$$\tilde{h}_p: U_p/\Gamma \to \mathbf{R}$$

approximating G/Γ restricted on U_p/Γ. This $\tilde{h}_p$ induces an f-invariant N-coherent Morse extension $h_p: M_p \to \mathbf{R}$ of $h_1^* \mid T_\rho \cap M_p$. h_p evidently still approximates $G \mid M_p$.

Step 3. If $\nu \neq p$, we extend h_p to an f-invariant Morse function H_p defined on a tubular neighborhood $T_{\rho_p}(M_p)$ of M_p by considering $h_p^*: T_{\rho_p}(M_p) \to \mathbf{R}$, and then patching h_p^* and h_1^* together near N as follows.

$$\begin{aligned} H_p(x) &= h_1^*(x), & x &\in T_\eta \cap T_{\rho_p}(M_p), \\ &= h_p^*(x) + \varphi_\eta(r(x) - \eta)(h_1^*(x) - h_p^*(x)), & x &\in (T_{2\eta} - T_\eta) \cap T_{\rho_p}(M_p), \\ &= h^*(x), & x &\in T_{\rho_p}(M) - T_{2\eta}, \end{aligned}$$

where $\eta = \rho/3$ and $r(x)$ denotes the distance from x to N.

By taking ρ_p sufficiently small, h_1^* and h_p^* as well as their derivatives will differ from each other only by a small amount in the patching strip. This guarantees that no critical point of H_p will appear in the strip. Clearly H_p approximates G. H_p is also f-invariant, since h_1^* and h_p^* are f-invariant and φ_ε is symmetric with respect to 0.

Step 4. For M_k, we assume according to the induction hypothesis that for each divisor l of k, H_l has been constructed. By the Lemma 1, we extend the function

$$\bigcup_l H_l \mid M_k \cap \left(\bigcup_l T_{\rho_l}(M_l)\right)$$

to an f-invariant N-coherent Morse function $h_k : M_k \to \mathbf{R}$ in the way similar to that described in Step 2. h_k approximates G again. If $k < \nu$, we construct again h_k^* and patch together h_k^* and h^*_l, for all divisors l of k, as in Step 2 to obtain H_k. If $k = \nu$, we take $F = h_\nu$. This completes the construction of F.

Remark. Such F is indeed M_l-coherent for all l.

5. **The inequality and its applications.** In general, for $Y \subset X \subset M$, let

$$\beta_q(X, Y) = \text{the Betti number of the pair } (X, Y),$$

$$\lambda_q(X, Y) = \text{the trace of } f_* \text{ on } H_q(X, Y),$$

and let

$$B_q(X, Y) = \beta_q(X, Y) - \beta_{q-1}(X, Y) + \cdots + (-1)^q \beta_0(X, Y),$$

$$\Lambda_q(X, Y) = \lambda_q(X, Y) - \lambda_{q-1}(X, Y) + \cdots + (-1)^q \lambda_0(X, Y).$$

We fix an f-invariant N-coherent Morse function F chosen arbitrarily. For a real number a, let M^a be the set $\{x \in M \mid F(x) \leq a\}$.

Let all the critical values c_α's of F be ordered such that $c_1 > c_2 > \cdots > c_\mu$. Let $p_1^\alpha, \ldots, p_l^\alpha, \ldots, p_k^\alpha$ be all the critical points of F with critical value c_α and of indices $\nu_1^\alpha, \ldots, \nu_l^\alpha, \ldots, \nu_k^\alpha$ respectively, where $p_1^\alpha, \ldots, p_l^\alpha$ are precisely the ones contained in N. (l and k depend on α. The superscript α will be omitted everywhere when no confusion can occur.)

For each p_j, $1 \leq j \leq k$, there is an N-coherent coordinate neighborhood (x_i) of p_j. Let e_j be the ν_j-cell $\{x_{\nu_j+1} = x_{\nu_j+2} = \cdots = x_m = 0\}$. Consider numbers a_0, $a_1, \ldots, a_\mu$ such that

$$a_0 > c_1 > a_1 > c_2 > \cdots > a_{\mu-1} > c_\mu > a_\mu.$$

When a_α is chosen sufficiently close to c_α, we can have

(1) e_j's are disjoint and $\partial e_j \subset M^{a_\alpha}$;

(2) $\{(e_j, \partial e_j) \mid j = 1, \ldots, l\}$ and $\{(e_j, \partial e_j) \mid j = 1, \ldots, l, \ldots, k\}$ are respectively the generators of the homology groups $H(N^{a_{\alpha-1}}, N^{a_\alpha})$ and $H(M^{a_{\alpha-1}}, M^{a_\alpha})$; and

(3) for $1 \le j \le l$, f is the identity map on e_j and for $l < j \le k$, $f_*(e_j, \partial e_j) = (e_i, \partial e_i)$ with $i \ne j$, where f_* is the induced map of f on $H(M^{a_{\alpha-1}}, M^{a_\alpha})$.

It follows that for each q and α both of $\beta_q(N^{a_{\alpha-1}}, N^{a_\alpha})$ and $\lambda_q(M^{a_{\alpha-1}}, M^{a_\alpha})$ are equal to the number of e_j's with $\nu_j = q$ and $1 \le j \le l$. Hence we have

$$\beta_q(N^{a_{\alpha-1}}, N^{a_\alpha}) = \lambda_q(M^{a_{\alpha-1}}, M^{a_\alpha}),$$
$$B_q(N^{a_{\alpha-1}}, N^{a_\alpha}) = \Lambda_q(M^{a_{\alpha-1}}, M^{a_\alpha}).$$

From the exactness of

$$0 \to \partial_*(H_{q+1}(N, N^{a_{\alpha-1}})) \to H_q(N^{a_\alpha}, N^{a_{\alpha-1}}) \to H_q(N, N^{a_\alpha}) \to H_q(N, N^{a_{\alpha-1}}) \to \cdots,$$

we have

$$B_q(N, N^{a_\alpha}) = B_q(N^{a_{\alpha-1}}, N^{a_\alpha}) + B_q(N, N^{a_{\alpha-1}}) - \varepsilon_{q,\alpha}$$

where $\varepsilon_{q,\alpha}$ is the rank of $\partial_*(H_{q+1}(N, N^{a_{\alpha-1}}))$. Similarly, we have

$$\Lambda_q(M, M^{a_\alpha}) = \Lambda_q(M^{a_{\alpha-1}}, M^{a_\alpha}) + \Lambda_q(M, M^{a_{\alpha-1}}) - \eta_{q,\alpha}$$

where $\eta_{q,\alpha}$ is the trace of f_* on $\partial_*(H_{q+1}(M, M^{a_{\alpha-1}}))$. By induction we have

$$B_q(N) = \sum_\alpha B_q(N^{a_\alpha}, N^{a_{\alpha-1}}) - \sum_\alpha \varepsilon_{q,\alpha}$$

and

$$\Lambda_q(f) = \sum_\alpha \Lambda_q(M^{a_\alpha}, M^{a_{\alpha-1}}) - \sum_\alpha \eta_{q,\alpha}.$$

The well-known Morse inequality states that given an arbitrary Morse function on M, we have

$$B_q(M) \le C_q \stackrel{\text{def}}{=} c_q - c_{q-1} + \cdots + (-1)^q c_0$$

where c_q denotes the number of critical points of the Morse function with index q. The difference $C_q - B_q(M)$ is given by

$$\sum_\alpha \operatorname{rank}[\partial_*(H_{q+1}(M^{a_\alpha}, M^{a_{\alpha-1}}))]$$

if we adopt the subdivision of M according to the Morse function as we did in the above.

Definition 3. We call the difference $C_q - B_q(M)$ the *qth Morse difference.*

We denote the qth Morse difference of F by $\delta_q(F)$. However,

$$|\eta_{q,\alpha} - \varepsilon_{q,\alpha}| \leq \operatorname{rank}[\partial_*(H_{q+1}(M^{a_\alpha}, M^{a_{\alpha-1}}))].$$

Therefore we obtain

Theorem 3. *Given a periodic transformation f of a compact smooth m-dimensional manifold M with fixed point set N, we have the inequality*

$$|\Lambda_q(f) - B_q(N)| \leq \delta_q(F)$$

for each $q = 0, \ldots, m$ *and each f-invariant N-coherent Morse function F, where*

$$\Lambda_q(f) = \sum_{r=0}^{q} (-1)^{q-r} \text{ trace of } f_* \text{ on } H_r(M),$$
$$B_q(N) = \sum_{r=0}^{q} (-1)^{q-r} \text{ rth Betti number of } N,$$

and $\delta_q(F)$ *is the qth Morse difference of F.*

As corollaries we obtain a fixed point set theorem.

Theorem 4. *Given a periodic transformation f of a compact smooth manifold M, if* $|\Lambda_q(f)| > \delta_q(F)$ *for some* $q = 1, \ldots, m$ *and some f-invariant Morse function F on M, then f has a fixed point.*

Proof. Suppose f is fixed point free. Then every Morse function is N-coherent. Also $B_q(N) = 0$. These lead to a contradiction.

Remark 2. In particular when $q = m$, Λ_m is the usual Lefschetz number and $\delta_m(F) = 0$ for all F. Therefore this corollary is a generalization of the Lefschetz fixed point theorem for a periodic map.

Remark 3. Such a fixed point theorem based on Λ_q and $\delta_q(F)$ for arbitrary q and F gives the best possible estimation. In fact, let $T^2 = S^1 \times S^1 = \{e^{i\theta}, e^{i\varphi}) \mid 0 \leq \theta, \varphi < 2\pi\}$ and consider $f\colon (e^{i\theta}, e^{i\varphi}) \to (e^{i\theta}, e^{-i\varphi})$ and $F(e^{i\theta} + e^{i\varphi}) = \cos\theta + \cos 2\varphi$. Then F is an f-invariant Morse function with $\Lambda_1 = 1 = \delta_1(F)$ but f has no fixed point.

Since $\delta_m(F) = 0$, we obtain

Corollary 1. *Given a periodic transformation f on a compact smooth manifold* M^m *with fixed point set N, we have the Lefschetz number of f equal to the Euler number of the fixed point set N and therefore equal to the integral over N of the restricted "intrinsic curvature" in the sense of Chern* [1].

This statement can be regarded as a generalization of the Gauss-Bonnet theorem. A stronger result for any isometry can be proven rather directly by Mayer-Vietoris sequence applying on a tubular neighborhood of N. However, the above approach using the viewpoint of Morse theory may help one to have better geometric insight.

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