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A PROOF THAT $\mathcal{H}^2$ AND $\mathcal{T}^2$ ARE DISTINCT MEASURES(1)

BY

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ABSTRACT. It is proven that there exists a subset E of $\mathbf{R}^3$ such that the two-dimensional $\mathcal{T}$ measure of E is less than its two-dimensional Hausdorff measure. E is the image under the usual isomorphism of $\mathbf{R} \times \mathbf{R}^2$ onto $\mathbf{R}^3$ of the Cartesian product of $\{x: -4 \le x \le 4\}$ and a Cantor type subset of $\mathbf{R}^2$; the latter term in this product is the intersection of a decreasing sequence, every member of which is the union of certain closed circular disks.

1. **Introduction.** To any positive integers m, n with $m \leq n$ there correspond several m-dimensional measures over $\mathbf{R}^n$. These measures were studied extensively by H. Federer in [2]. Three of them are the m-dimensional Carathéodory, $\mathcal{T}$ and Hausdorff measures, which are denoted by $\mathcal{C}^m$, $\mathcal{T}^m$ and $\mathcal{H}^m$ respectively. It is known that $\mathcal{C}^m(S) \leq \mathcal{T}^m(S) \leq \mathcal{H}^m(S)$ for all $S \subset \mathbf{R}^n$ and that $\mathcal{C}^m(S) = \mathcal{T}^m(S) = \mathcal{H}^m(S)$ if $m = 1$, $m = n$, or S is m-rectifiable [2, 2.10.6, 2.10.4]. However, it was shown by G. Freilich [3] and E. F. Moore [4] that $\mathcal{C}^2$ and $\mathcal{H}^2$ are distinct measures over $\mathbf{R}^3$; more recently the author [1] established that $\mathcal{C}^2$ and $\mathcal{T}^2$ are also distinct over $\mathbf{R}^3$.

In this paper we prove (Theorem 5.4) that there also exists $E \subset \mathbf{R}^3$ satisfying $\mathcal{T}^2(E) < \mathcal{H}^2(E)$. A precise definition of E is given in §2, but roughly this set is the image under the usual isomorphism of $\mathbf{R} \times \mathbf{R}^2$ onto $\mathbf{R}^3$ of the Cartesian product of $\{x: -4 \leq x \leq 4\}$ and a Cantor type subset of $\mathbf{R}^2$; the latter term in this product is the intersection of a decreasing sequence, every member of which is the union of certain closed circular disks.

2. **Preliminaries.** In general we adopt in this paper the notation and terminology of [2]. Presented in this section are additional definitions that we use.

Define $p: \mathbf{R}^3 \to \mathbf{R}^3$, $q: \mathbf{R}^3 \to \mathbf{R}^3$, $\iota: \mathbf{R} \times \mathbf{R}^2 \to \mathbf{R}^3$, $p(x_1, x_2, x_3) = (x_1, 0, 0)$, $q(x_1, x_2, x_3) = (0, x_2, x_3)$, $\iota(x_1, (x_2, x_3)) = (x_1, x_2, x_3)$ for $x_1, x_2, x_3 \in \mathbf{R}$.

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To define E first inductively define families $G_0, G_1, G_2, \cdots$ of closed circular disks contained in $\mathbf{R}^2 = \mathbf{C}$ by taking

$$G_0 = \{\mathrm{B}(0, \tfrac{1}{2})\},$$

$$G_n = \{\mathrm{B}[z + 0.99r \exp(0.02\pi k i), 0.01r]: \mathrm{B}(z, r) \in G_{n-1}, k = 1, \cdots, 100\}$$

for $n \geq 1$; then let $E = \iota(\{x: -4 \leq x \leq 4\} \times \bigcap_{n=0}^{\infty} \bigcup G_n)$.

Let $C = q(E)$, $K_n = \{\iota(\{0\} \times S): S \in G_n\}$.

For $\alpha \geq 0$, n a nonnegative integer define

$$\zeta(\alpha, n) = \{q^{-1}(C) \cap \iota(\{x: \beta \leq x \leq \beta + \alpha\} \times S): \beta \in \mathbf{R}, S \in G_n\}.$$

If $S \subset E$ and $\operatorname{diam} q(S) > 0$, then let

$$\eta(S) = \sup\{n: q(S) \subset T \text{ for some } T \in K_n\}$$

and take $\mu(S)$ to be that element of $K_{\eta(S)}$ containing $q(S)$.

For $S \subset \mathbf{R}^3$, $a \in \mathbf{R}^3$ define $\xi(S, a) = S \cap q^{-1}\{q(a)\}$.

Let $S - T = \{x - y: x \in S, y \in T\}$ for $S, T \subset \mathbf{R}^n$.

Finally, for $\emptyset \neq S \subset \mathbf{R}^n$ let

$$b^1(S) = \operatorname{diam} S, \quad b^2(S) = (\pi/4)(\operatorname{diam} S)^2 \quad \text{and}$$

$$t^2(S) = (\pi/4) \sup\{|(a_1 - b_1) \wedge (a_2 - b_2)|: a_1, b_1, a_2, b_2 \in S\}.$$

These are the gauge functions used in defining $\mathcal{H}^1, \mathcal{H}^2$ and $\mathcal{T}^2$, respectively [2, 2.10.1–2.10.3].

3. Some lemmas. We prove here several results for use in §5.

3.1. Lemma. *If $D \subset K_n$, $2 \leq \operatorname{card} D \leq 51$ and $\eta(\bigcup D) = n - 1$, then there exist $A, B \in D$ such that*

$$\operatorname{dist}(A, B) \geq 10^{-2n}(99 \sin[(\operatorname{card} D - 1)0.01\pi] - 1).$$

Proof. The conclusion follows from the observation that for some $A, B \in D$ the distance between the centers of A and B is at least $10^{-2n} 99 \sin[(\operatorname{card} D - 1)0.01\pi]$.

3.2. Lemma. *If $A \subset E$ and* $\operatorname{diam} q(A) > 0$, *then* $\mathcal{H}^1[q(A)] \leq \operatorname{diam} \mu(A)$.

Proof. For any integer $m \geq \eta(A)$ we let $W_m = K_m \cap \{S: S \cap q(A)\} \neq \emptyset$ and obtain our assertion by noting that $q(A) \subset \bigcup W_m$ and $\Sigma_{S \in W_m} b^1(S) \leq 10^{-2\eta(A)} = \operatorname{diam} \mu(A)$.

3.3. Lemma. $\mathcal{H}^1(C) > 0$.

Proof. Consider any countable covering of C consisting of nonempty subsets of C that are open in C and let W be a finite subcovering. Since $\{\mu(S): S \in W\}$ is a covering of C, and $T \cap C \neq \emptyset$ for any $T \in \bigcup_{i=1}^{\infty} K_i$, it follows that $\Sigma_{S \in W} 10^{-2\eta(S)} \geq 1$. Using this result, Lemma 3.1, and the fact that $\operatorname{card}(K_{\eta(S)+1} \cap \{T: T \cap S \neq \emptyset\}) \geq 2$ for all $S \in W$, we deduce that

$$\sum_{S \in W} b^1(S) \geq \sum_{S \in W} 10^{-2\eta(S)-2}[99 \sin(0.01\pi) - 1] \geq 0.99 \sin(0.01\pi) - 0.01;$$

hence $\mathcal{H}^1(C) > 0$.

3.4. **Corollary.** $0 < \mathcal{H}^2(E) < \infty$.

Proof. We combine Lemmas 3.2, 3.3 and [2, 2.10.45].

3.5. **Lemma.** *If $A_1 \in \zeta(\alpha, n)$, $A_2 \in \zeta(10^{2(n-m)}\alpha, m)$, and B_1 is a nonempty closed subset of A_1, then there exists a closed subset B_2 of A_2 such that*

(i) $b^2(B_2) = 10^{4(n-m)} b^2(B_1)$,

(ii) $t^2(B_2) = 10^{4(n-m)} t^2(B_1)$,

(iii) $\mathcal{H}^2(B_2) = 10^{4(n-m)} \mathcal{H}^2(B_1)$.

Proof. Let c_j denote the center of A_j for $j = 1, 2$. Let $f: \mathbf{R}^3 \to \mathbf{R}^3$ be defined by $f(x) = 10^{2(n-m)}(x - c_1) + c_2$ for $x \in \mathbf{R}^3$. Let $B_2 = f(B_1)$. Then clearly (i) and (ii) hold. Furthermore, since $\operatorname{Lip} f = 10^{2(n-m)} = 1/\operatorname{Lip}(f^{-1})$, (iii) follows from [2, 2.10.11].

3.6. **Corollary.** *If $A \in \zeta(8 \cdot 10^{-2n}, n)$, B_1 is a closed subset of A and $b^2(B_1) > 0$, then there exists a closed subset B_2 of E such that $\eta(B_2) = 0$ and*

$$\mathcal{H}^2(B_2)/b^2(B_2) = \mathcal{H}^2(B_1)/b^2(B_1).$$

3.7. **Corollary.** *If $A \in \zeta(\alpha, n)$ then $\mathcal{H}^2(A) = 10^{-2n}\alpha\mathcal{H}^2(E)/8$.*

Proof. We note that $\mathcal{H}^2(S) = 10^{-4n}\mathcal{H}^2(E)$ for $S \in \zeta(8 \cdot 10^{-2n}, n)$ by Lemma 3.5(iii), and combine this with [2, 2.10.45] to obtain our conclusion.

4. **A key lemma.** Our main goal here is to prove Lemma 4.5 for later use in the proof of Theorem 5.3. Throughout this section we assume that $A \subset E$ is such that $q(A) = C$, and $-x \in A$ for all $x \in A$, and let $d = \operatorname{diam} A$.

4.1. *Notation.* For $a \in S \subset E$ let

$$\lambda(S, a) = \{(u, v): u \in \xi(S, a) - S,\ v \in S - S,$$
$$|u \wedge v| \geq (\operatorname{diam} S)^2 - 10^{-18} \text{ and } |q(u) \wedge q(v)| \leq 10^{-9} \operatorname{diam} S\}.$$

4.2. **Remark.** If $a \in S \subset E$ and $(v_1, v_2) \in \lambda(S, a)$, then $|v_j|^2 \geq (\operatorname{diam} S)^2 - 2 \cdot 10^{-18}$ for $j = 1, 2$.

4.3. **Lemma.** *If $a_1, a_2, b_1, b_2 \in A$, $(v_1, v_2) = (a_1 - b_1, a_2 - b_2) \in \lambda(A, a_1)$, and $j = 1$ or $j = 2$, then*

(i) $$|q(2a_j) - q(v_j)| \le 2 \cdot 10^{-9},$$

(ii) $$||q(2a_j)|^2 - |q(v_j)|^2| \le 10^{-8}.$$

Proof. It follows from Remark 4.2 that

$$2a_j \cdot b_j = |a_j|^2 + |b_j|^2 - |v_j|^2 \le |a_j|^2 + |b_j|^2 - d^2 + 2 \cdot 10^{-18}.$$

Furthermore, since $-a_j, -b_j \in A$ we have that $|a_j| \le d/2$, $|b_j| \le d/2$. Together these results yield (i) because

$$|2a_j - v_j|^2 = |a_j + b_j|^2 = |a_j|^2 + 2a_j \cdot b_j + |b_j|^2 \le 2 \cdot 10^{-18}.$$

We then deduce (ii) immediately from (i) by noting that

$$||q(2a_j)|^2 - |q(v_j)|^2| \le |q(2a_j) + q(v_j)| \cdot |q(2a_j) - q(v_j)| \le 10^{-8}.$$

4.4. **Lemma.** *If* $(v_1, v_2) \in \lambda(A, a)$ *for some* $a \in A$, *then*

$$||q(v_1)|^2 + |q(v_2)|^2 - d^2| \le 3 \cdot 10^{-6}.$$

Proof. From Remark 4.2, the inequality $d \le 10$ and the definition of $\lambda(A, a)$ we obtain that

$$\begin{aligned} |p(v_1) \cdot p(v_2)|^2 &\le [d^2 - |q(v_1)|^2][d^2 - |q(v_2)|^2] \\ &\le [|p(v_1)|^2 + 2 \cdot 10^{-18}]\,[|p(v_2)|^2 + 2 \cdot 10^{-18}] \\ &\le |p(v_1) \cdot p(v_2)|^2 + 5 \cdot 10^{-16}, \end{aligned}$$

$$|[|q(v_1)| \cdot |q(v_2)|]^2 - |q(v_1) \cdot q(v_2)|^2| = |q(v_1) \wedge q(v_2)|^2 \le 10^{-16},$$

$$|v_1 \cdot v_2| = [(|v_1| \cdot |v_2|)^2 - |v_1 \wedge v_2|^2]^{1/2} \le 2^{1/2}\, 10^{-8}.$$

We then use these results and the fact that $1 \le d \le 10$ to conclude that

$$\begin{aligned} &||q(v_1)|^2 + |q(v_2)|^2 - d^2| \\ &\le |[d^2 - |q(v_1)|^2][d^2 - |q(v_2)|^2] - [|q(v_1)| \cdot |q(v_2)|]^2| \\ &\le ||p(v_1) \cdot p(v_2)|^2 - |q(v_1) \cdot q(v_2)|^2| + 6 \cdot 10^{-16} \\ &= |v_1 \cdot v_2| \cdot |p(v_1) \cdot p(v_2) - q(v_1) \cdot q(v_2)| + 6 \cdot 10^{-16} \le 3 \cdot 10^{-6}. \end{aligned}$$

4.5. **Lemma.** $\lambda(A, a) = \emptyset$ *for some* $a \in A$.

Proof. Choose $\alpha, \beta, \gamma \in A$ satisfying $q(\alpha) = (0, 0.5, 0)$, $q(\beta) = (0, 0.49, 0)$, $q(\gamma) = (0, 0.4999, 0)$. We will obtain our conclusion by showing that if there exist $(u_1, u_2) \in \lambda(A, \alpha)$, $(v_1, v_2) \in \lambda(A, \beta)$, then $\lambda(A, \gamma) = \emptyset$.

To prove this we first note that $0.98 \leq |q(2x)| \leq 1$ for all $x \in A$ and then apply Lemmas 4.4, 4.3(ii) to obtain that

$$d^2 \geq |q(u_1)|^2 + |q(u_2)|^2 - 3 \cdot 10^{-6} \geq 1.9603969, \tag{1}$$

$$d^2 \leq |q(v_1)|^2 + |q(v_2)|^2 + 3 \cdot 10^{-6} \leq 1.9604031. \tag{2}$$

Next take any $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in \xi(A, \gamma) - A$, $\delta \in A$, $\tau = (\tau_1, \tau_2, \tau_3) \in \xi(A, \delta) - A$. To establish that $(\sigma, \tau) \notin \lambda(A, \gamma)$ we observe that if $|q(2\delta)| \leq 0.9802$ then

$$\big|\, |q(\sigma)|^2 + |q(\tau)|^2 - d^2 \big| \geq 4 \cdot 10^{-6}$$

by Lemma 4.3(ii) and (1), while if $|q(2\delta)| \geq 0.9998$ then Lemma 4.3(ii) and (2) yield the same conclusion; in either case $(\sigma, \tau) \notin \lambda(A, \gamma)$ by Lemma 4.4. On the other hand, if $0.9802 < |q(2\delta)| < 0.9998$, then Lemma 4.3(i) and the fact that $q(\delta)$ is not in any element of K_2 nearest to or furthest away from the origin are used to obtain that

$$|\tau_3| \geq 10^{-2} \sin(2 \cdot 10^{-2}\pi) - 2 \cdot 10^{-4} \geq 4 \cdot 10^{-4};$$

furthermore Lemma 4.3(i) also implies that $|\sigma_2| \geq 0.9997$, $|\sigma_3| \leq 2 \cdot 10^{-9}$; consequently $(\sigma, \tau) \notin \lambda(A, \gamma)$ in this case either since $|q(\sigma) \wedge q(\tau)| \geq 10^{-4}$.

5. **Final results.** Our main conclusion is Theorem 5.4. This result follows principally from Theorem 5.3, which in turn depends on Lemmas 4.5, 5.1 and 5.2.

5.1. **Lemma.** *If* A *is a closed subset of* E, diam $q(A) > 0$ *and* diam $\mu(A) \leq$ [diam $p(A)$]/3, *then* $\mathcal{H}^2(A) \leq 2b^2(A)/3$.

Proof. We use [2, 2.10.45] and Lemma 3.2 to obtain that $\mathcal{H}^2(A) \leq (\pi/2)\mathcal{H}^1[p(A)]\mathcal{H}^1[q(A)] \leq (\pi/2)\mathcal{H}^1[p(A)]\,\mathrm{diam}\,\mu(A) \leq 2b^2(A)/3$.

5.2. **Lemma.** *If* A *is a closed subset of* E, diam $q(A) > 0$ *and* $(\mathrm{diam}\, A)^2 \leq 4 \cdot 10^{-2\eta(A)}/3$, *then* $\mathcal{H}^2(A)/b^2(A) \leq 0.992$.

Proof. By Lemma 5.1 we may assume that diam $\mu(A) >$ [diam $p(A)$]/3 and then by Corollary 3.6 further assume that $\eta(A) = 0$.

Let $d = \mathrm{diam}\, A$, $W = K_1 \cap \{S : S \cap q(A) \neq \emptyset\}$. Define $\psi(S) = p[q^{-1}(S) \cap A]$ for $S \in W$. Applying [2, 2.10.45] and Lemma 3.2 we then have that

$$\mathcal{H}^2(A)/b^2(A) \leq \frac{\pi}{2} \sum_{S \in W} \frac{\mathcal{H}^1[\psi(S)]\mathcal{H}^1[S \cap q(A)]}{b^2(A)} \leq 0.02 \sum_{S \in W} \frac{\mathcal{H}^1[\psi(S)]}{d^2}. \tag{3}$$

Let $n = \mathrm{card}\, W$. Let m be the greatest integer not exceeding $n/2$. Define $\rho(x) = 0.99 \sin(0.01\pi x) - 0.01$ for $x \in \mathbf{R}$. Using Lemma 3.1 we deduce that if $S_1, \cdots, S_{2m}$ is a sequence of distinct elements of W arranged in clockwise order, then

$$\mathcal{H}^1[\psi(S_i)] + \mathcal{H}^1[\psi(S_{i+m})] \le 2(d^2 - [\rho(m)]^2)^{1/2} \quad \text{for } i = 1, \cdots, m. \tag{4}$$

Furthermore, by Lemma 3.1 there exists $S \in W$ such that

$$\mathcal{H}^1[\psi(S)] \le (d^2 - [\rho(n-1)]^2)^{1/2}. \tag{5}$$

At this point we divide the proof into several cases and subcases, in each of which we show that $\mathcal{H}^2(A)/b^2(A) \le 0.992$.

We first consider the following two cases:

Case I. $n = 3$. The desired result is obtained by first observing that (3), (4) and (5) imply

$$\mathcal{H}^2(A)/b^2(A) \le [0.04(d^2 - [\rho(1)]^2)^{1/2}/d^2] + [0.02(d^2 - [\rho(2)]^2)^{1/2}/d^2]$$

and then maximizing separately both terms of the right-hand side of this inequality with respect to d for $d \ge \rho(2)$.

Case II. $n \ne 3$. Define $f\colon \mathbf{R} \cap \{x\colon x \ge \rho(m)\} \to \mathbf{R}$, $f(x) = 0.02n(x^2 - [\rho(m)]^2)^{1/2}/x^2$ for $x \ge \rho(m)$. We use (3), (4) and for n odd also (5) to obtain $\mathcal{H}^2(A)/b^2(A) \le f(d)$ and further observe that the absolute maximum for f occurs at $2^{1/2}\rho(m)$ and f is increasing on $\{x\colon \rho(m) \le x \le 2^{1/2}\rho(m)\}$. Then we divide the remainder of the proof into the following three subcases:

Case II.A. $2 \le n \le 96$ and n is even. Let $g\colon \mathbf{R} \cap \{x\colon \rho(x/2) \ne 0\} \to \mathbf{R}$, $g(x) = 0.01\,x/\rho(x/2)$ whenever $\rho(x/2) \ne 0$. Our conclusion is obtained by noting that $g(n) = f[2^{1/2}\rho(m)]$, g has no relative maximum on $\{x\colon 2 < x < 96\}$, $g(2) \le 0.95$ and $g(96) \le 0.99$.

Case II.B. $5 \le n \le 97$ and n is odd. Let $g\colon \mathbf{R} \cap \{x\colon \rho[(x-1)/2] \ne 0\} \to \mathbf{R}$, $g(x) = 0.01x/\rho[(x-1)/2]$ whenever $\rho[(x-1)/2] \ne 0$ and proceed as in Case II.A.

Case II.C. $n = 98$, 99, or 100. We observe that $f[(4/3)^{1/2}] \ge f(d)$ because $d \le (4/3)^{1/2} \le 2^{1/2}\rho(m)$, and compute $f[(4/3)^{1/2}] \le 0.91$ for $n = 98, 99, 100$.

5.3. **Theorem.** *There exists a nonempty closed subset M of E such that*

$$t^2(M) \le \mathcal{H}^2(M) - 10^{-44}\mathcal{H}^2(E). \tag{6}$$

Proof. By Corollary 3.4 and the definition of $\mathcal{H}^2$ there exists a countable covering W of E consisting of nonempty closed subsets of E for which

$$0 < \sum_{S \in W} b^2(S) \le (1 + 10^{-44})\mathcal{H}^2(E) \le (1 + 10^{-44}) \sum_{S \in W} \mathcal{H}^2(S);$$

hence there exists a nonempty closed subset F of E satisfying

$$0 < b^2(F) \le (1 + 10^{-44})\mathcal{H}^2(F). \tag{7}$$

Furthermore, it follows from Lemma 5.1 and Corollary 3.6 that we may assume $\eta(F) = 0$.

Next, let $d = \operatorname{diam} F$. We note that Lemmas 5.1, 5.2 imply $10 \geq d^2 \geq 4/3$.

Define $f: \mathbf{R}^3 \to \mathbf{R}$, $f(x_1, x_2, x_3) = x_1$ for $(x_1, x_2, x_3) \in \mathbf{R}^3$, and then let

$$F_1 = \{x\colon \operatorname{diam}(F \cup \{x\}) = d, \text{ and } f(x) \geq f(y) \text{ for some } y \in \xi(F, x)\},$$

$$F_2 = \{x\colon \operatorname{diam}(F_1 \cup \{x\}) = d, \text{ and } f(x) \leq f(y) \text{ for some } y \in \xi(F_1, x)\},$$

$$F_3 = F_2 \cap \{x\colon p(x) \in p[\xi(F_2, y)] \text{ for all } y \in F_2 \text{ satisfying } |q(x-y)| \leq 10^{-44}\},$$

$$F_4 = \{x\colon \operatorname{dist}(x, C) \leq (\mathcal{H}^1[\xi(F_3, x)] + \mathcal{H}^1[\xi(F_3, -x)])/4\},$$

$$F_5 = \{x\colon \operatorname{dist}(x, C) \leq \mathcal{H}^1[\xi(F_4, x)]\}.$$

Applying Lemma 4.5 we choose $a \in F_4$ for which $\lambda(F_4, a) = \emptyset$, and let

$$M = F_5 \cup \{x\colon 0 \leq f(x) - \mathcal{H}^1[\xi(F_5, a)]/2 \leq 10^{-20},\ q(x) \in C \text{ and } |q(x-a)| \leq 10^{-22}\}.$$

We observe that since $d \leq 10^{1/2}$, clearly $M \subset E$. Furthermore, to establish (6) it need only be proven that

$$t^2(M) \leq 2b^2(F_4), \tag{8}$$

$$b^2(F_4) \leq b^2(F), \tag{9}$$

$$\mathcal{H}^2(M) \geq 2\mathcal{H}^2(F) + 3 \cdot 10^{-44}\mathcal{H}^2(E) \tag{10}$$

since the inequalities (8), (9), (7), $\mathcal{H}^2(F) \leq \mathcal{H}^2(E) < \infty$ and (10) then yield this conclusion.

To obtain (8) we first define $g\colon F_5 - F_5 \to F_4 - F_4$ by $g(x) = p(x)/2 + q(x)$ for $x \in F_5 - F_5$ and observe that for $x_1, x_2 \in F_5 - F_5$

$$\begin{aligned} &|x_1 \wedge x_2| \\ &= |2p[g(x_1)] \wedge q[g(x_2)] + 2p[g(x_2)] \wedge q[g(x_1)] + q[g(x_1)] \wedge q[g(x_2)]| \qquad (11)\\ &= (4|g(x_1) \wedge g(x_2)|^2 - 3|q[g(x_1)] \wedge q[g(x_2)]|^2)^{1/2}. \end{aligned}$$

We next take any $u_1, u_2 \in M - M$ and consider the following two possibilities: If $u_1, u_2 \in F_5 - F_5$ then $(\pi/4)|u_1 \wedge u_2| \leq 2b^2(F_4)$ by (11).

On the other hand, suppose at least one of u_1, u_2, say u_1 for the sake of argument, is not in $F_5 - F_5$. Then $u_1 = v_1 + w_1$, $u_2 = v_2 + w_2$, where $v_1 \in \xi(F_5, a) - F_5$, $v_2 \in F_5 - F_5$, $|w_1| \leq 2 \cdot 10^{-20}$, $|w_2| \leq 2 \cdot 10^{-20}$; together these relations yield

$$|u_1 \wedge u_2| \le |v_1 \wedge v_2| + |v_1 \wedge w_2| + |v_2 \wedge w_1| + |w_1 \wedge w_2|$$

$$\le |v_1 \wedge v_2| + 6 \cdot 10^{-20} \operatorname{diam} F_5 \le |v_1 \wedge v_2| + 6 \cdot 10^{-19}. \tag{12}$$

Finally, using (11) and the fact that $(g(v_1), g(v_2)) \notin \lambda(F_4, a)$ by the choice of a, we find that

$$|v_1 \wedge v_2| \le 2(\operatorname{diam} F_4)^2 - 7 \cdot 10^{-19}$$

and combine this with (12) to conclude (8).

To deduce (9) we let $\delta = \operatorname{diam} F_3$, take any $x, y \in F_4$ and observe that

$$\mathcal{H}^1[\xi(F_3, x)] + \mathcal{H}^1[\xi(F_3, y)] \le \max\{2[\delta^2 - |q(x-y)|^2]^{1/2}, \delta\}.$$

We then use this relation twice, the second time with x, y replaced by $-x, -y$, and also the inequalities $\delta \le d$, $d^2 \ge 4/3$, to conclude that

$$(\operatorname{diam}[\xi(F_4, x) \cup \xi(F_4, y)])^2$$

$$\le (\mathcal{H}^1[\xi(F_3, x)] + \mathcal{H}^1[\xi(F_3, y)] + \mathcal{H}^1[\xi(F_3, -x)] + \mathcal{H}^1[\xi(F_3, -y)])^2/16$$

$$+ |q(x-y)|^2$$

$$\le \max\{\delta^2 - |q(x-y)|^2, \delta^2/4\} + |q(x-y)|^2 \le d^2.$$

To prove (10) we first note that clearly $\mathcal{H}^2(F_2) \ge \mathcal{H}^2(F)$ and that $\mathcal{H}^2(F_5) = 2\mathcal{H}^2(F_3)$ by [2, 2.10.45]. Hence it suffices to show that

$$\mathcal{H}^2(F_3) \ge \mathcal{H}^2(F_2) - 5 \cdot 10^{-45}\mathcal{H}^2(E), \tag{13}$$

$$\mathcal{H}^2(M \sim F_5) \ge 10^{-43}\mathcal{H}^2(E). \tag{14}$$

To obtain these last two inequalities, we first show that if α and β are endpoints of $\xi(F_2, \alpha)$ and $\xi(F_2, \beta)$ respectively, $r > 0$, $|q(\alpha - \beta)| \le r$ and

$$0 < |p(\alpha - \beta)| = \min\{|p(\alpha - x)| : x \in \xi(F_2, \beta)\},$$

then

$$|p(\alpha - \beta)| \le 2r. \tag{15}$$

To establish (15) we choose $\gamma \in F_2$ satisfying $|\beta - \gamma| = d$, and $p(\alpha - \beta) \cdot p(\beta - \gamma) \ge 0$, note that $|p(\beta - \gamma)| \ge 1/2$ since $d^2 \ge 4/3$, and then find that

$$d^2 \ge |\alpha - \gamma|^2 = |(\alpha - \beta) + (\beta - \gamma)|^2$$

$$\ge 2p(\alpha - \beta) \cdot p(\beta - \gamma) + 2q(\alpha - \beta) \cdot q(\beta - \gamma) + |\beta - \gamma|^2$$

$$\ge |p(\alpha - \beta)| - 2r + d^2.$$

We then deduce (13) by using (15) with $r = 10^{-44}$ to obtain that $F_2 \sim F_3$ is contained in the union of $2 \cdot 10^{44}$ elements of $\zeta(2 \cdot 10^{-44}, 22)$, and combining this with Corollary 3.7.

To prove (14) we consider any $b \in F_5$ with $|q(a-b)| \leq 10^{-22}$ and apply (15), first with $r = 10^{-44}$ and then with $r = 10^{-22}$ to obtain that

$$\begin{aligned}
&\mathcal{H}^1[\xi(F_5, b)] - \mathcal{H}^1[\xi(F_5, a)] \\
&= \mathcal{H}^1[\xi(F_3, b)] - \mathcal{H}^1[\xi(F_3, a)] + \mathcal{H}^1[\xi(F_3, -b)] - \mathcal{H}^1[\xi(F_3, -a)] \\
&\leq \mathcal{H}^1[\xi(F_2, b)] - \mathcal{H}^1[\xi(F_2, a)] + \mathcal{H}^1[\xi(F_2, -b)] - \mathcal{H}^1[\xi(F_2, -a)] + 8 \cdot 10^{-44} \\
&\leq 9 \cdot 10^{-22}.
\end{aligned}$$

Consequently, $M \sim F_5$ contains an element of $\zeta(9 \cdot 10^{-21}, 11)$ and then (14) follows from Corollary 3.7.

5.4. **Theorem.** $\mathcal{T}^2(E) < \mathcal{H}^2(E)$.

Proof. Given any $\delta > 0$ choose a positive integer n satisfying $10^{-2n+1} < \delta$. Since E is contained in the union of 10^{4n} elements of $\zeta(8 \cdot 10^{-2n}, n)$, Theorem 5.3 and Lemma 3.5(ii), (iii) imply there exists a family W_1 consisting of 10^{4n} nonempty closed subsets of E of diameter less than δ such that

$$\sum_{S \in W_1} t^2(S) \leq \mathcal{H}^2\left(\bigcup W_1\right) - 10^{-44}\mathcal{H}^2(E). \tag{16}$$

Furthermore, since $\mathcal{T}^2(E \sim \bigcup W_1) \leq \mathcal{H}^2(E \sim \bigcup W_1)$ and $\mathcal{H}^2(E) > 0$, there also exists a countable family W_2 of nonempty closed sets of diameter less than δ covering $E \sim \cup W_1$ for which

$$\sum_{S \in W_2} t^2(S) \leq \mathcal{H}^2\left(E \sim \bigcup W_1\right) + 5 \cdot 10^{-45}\mathcal{H}^2(E). \tag{17}$$

Then $W_1 \cup W_2$ is a countable covering of E by nonempty closed sets of diameter less than δ which by (16) and (17) satisfies

$$\sum_{S \in W_1 \cup W_2} t^2(S) \leq (1 - 5 \cdot 10^{-45})\mathcal{H}^2(E);$$

hence $\mathcal{T}^2(E) \leq (1 - 5 \cdot 10^{-45})\mathcal{H}^2(E)$. Finally, this last inequality and Corollary 3.4 yield the desired conclusion.

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