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A MULTIPLIER THEOREM FOR FOURIER TRANSFORMS

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ABSTRACT. A function f analytic in the upper half-plane Π^+ is said to be of class $E_p(\Pi^+)$ $(0 < p < \infty)$ if there exists a constant C such that $\int_{-\infty}^{\infty} |f(x+iy)|^p dx \le C < \infty$ for all $y > 0$. These classes are an extension of the H_p spaces of the unit disc U. For f belonging to $E_p(\Pi^+)$ $(0 < p \le 2)$, there exists a Fourier transform $\hat{f}$ with the property that $f(z) = (2\pi)^{-1} \int_0^\infty \hat{f}(t) e^{izt} dt$. This makes it possible to give a definition for the multiplication of $E_p(\Pi^+)$ $(0 < p \le 2)$ into $L_q(0,\infty)$ that is analogous to the multiplication of $H_p(U)$ into l_q. In this paper, we consider the case $0 < p < 1$ and $p \le q$ and derive a necessary and sufficient condition for multiplying $E_p(\Pi^+)$ into $L_q(0,\infty)$.

1. **Introduction.** A function f analytic in the unit disc U is said to be of class $H_p(U)$ if there exists a constant C such that $\int_0^{2\pi} |f(re^{i\theta})|^p d\theta \le C < \infty$ for all $r < 1$. For these classes there exists a rich and varied theory which is described in Duren's book [2]. Among the concepts studied is that of multipliers from $H_p(U)$ to l_q.

Definition 1. A sequence $\{\lambda_n\}$ is said to multiply $H_p(U)$ into l_q $(0 < q < \infty)$, if for each $f(z) = \sum a_n z^n$ belonging to $H_p(U)$, $\sum |a_n|^q |\lambda_n|^q < \infty$.

Duren and Shields have shown that a necessary and sufficient condition for $\{\lambda_n\}$ to multiply $H_p(U)$ $(0 < p < 1)$ into l_q $(p \le q < \infty)$ is that

$$\sum_{n=1}^{N} n^{q/p} |\lambda_n|^q = O(N^q) \qquad [2], [3].$$

It is our aim in this paper to consider classes of functions analytic in the upper half-plane Π^+, which are analogous to the classes $H_p(U)$, and to prove a result similar to that of Duren and Shields.

2. **The main result.**

Definition 2. A function f analytic in Π^+ is said to be of class $E_p(\Pi^+)$ $(0 < p < \infty)$ if there exists a constant C such that

$$M_p(y,f) = \left\{ \int_{-\infty}^{\infty} |f(x+iy)|^p dx \right\}^{1/p} \le C < \infty$$

for all $0 < y < \infty$.

The expression $M_p(y,f)$ is called a pth mean of f. Also the expression $M_\infty(y,f) = \sup_{-\infty < x < \infty} |f(x+iy)|$ is a pth mean of f and, if $M_\infty(y,f)$ is bounded, f is said to belong to $E_\infty(\Pi^+)$.

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Definition 3. If f belongs to $E_p(\Pi^+)$ $(0 < p < 1)$, then the Fourier transform of f is

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x + iy)e^{-i(x+iy)t}dx \qquad [6].$$

A proof of the fact that $\hat{f}$ exists and is independent of y is given in §5. In addition, the facts $\hat{f}(t)$ is continuous, $\hat{f}(t) = 0$ for $t \leq 0$, and

$$f(z) = (2\pi)^{-1} \int_0^{\infty} \hat{f}(t)e^{izt}dt,$$

are proved there.

Definition 4. Let $\phi(t)$ be a function measurable on $(0,\infty)$. Then $\phi(t)$ is said to multiply $E_p(\Pi^+)$ $(0 < p < 1)$ into $L_q(0,\infty)$ $(0 < q < \infty)$, if for each $f(z) = (2\pi)^{-1} \int_0^{\infty} \hat{f}(t)e^{izt}dt$ belonging to $E_p(\Pi^+)$,

$$\int_0^{\infty} |\phi(t)|^q |\hat{f}(t)|^q dt < \infty.$$

We now state the main result.

Theorem A. *Let $\phi(t)$ be a function measurable on $(0,\infty)$. Then $\phi(t)$ multiplies $E_p(\Pi^+)$ into $L_q(0,\infty)$ $(p \leq q)$ if and only if*

(1) $$\int_0^{X} t^{q/p}|\phi(t)|^X dt \leq KX^q,$$

where K is a positive constant.

The proof of Theorem A requires the use of two other results.

Theorem B. *If $0 < p < q \leq \infty$, f belongs to $E_p(\Pi^+)$, $\alpha = 1/p - 1/q$, and $\lambda \geq p$, then $\int_0^{\infty} y^{\lambda\alpha-1}M_q^{\lambda}(y,f)\,dy < \infty$.*

The second of these results needs some introduction. If f belongs to $E_p(\Pi^+)$ $(0 < p < \infty)$, then $\lim_{y\to 0} f(x + iy) = f(x)$ exists a.e. and

$$\rho(f,g) = \int_0^{\infty} |f(x) - g(x)|^p dx,$$

where f and g belong to $E_p(\Pi^+)$, is a translation invariant metric on $E_p(\Pi^+)$. Moreover, under this metric, $E_p(\Pi^+)$ $(0 < p < \infty)$ is a complete topological vector space. In other words, $E_p(\Pi^+)$ $(0 < p < \infty)$ is an F-space [1], [2], [5]. Finally, we say that an operator Λ from $E_p(\Pi^+)$ into $L_q(0,\infty)$ is bounded if there exists a constant K such that $\|\Lambda(f)\|_q < K\|f\|_p$, where $\|f\|_p = \{\int_0^{\infty} |f(x)|^p dx\}^{1/p}$.

Theorem C. *Let $\phi(t)$ be a function measurable on $(0,\infty)$. If $\phi(t)$ multiplies $E_p(\Pi^+)$ $(0 < p < 1)$ into $L_q(0,\infty)$ then the operator $\Lambda(f)(t) = \phi(t)\hat{f}(t)$ is bounded.*

We defer, for now, the proofs of Theorem B and Theorem C in order to give an immediate proof of Theorem A.

Proof of Theorem A. We begin by showing that (1) is necessary. So let us consider the function

$$F(z) = F_\rho(z) = (2\pi)^{-1} \int_0^\infty t^{1/p} e^{-\rho t} e^{itz}\, dt.$$

Since the Laplace transform of t^{u-1} $(u > 0)$ is $\Gamma(u)/s^u$, where s is a complex number with Re $s > 0$, we see that setting $u = 1 + 1/p$ and $s = -iz + \rho$ gives

$$F(z) = \Gamma(1 + 1/p)/(\rho - iz)^{1+1/p}.$$

From this it follows that $F(z)$ belongs to $E_p(\Pi^+)$ and $\|F\|_p = M/\rho$. But by Theorem C there exists a constant K such that

$$\|\Lambda(f)\|_q \le K\|F\|_p,$$

so $\|\hat{F}(t)\phi(t)\|_q \le KM/\rho$. Thus, our next step is to find $\hat{F}(t)$. However, $F(x + iy) = F_y(x)$ is in $L_1(-\infty, \infty)$ and is the Fourier transform of

$$\begin{aligned} g(t) &= (2\pi)^{-1} t^{1/p} e^{-\rho t} e^{-yt} \qquad &&\text{if } t \ge 0,\\ &= 0 &&\text{if } t < 0, \end{aligned}$$

which also belongs to $L_1(-\infty, \infty)$. Hence $\hat{F}(t)e^{-yt} = \hat{F}_y(t) = 2\pi g(t)$ or $\hat{F}(t) = t^{1/p}e^{-\rho t}$ if $t \ge 0$ and zero if $t < 0$ [7]. Consequently,

$$\int_0^\infty t^{q/p} |\phi(t)|^q e^{-q\rho t}\, dt \le K^q M^q/\rho^q$$

and this implies that

$$\int_0^X e^{-q\rho X} t^{q/p} |\phi(t)|^q\, dt \le K^q M^q/\rho^q$$

for $X > 0$. So taking $\rho = 1/X$, we find

$$\int_0^X t^{q/p} |\phi(t)|^q\, dt \le KM^q e^q X^q.$$

To prove that (1) is sufficient, we begin by considering the integral

$$\int_0^\infty t^{q/p} |\phi(t)|^q e^{-yt}\, dt \qquad (y > 0).$$

Letting $S(t) = \int_0^t \tau^{q/p} |\phi(\tau)|^q\, dt$ and integrating by parts we find that when we use the estimate $S(t) \le Kt^q$, the integral is less than or equal to $Ky \int_0^\infty t^q e^{-yt}\, dt = K\Gamma(q + 1)/y^q$. Hence

$$y^q \int_0^\infty t^{q/p} |\phi(t)|^q e^{-yt}\, dt \le C < \infty$$

for $y > 0$. Next we note that for $\gamma = q(1/p - 1)$, Theorem B implies that for f belonging to $E_p(\Pi^+)$ $(0 < p < 1)$,

$$\int_0^\infty y^{\gamma-1} M_1^q(y,f)\,dy < \infty.$$

Thus for each f belonging to $E_p(\Pi^+)$

$$\int_0^\infty y^{\gamma-1} M_1^q(y,f)\left[y^q \int_0^\infty t^{q/p}|\phi(t)|^q e^{-yt}\,dt\right] dy < \infty,$$

or using Fubini's theorem

$$\int_0^\infty \int_0^\infty t^{q/p}|\phi(t)|^q y^{\gamma+q-1} M_1^q(y,f) e^{-yt}\,dy\,dt < \infty.$$

But from the definition of the Fourier transform for f, we have $|\hat{f}(t)|e^{-yt} \leq M_1(y,f)$. Thus

$$\int_0^\infty |\phi(t)|^q |\hat{f}(t)|^q t^{q/p} \int_0^\infty y^{\gamma+q-1} e^{-(q+1)yt}\,dy\,dt < \infty,$$

or

$$\frac{\Gamma(q/p)}{(q+1)^{q/p}} \int_0^\infty |\phi(t)|^q |\hat{f}(t)|^q\,dt < \infty. \qquad \square$$

Theorem A has the following interesting corollary.

Corollary. *If f belongs to $E_p(\Pi^+)$ $(0 < p < 1)$, then* $\int_0^\infty |\hat{f}(t)|^p t^{p-2}\,dt < \infty$.

This is an extension of the following results.

Theorem (Hardy-Littlewood-Titchmarsh). *If f belongs to $E_p(\Pi^+)$ $(1 < p \leq 2)$, then* $\int_0^\infty |\hat{f}(t)|^p t^{p-2}\,dt < \infty$ [8].

Theorem (Hille-Tamarkin). *If f belongs to $E_1(\Pi^+)$, then* $\int_0^\infty |\hat{f}(t)|/t\,dt < \infty$ [4].

3. **The proof of Theorem B.** This proof is a consequence of several other theorems.

Theorem 1. *Let $u(z)$ be a nonnegative subharmonic function defined on Π^+ and suppose*

$$\int_{-\infty}^\infty u(x+iy)\,dx \leq C/y^\alpha \qquad (y > 0),$$

where $\alpha \geq 0$. Then there exists a constant $K = K(\alpha)$ such that $u(x_0 + iy_0) \leq KC/y_0^{\alpha+1}$ for each point $z_0 = x_0 + iy_0$ $(y_0 > 0)$.

Proof. The case $\alpha = 0$ was proved by Krylov [5]. So assume $\alpha > 0$. Then setting $y_1 = y_0/2$ and $u_{y_1}(z) = u(x + i(y + y_1))$, we find

$$\int_{-\infty}^\infty u_{y_1}(x+iy)\,dx \leq C/y_1^\alpha \qquad (y > 0).$$

Hence, by the case $\alpha = 0$, we have $u_{y_1}(x_0 + iy_2) \leq KC/y_1^\alpha y_2$ $(y_2 > 0)$, and putting $y_1 = y_2 = y_0/2$,

$$u(x_0 + iy_0) \leq 2^{\alpha+1}KC/y_0^{\alpha+1}. \qquad \square$$

Theorem 2. *Suppose $f(z)$ is analytic in Π^+ and*

$$(1) \qquad M_p(y,f) \leq C/y^\beta \qquad (0 < p < \infty, \beta \geq 0).$$

Then there exists a constant $K = K(\beta,p,q)$ such that

$$(2) \qquad M_q(y,f) \leq KC/y^{\beta+1/p-1/q} \qquad (p < q \leq \infty).$$

Proof. It suffices to consider the case $q = \infty$. For suppose (2) has been proven for $q = \infty$ and $K \geq 1$ (which we may assume without loss of generality). Then

$$\begin{aligned} M_q(y,f) &= \left\{\int_\infty^\infty |f(x+iy)|^p |f(x+iy)|^{q-p} dx\right\}^{1/q} \\ &\leq [M_\infty(y,f)]^{q-p/q}[M_p(y,f)]^{p/q} \\ &\leq K^{q-p/q}C/y^\lambda, \end{aligned}$$

where $\lambda = \beta + 1/p - 1/q$. Now to derive the theorem for $q = \infty$, let $u(z)$ be the nonnegative subharmonic function $|f(z)|^p$ and $\alpha = \beta p$. Then Theorem 1 implies

$$|f(x_0 + iy_0)|^p \leq KC/y_0^{\beta p+1},$$

which is equivalent to (2). $\square$

Theorem 3. *Suppose f belongs to $E_p(\Pi^+)$. Then for $1 < p < \infty$, $-1 < b$, and $1 < a < \infty$,*

$$(3) \qquad \int_0^\infty y^b M_p^a(y,f)\,dy \leq C\int_0^\infty y^{a+b} M_p^a(y,f')\,dy,$$

where $C = C(a,b)$ is independent of f.

Proof. We begin by assuming that f is analytic in the closed upper half-plane. Then integrating by parts we find

$$\begin{aligned} \int_0^{y_0} y^b M_p^a(y,f)\,dy &= \frac{y_0^{b+1}}{b+1} M_p^a(y_0,f) \\ &\quad - \frac{1}{b+1}\int_0^{y_0} y^{b+1}\frac{\partial}{\partial y}\{M_p^a(y,f)\}\,dy. \end{aligned}$$

Thus our next step is to estimate $|(\partial/\partial y)M_p^a(y,f)|$. But

$$(\partial/\partial y)M_p^a(y,f) = (a/p)M_p^{a-p}(y,f)(\partial/\partial y)M_p^p(y,f),$$

so we need to estimate $|(\partial/\partial y)M_p^p(y,f)|$.

However,

$$\left|\frac{\partial}{\partial y}|f(x+iy)|^p\right| = p|f(x+iy)|^{p-1}\left|\frac{\partial}{\partial y}|f(x+iy)|\right|$$

and

$$\frac{||f(x+iy_1)|-|f(x+iy_2)||}{|y_1-y_2|} \le \frac{|f(x+iy_1)-f(x+iy_2)|}{|y_1-y_2|}$$

implies

$$|(\partial/\partial y)|f(x+iy)|| \le |f'(x+iy)|,$$

so

$$\left|\frac{\partial}{\partial y}|f(x+iy)|^p\right| \le p|f(x+iy)|^{p-1}|f'(x+iy)|.$$

Thus Hölder's inequality implies

$$|(\partial/\partial y)M_p^p(y,f)| \le pM_p^{p-1}(y,f)M_p(y,f')$$

and this implies

$$|(\partial/\partial y)M_p^a(y,f)| \le aM_p^{a-1}(y,f)M_p(y,f').$$

But now we have

$$\left|\int_0^{y_0} y^{b+1}\frac{\partial}{\partial y}\{M_p^a(y,f)\}\,dy\right| \le a\int_0^{y_0} y^{b+1}M_p^{a-1}(y,f)M_p(y,f')\,dy$$

$$\le a\left\{\int_0^{y_0} y^b M_p^a(y,f)\,dy\right\}^{1-1/a}\left\{\int_0^{y_0} y^{a+b}M_p^a(y,f')\,dy\right\}^{1/a},$$

where we have used Hölder's inequality again. Hence

$$\left\{\int_0^{y_0} y^b M_p^a(y,f)\,dy\right\}^{1/a} \tag{4}$$

$$\le \left(\frac{y_0^{b+1}}{b+1}\right)^{1/a} M_p(y_0,f) + \frac{a}{b+1}\left\{\int_0^{y_0} y^{a+b}M_p^a(y,f')\,dy\right\}^{1/a},$$

where we have used the estimate

$$\int_0^{y_0} y^b M_p^a(y,f)\,dy \ge \frac{y_0^{b+1}M_p^a(y_0,f)}{b+1}.$$

which follows from the fact that the means $M_p(y,f)$ are nonincreasing functions of y [5].

From (4), it is clear that in order to complete the proof for this case, we need only show that $y_0^{b+1} M_p^a(y_0,f)$ tends to zero as y_0 tends to infinity. But using Theorem 2, it is easy to see that $f(x + iy_0) = -i \int_{y_0}^{\infty} f'(x + iy)\, dy$ and applying Minkowski's inequality, we find

$$M_p(y_0,f) \leq \int_{y_0}^{\infty} M_p(y,f')\, dy.$$

So suppose $r > 1$. Then

$$M_p^a(y_0,f) \leq [C(y_0)]^a \left[\frac{1}{r-1} \int_{y_0}^{\infty} y^r M_p(y,f') \frac{d(-1/y^{r-1})}{C(y_0)}\right]^a,$$

where $C(y_0) = \int_{y_0}^{\infty} d(-1/y^{r-1}) = 1/y_0^{r-1}$, and Jensen's inequality gives

$$M_p^a(y_0,f) \leq [C(y_0)]^{a-1} \frac{1}{(r-1)^{a-1}} \int_{y_0}^{\infty} y^{ar-r} M_p^a(y,f')\, dy.$$

Hence setting $r = (a + b)/(a - 1)$, we have

$$y_0^{b+1} M_p^a(y_0,f) \leq \frac{1}{((b+1)/(a-1))^{a-1}} \int_{y_0}^{\infty} y^{a+b} M_p^a(y,f')\, dy,$$

from which it follows that $y_0^{b+1} M_p^a(y_0,f)$ tends to zero as y_0 tends to infinity.

Finally we remove the restriction that f is analytic in the closed upper half-plane. Since $f_y(z) = f(z + iy)$ is analytic in the closed upper half-plane, the theorem holds for $f_y(z)$. Thus the result for $f(z)$ follows from letting y tend to zero and applying the monotone convergence theorem. □

These three theorems have prepared the way for a proof of Theorem B.

Proof of Theorem B. We first reduce the theorem to the case $\lambda = p = 2$. By Theorem 2

$$M_p^\lambda(y,f) \leq K^{\lambda-p} M_q^p(y,f)/y^{\alpha(\lambda-p)},$$

so

$$\int_0^{\infty} y^{\alpha\lambda-1} M_q^\lambda(y,f)\, dy \leq K^{\lambda-p} \int_0^{\infty} y^{\alpha p-1} M_q^p(y,f)\, dy.$$

Hence we can assume $\lambda = p$. Next assume the theorem is true for $\lambda = p = 2$ and $f(z) \neq 0$ in Π^+ and belongs to $E_p(\Pi^+)$. Then $g(z) = [f(z)]^{p/2}$ belongs to $E_2(\Pi^+)$ and

$$\int_0^{\infty} y^{-p/q} M_q^p(y,f)\, dy = \int_0^{\infty} y^{-2/s} M_s^2(y,g)\, dy < \infty,$$

where $s = 2q/p > 2$. In case $f(z)$ has zeros in Π^+, it is possible to write it as a sum of two nonzero functions in $E_p(\Pi^+)$ [2] and still show that it suffices to take $p = 2$.

So let $f \in E_2(\Pi^+)$. Then using the Paley-Wiener theorem [7], we can write

$$f(z) = \frac{1}{2\pi}\int_0^\infty \hat{f}(t)e^{izt}\,dt,$$

where $\hat{f}(t)$ is the Fourier transform of the boundary function $f(x)$ of $f(z)$. Also

$$f'(z) = \frac{1}{2\pi}\int_0^\infty t\hat{f}(t)e^{izt}\,dt.$$

Next we assume $2 < q < \infty$. Then by Theorem 3

$$\int_0^\infty y^{-2/q}M_q^2(y,f)\,dy \leq C\int_0^\infty y^{2-2/q}M_q^2(y,f')\,dy,$$

and by Theorem 2 $M_q(y,f') \leq Ky^{1/q-1/2}M_2(y/2,f')$, so

$$\int_0^\infty y^{-2/q}M_q^2(y,f)\,dy \leq CK\int_0^\infty yM_2^2(y/2,f')\,dy.$$

Finally, by Plancherel's theorem [7], we find

$$\begin{aligned}\int_0^\infty y^{-2/q}M_q^2(y,f)\,dy &\leq \frac{CK}{2\pi}\int_0^\infty y\int_0^\infty |\hat{f}(t)|^2t^2e^{-yt}\,dt\,dy\\ &= \frac{CK}{2\pi}\int_0^\infty |\hat{f}(t)|t^2\int_0^\infty ye^{-yt}\,dy\,dt\\ &= \frac{CK}{2\pi}\int_0^\infty |\hat{f}(t)|^2\,dt\\ &= CK\int_0^\infty |f(x)|^2\,dx < \infty.\end{aligned}$$

If $q = \infty$, then the estimate

$$M_\infty^2(y,f) \leq KM_r^2(y/2,f)/y^{2/r}$$

for some $r > 2$ can be used to derive the desired results. □

4. The proof of Theorem C. Since $E_p(\Pi^+)$ is an F-space under the metric $\rho(f,g) = \int_{-\infty}^{\infty} |f(x) - g(x)|^p\,dx$, we can use the closed graph theorem. Thus we need to show that Λ is a closed operator. So let $\{f_n\}$ be a sequence which converges in $E_p(\Pi^+)$ to f and also suppose $\Lambda(f_n)(t) = \phi(t)\hat{f}_n(t)$ converges to $g(t)$ in $L_q(0,\infty)$. Then we need to show that $\Lambda(f)(t) = g(t)$ a.e.

Considering the sequence $\{f_n\}$ and f first, we find by Theorem 2 that

$$\left\{\int_{-\infty}^\infty |f_n(x+iy_0) - f(x+iy_0)|^2\,dx\right\}^{1/2} \leq \frac{K\|f_n - f\|_p}{y_0^{1/p-1/2}},$$

where $y_0 > 0$. Thus $f_{y_0,n}(z) = f_n(z + iy_0)$ converges to $f_{y_0}(z) = f(z + iy_0)$ in $E_2(\Pi^+)$. Moreover, it is easy to see that the Fourier transform of $f_{n,y}(x)$ is $\hat{f}_n(t)e^{-y_0 t}$, while the Fourier transform of $f_{y_0}(x)$ is $f(t)e^{-y_0 t}$. Consequently, Plancherel's theorem [7] implies that $\hat{f}_n(t)e^{-y_0 t}$ converges to $\hat{f}(t)e^{-y_0 t}$ in $L_2(0, \infty)$. Hence, there exists a subsequence $\{\hat{f}_k(t)\}$ of $\{\hat{f}_n(t)\}$ converging to $\hat{f}(t)$ a.e. But the sequence $\{\Lambda(f_k)\}$ also converges to $g(t)$ in $L_q(0, \infty)$. Therefore, there exists a subsequence of $\{\Lambda(f_k)\}$, which we also denote by $\{\Lambda(f_k)\}$, converging to $g(t)$ a.e. Thus $\{\phi(t)f_k(t)\}$ converges to $\phi(t)\hat{f}(t)$ a.e. and also to $g(t)$ a.e., which implies

$$\phi(t)\hat{f}(t) = g(t) \quad \text{a.e.} \qquad \square$$

5. **Fourier transform.** The Fourier transform defined in §2 certainly exists since Theorem 2 implies that $f_y(x) = f(x + iy)$ belongs to $L_1(-\infty, \infty)$. In fact, if C is a constant such that $M_p(y, f) \leq C$ for $y > 0$, then there exists a constant $K = K(0, p, 1)$ such that

$$\int_{-\infty}^{\infty} |f(x + iy)|\, dx \leq CK/y^{1/p-1} \tag{1}$$

for $y > 0$.

To see that $\hat{f}$ is independent of y, fix $0 < y_1 < y_2 < \infty$ and for each $\alpha > 0$ let Γ_α be the rectangular contour with vertices $\pm\alpha + iy_1$ and $\pm\alpha + iy_2$. By Cauchy's theorem

$$\int_{\Gamma_\alpha} f(z)e^{-itz}\, dz = 0. \tag{2}$$

Next let $I = [y_1, y_2]$ and put

$$\Phi(\beta) = i\int_I f(\beta + iu)e^{-it\beta}e^{tu}\, du.$$

Then $|\Phi(\beta)| \leq e^{ty_2}\int_{y_1}^{y_2} |f(\beta + iu)|\, du$. Now if we let

$$\Psi(\beta) = \int_{y_1}^{y_2} |f(\beta + iu)|\, du,$$

then Fubini's theorem and (1) imply

$$\int_{-\infty}^{\infty} \Psi(\beta)d\beta = \int_{y_1}^{y_2}\int_{-\infty}^{\infty} |f(\beta + iy)|\, d\beta\, dy \leq \frac{CK}{y_1^{1/p-1}}(y_2 - y_1).$$

Thus there exists a sequence $\{\alpha_j\}$ such that $\alpha_j \to \infty$ as $j \to \infty$ and $\Psi(\alpha_j) + \Psi(-\alpha_j) \to 0$ as $j \to \infty$. Hence we have

$$\Phi(\alpha_j) \to 0 \quad \text{and} \quad \Phi(-\alpha_j) \to 0 \tag{3}$$

as $j \to \infty$. Now combining (1), (2), and (3), we find

$$\int_{-\infty}^{\infty} f(x + iy_1)e^{-i(x+iy_1)t}\, dx = \int_{-\infty}^{\infty} f(x + iy_2)e^{-i(x+iy_2)t}\, dx, \tag{4}$$

i.e., $\hat{f}$ is independent of y.

If we let $f_y(z) = f(z + iy)$, then (4) becomes

$$\hat{f}(t) = e^{y_1 t}\hat{f}_{y_1}(t) = e^{y_2 t}\hat{f}_{y_2}(t).$$

Since $\hat{f}_y$ is the Fourier transform of an $L_1(-\infty, \infty)$ function, it is continuous and hence $\hat{f}$ is continuous.

Using (1) again, we see that

$$|\hat{f}(t)|e^{-ty} = |\hat{f}_y(t)| \le \|f_y\|_1 \le CK/y_0^{1/p-1}$$

for a fixed $y_0 < y$. Thus if we fix $t < 0$ and let $y \to \infty$, we find $\hat{f}(t) = 0$. Hence $\hat{f}(t)$ is identically zero on $(0,\infty)$ and by continuity it is zero at $t = 0$. Also note $f_y(t) \equiv 0$ on $(-\infty, 0]$.

As we have noted, $\hat{f}(t) = f_y(t)e^{yt}$, so $\hat{f}_y(t) = \hat{f}(t)e^{-yt} = \hat{f}_{y_0}(t)e^{(y_0-y)t}$, and letting $y_0 = y/2$, we have

$$\begin{aligned} \int_0^\infty |\hat{f}_y(t)|\,dt &\le \|f_{y_0}\|_1 \int_0^\infty e^{(y_0-y)t}\,dt \\ &\le \frac{KC}{y_0^{1/p-1}}\frac{1}{y-y_0} \\ &= \frac{2^{1/p}KC}{y^{1/p}}. \end{aligned}$$

Hence for $y > 0$, $\hat{f}_y$ belongs to $L_1(-\infty, \infty)$ and we can apply the inversion theorem [7], to find

$$\begin{aligned} f(z) = f_y(x) &= (2\pi)^{-1}\int_0^\infty \hat{f}_y(t)e^{itx}\,dt \\ &= (2\pi)^{-1}\int_0^\infty \hat{f}(t)e^{-ty}e^{itx}\,dt \\ &= (2\pi)^{-1}\int_0^\infty \hat{f}(t)e^{itz}\,dt. \end{aligned}$$

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