WEIGHTED JOIN-SEMIlattICES AND TRANSVERSAL MATROIDS

BY

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ABSTRACT. We investigate join-semilattices in which each element is assigned a nonnegative weight in a strictly increasing way. A join-subsemilattice of a Boolean lattice is weighted by cardinality, and we give a characterization of these in terms of the notion of a spread. The collection of flats with no coloops (isthmuses) of a matroid or pregeometry, partially ordered by set-theoretic inclusion, forms a join-semilattice which is weighted by rank. For transversal matroids these join-semilattices are isomorphic to join-subsemilattices of Boolean lattices. Using a previously obtained characterization of transversal matroids and results on weighted join-semilattices, we obtain another characterization of transversal matroids. The problem of constructing a transversal matroid whose join-semilattice of flats is isomorphic to a given join-subsemilattice of a Boolean lattice is then investigated.

1. Introduction. We are motivated by a recent study [3] of transversal matroids in which a characterization of transversal matroids is given in terms of the join-semilattice of flats with no coloops (isthmuses). The characterization depends on an algorithm used to construct a family of flats. We consider here the more general situation of a join-semilattice J in which each element has assigned to it a nonnegative weight where the only assumption is that weight is a strictly increasing function on J . We introduce the concept of a spread for such a weighted join-semilattice, show a spread is unique if it exists at all, and give a characterization of join-semilattices of a finite Boolean lattice which are weighted by cardinality. Making use of the characterization of transversal matroids given in [3], we then give an alternate characterization which has the advantage of not depending on an algorithm. Finally we consider the problem of constructing a transversal matroid such that its weighted join-semilattice of flats with no coloops is isomorphic to a given join-subsemilattice of a Boolean lattice.

A family of objects differs from a set in that the objects may be repeated. We use parentheses $()$ to denote families and braces $\{ \}$ to denote sets. If E is a

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set and $\mathfrak{U} = (a_i; i \in I)$ is a family of elements of E , then for $x \in E$ we define the *multiplicity* of x in $\mathfrak{U}$ by

$$m(\mathfrak{U}, x) = |\{i \in I: a_i = x\}|.$$

This multiplicity may be 0. If $\mathfrak{U}_1, \mathfrak{U}_2$ are two families of elements of E , we write $\mathfrak{U}_1 \leq \mathfrak{U}_2$ provided $m(\mathfrak{U}_1, x) \leq m(\mathfrak{U}_2, x)$ for all $x \in E$. We consider two families to be identical if $m(\mathfrak{U}_1, x) = m(\mathfrak{U}_2, x)$ for all $x \in E$. The cardinality $\|\mathfrak{U}\|$ of a family $\mathfrak{U}$ is defined by $\|\mathfrak{U}\| = |I|$.

2. Weighted join-semilattices. A *join-semilattice* is a partially ordered set J such that each pair a, b of elements of J has a least upper bound, which is denoted by $a \vee b$. A nonempty collection of subsets of a set E which is closed under union is a join-semilattice; the partial order is set-theoretic inclusion. Such a join-semilattice is a join-subsemilattice of the Boolean lattice $B(E)$ of all subsets of E . A finite join-semilattice has a maximal element which we usually denote by 1; it need not have a minimal element but, if it does, it is usually denoted by 0. It is well known that a finite join-semilattice with a minimal element is a lattice [1].

If P is a partially ordered set, then a mapping ω from P to the nonnegative integers is a *weighting* of P if $a < b$ implies $\omega(a) < \omega(b)$ ($a, b \in P$). A weighting of P need not be a grading [1] of P , for we do not assume that $\omega(b) = \omega(a) + 1$ if b covers a . If E is a set and P is a collection of subsets of E partially ordered by set-theoretic inclusion, then $\omega(A) = |A|$ defines a weighting of P . A partially ordered set with a specific weighting is called a *weighted partially ordered set*.

Let P be a weighted partially ordered set with a maximal element 1 and set $\omega(1) = r$. Thus any chain has length at most r . A *spread* of P is a family $\mathfrak{U} = (a_i; i \in I)$ of elements of P such that the following condition holds: if $x \in P$ with $\omega(x) = r - k$ ($0 \leq k \leq r$), then

$$(2.0.1) \quad |\{i \in I: a_i \geq x\}| = k.$$

In other words, exactly k members of $\mathfrak{U}$ are greater than or equal to x if $\omega(x) = r - k$. If P has a minimal element 0, then clearly the cardinality $\|\mathfrak{U}\|$ of the spread $\mathfrak{U}$ equals $r - \omega(0)$. Thus if $\omega(0) = 0$, a spread of P has exactly $r = \omega(1)$ elements. The notion of a spread arose in a characterization of transversal matroids [3], which will be fully explained in due course.

Theorem 2.1. *If the weighted partially ordered set P with maximal element has a spread, the spread is unique.*

Proof. Suppose $\mathfrak{U}$ is a spread of P , and let $\omega(1) = r$. Let $a \in P$ with $\omega(a) = r - k$. Then from (2.0.1) we conclude that the multiplicity of a in $\mathfrak{U}$ is given by

$$(2.1.1) \quad m(\mathcal{U}, a) = k - \sum_{x > a} m(\mathcal{U}, x).$$

This implies, in particular, that $\sum_{x > a} m(\mathcal{U}, x) \leq k$. The equation (2.1.1) along with $m(\mathcal{U}, 1) = 0$ furnishes a recursion formula for the multiplicities of elements of P in $\mathcal{U}$. Since these multiplicities are uniquely determined, so is the spread $\mathcal{U}$.

The proof of Theorem 2.1 gives an algorithm for determining a spread if one exists. If there is no spread, then there is some element a of P with $\omega(a) = r - k$ such that $\sum_{x > a} m(\mathcal{U}, x) > k$.

The next theorem furnishes examples of weighted partially ordered sets which have spreads. Theorem 2.3 will then show that these examples are not very special.

Theorem 2.2. *Let J be a join-subsemilattice of the Boolean lattice on a set E with $|E| = r$. Suppose E is the maximal element of J , and let J be weighted by the cardinality function. Then*

$$(2.2.1) \quad J \text{ has a spread } \mathcal{U}, \text{ and } \|\mathcal{U}\| = r - \left| \bigcap \{A : A \in J\} \right|.$$

$$(2.2.2) \quad m(\mathcal{U}, A) = \left| \left(\bigcap X : A \subsetneq X \in J \right) \right| - |A| \quad (A \in J).$$

Proof. The assumption that E is the maximal element of J is one of convenience. If this were not the case, E would be replaced by a smaller set.

Let $A \in J$ with $|A| = r - k$. We need to show that $m(\mathcal{U}, A) = k - \sum_{A \subsetneq X \in J} m(\mathcal{U}, X)$ and $m(\mathcal{U}, E) = 0$ allows us to define $\mathcal{U}$ recursively. This is surely true for $k = 0$ (that is, $A = E$). We proceed by induction on k . Let $A \in J$ with $|A| = r - k$. Let $X_1, \dots, X_t$ be the members of J which strictly contain A . Thus $|X_i| = r - k_i$ where $k_i < k$ ($1 \leq i \leq t$). Since J is a join-semilattice, $X_{i_1} \cup \dots \cup X_{i_s} \in J$ whenever $1 \leq i_1 < \dots < i_s \leq t$; let $|X_{i_1} \cup \dots \cup X_{i_s}| = r - k_{i_1 \dots i_s}$ where $k_{i_1 \dots i_s} < k$. Thus by induction the recursion has produced $k_{i_1 \dots i_s}$ members of $\mathcal{U}$ that contain $X_{i_1} \cup \dots \cup X_{i_s}$. Hence by the principle of inclusion-exclusion exactly

$$n = \sum_{1 \leq i \leq t} k_i - \sum_{1 \leq i_1 < i_2 \leq t} k_{i_1 i_2} + \sum_{1 \leq i_1 < i_2 < i_3 \leq t} k_{i_1 i_2 i_3} - \dots$$

members of $\mathcal{U}$ that strictly contain A have been produced. But

$$\begin{aligned}
n &= \sum_{1 \leq i \leq t} |E \setminus X_i| - \sum_{1 \leq i_1 < i_2 \leq t} |E \setminus (X_{i_1} \cup X_{i_2})| \\
&\quad + \sum_{1 \leq i_1 < i_2 < i_3 \leq t} |E \setminus (X_{i_1} \cup X_{i_2} \cup X_{i_3})| - \dots \\
&= \left| \bigcup_{i=1}^t (E \setminus X_i) \right| = \left| E \setminus \bigcap_{i=1}^t X_i \right| = r - \left| \bigcap_{i=1}^t X_i \right|.
\end{aligned}$$

Since $A \subseteq X_i$ ($1 \leq i \leq t$), $A \subseteq \bigcap_{i=1}^t X_i$ so that $|\bigcap_{i=1}^t X_i| \geq r - k$ and $n \leq r - (r - k) = k$. Thus we can define $m(\mathfrak{U}, A)$ by $k - n \geq 0$. But $k - n = k - (r - |\bigcap_{i=1}^t X_i|) = |\bigcap_{i=1}^t X_i| - (r - k) = |\bigcap_{i=1}^k X_i| - |A|$. Thus we have proved that J has a spread and that (2.2.2) is satisfied.

We have yet to prove that $\|\mathfrak{U}\| = r - |\bigcap\{A : A \in J\}|$. This is surely true if $\emptyset \in J$. If $\emptyset \notin J$, then $J^* = J \cup \{\emptyset\}$ is a join-subsemilattice of the Boolean lattice on E and has a spread $\mathfrak{U}^*$ where $\|\mathfrak{U}^*\| = r$. But

$$\|\mathfrak{U}\| = \|\mathfrak{U}^*\| - m(\mathfrak{U}^*, \emptyset) = r - \left| \left(\bigcap X : X \in J \right) \right|$$

according to (2.2.2), and this establishes the formula for $\|\mathfrak{U}\|$.

Corollary 2.3. *If $\mathfrak{U} = (A_i : i \in I)$ is the spread of the join-subsemilattice J of the Boolean lattice on E , then for $A \in J$,*

$$(2.3.1) \quad \left(\bigcap X : A \subseteq X \in J \right) = \left(\bigcap A_i : A \subseteq A_i, i \in I \right).$$

Proof. The set on the right side of (2.3.1) surely contains that on the left. Suppose now that $A \subseteq X \in J$ but $X \neq A_i$ ($i \in I$). Then arguing by induction ($|X| > |A|$), and using (2.2.2) we conclude that

$$X = \left(\bigcap Y, X \subseteq Y \in J \right) = \left(\bigcap A_i : X \subseteq A_i, i \in I \right).$$

Thus if $A \subseteq X \in J$, then either $X = A_i$ for some $i \in I$ or else there is $J \subseteq I$ such that $X = \bigcap_{i \in J} A_i$. Since $A \subseteq X$, $A \subseteq A_i$ ($i \in J$) and this establishes (2.3.1).

Consider the join-semilattice J of Theorem 2.2 and its spread $\mathfrak{U} = (A_i : i \in I)$. Let $A \in J$ and let $\mathfrak{U}_A$ be the subfamily of $\mathfrak{U}$ consisting of all members A_i of $\mathfrak{U}$ with $A \subseteq A_i$. $\mathfrak{U}_A$ is the spread of the interval $[A, E]$ of J . Then for $A, B \in J$, $A \subseteq B$ if and only if $\mathfrak{U}_B \subseteq \mathfrak{U}_A$. For if $A \subseteq B$, then surely $\mathfrak{U}_B \subseteq \mathfrak{U}_A$. On the other hand if $\mathfrak{U}_B \subseteq \mathfrak{U}_A$, then $B \in \mathfrak{U}_B$ implies $B \in \mathfrak{U}_A$ so that $A \subseteq B$, while $B \notin \mathfrak{U}_B$ implies $B \notin \mathfrak{U}$ (i.e. $m(\mathfrak{U}, B) = 0$), which by (2.2.2) and (2.3.1) implies

$$B = \bigcap (A_i : A_i \in \mathfrak{U}_B) \supseteq \bigcap (A_i : A_i \in \mathfrak{U}_A) \supseteq A.$$

Thus $A \subseteq B$. Hence the partial order of J is determined by the partial order on the $\mathfrak{U}_A$ ($A \in J$).

If J_1 and J_2 are two join-semilattices, an injection $\sigma: J_1 \rightarrow J_2$ is a *semilattice monomorphism* if $\sigma(a \vee b) = \sigma(a) \vee \sigma(b)$ for all $a, b \in J_1$. We shall be interested now in weighted semilattices which are isomorphic where the isomorphism preserves weights.

Theorem 2.4. *Let J be a weighted join-semilattice with $\omega(1) = r$. Let E be a set with $|E| = r$. Then there is a semilattice monomorphism $\sigma: J \rightarrow B(E)$, the Boolean lattice on E , such that $\omega(a) = |\sigma(a)|$ for all $a \in J$ if and only if J has a spread of at most r elements.*

Proof. By Theorem 2.2 if such a σ exists, J has a spread with at most r elements. Suppose now J has a spread $\mathfrak{U} = (a_i : i \in I)$ with $\|\mathfrak{U}\| = |I| \leq r$. For $a \in J$, let $I_a = \{i \in I : a_i \geq a\}$. Thus if $\omega(a) = r - k$, $|I_a| = k$. We define a map $r: J \rightarrow B(I)$ by $r(a) = I \setminus I_a$. Thus for $a, b \in J$, $r(a \vee b) = I \setminus I_{a \vee b}$. But $I_{a \vee b} = I_a \cap I_b$; for if $a_i \geq a \vee b$, then $a_i \geq a$, b so that $I_{a \vee b} \subseteq I_a \cap I_b$ while if $a_i \in I_a \cap I_b$, then $a_i \geq a$, b and thus $a_i \geq a \vee b$ so that $I_a \cap I_b \subseteq I_{a \vee b}$. This means that

$$r(a \vee b) = I \setminus (I_a \cap I_b) = (I \setminus I_a) \cup (I \setminus I_b) = r(a) \cup r(b).$$

Suppose that for some $a, b \in J$ with $a \neq b$, we have $r(a) = r(b)$. We may suppose that $b \not\leq a$ so that $a \vee b > a$. Then, by the above, $r(a \vee b) = r(a) \cup r(b) = r(a)$. Thus $I_{a \vee b} = I_a$. But since $\omega(a \vee b) > \omega(a)$, $|I_{a \vee b}| < |I_a|$, and we have a contradiction. Thus r is a semilattice monomorphism from J to $B(I)$. We calculate that for $a \in J$

$$|r(a)| = |I| - |I_a| = |I| - (r - \omega(a)) = \omega(a) - (r - |I|),$$

where $r - |I| \geq 0$. Let $t = r - |I|$ and let I^* be a t element set with $I \cap I^* = \emptyset$. Then $|I \cup I^*| = r$ and $\sigma: J \rightarrow B(I \cup I^*)$ defined by $\sigma(a) = r(a) \cup I^*$ is a semilattice isomorphism with $|\sigma(a)| = |r(a)| + t = \omega(a)$. This completes the proof of the theorem.

3. Application to transversal matroids. A characterization of transversal matroids is given by Brualdi and Dinolt [3]. We shall use the result of §2 to give an alternate formulation of it. But first we review briefly matroids, in general, and transversal matroids, in particular; for further details we refer the reader to [3] and the references contained within.

Let E be a finite set. A *matroid* [5] M on E (or *combinatorial pregeometry* [4]) is a nonempty collection of subsets of E , called *independent sets* such that

(i) a subset of an independent set is independent (thus $\emptyset \in \mathcal{M}$) (ii) $A_1, A_2 \in \mathcal{M}$ with $|A_1| < |A_2|$ imply $A_1 \cup \{x\} \in \mathcal{M}$ for some $x \in A_2 \setminus A_1$. Each subset X of E has a well-defined *rank* $\rho(X)$ which equals the common cardinality of all maximal independent sets contained in X . The rank of the matroid $\mathcal{M}$ equals $\rho(E)$. For $X \subseteq E$, $\mathcal{M}_X = \{A: A \in \mathcal{M}, A \subseteq X\}$ is a matroid, called the *restriction* of $\mathcal{M}$ to A . A closure relation can be defined on the subsets of E by defining $\bar{X}$ to be the largest subset of E containing X which has the same rank as X . Those subsets F of E with $\bar{F} = F$ are called *flats*. The collection $\mathcal{L}(\mathcal{M})$ of flats of $\mathcal{M}$, partially ordered by set-theoretic inclusion, form a geometric lattice [4]. If $X \subseteq E$, then $x \in X$ is a *coloop* or *isthmus* of X if $\rho(X \setminus \{x\}) = \rho(X) - 1$. The collection $\mathcal{F}(\mathcal{M})$ of flats of $\mathcal{M}$ which have no coloops forms a join-subsemilattice of $\mathcal{L}(\mathcal{M})$. Given a pair F_1, F_2 of flats in $\mathcal{F}(\mathcal{M})$ with $F_1 \subseteq F_2$, such that no other flat of $\mathcal{F}(\mathcal{M})$ lies between F_1 and F_2 , then the interval $[F_1, F_2]$ of $\mathcal{L}(\mathcal{M})$ consists of all sets of the form $F_1 \cup A$ where $A \subseteq F_2 \setminus F_1$, $|A| \leq \rho(F_2) - \rho(F_1) - 1$, along with F_2 . Thus the flats of $\mathcal{F}(\mathcal{M})$, given as subsets of E with their rank, determine all flats of $\mathcal{L}(\mathcal{M})$ as sets and thus the partial order of $\mathcal{L}(\mathcal{M})$; that is, they determine the lattice $\mathcal{L}(\mathcal{M})$.

A matroid $\mathcal{M}$ on E is a *transversal matroid* provided there is a family $(A_1, \dots, A_n)$ of subsets of E such that $\mathcal{M} = \mathcal{M}(A_1, \dots, A_n)$, the collection of partial transversals of $(A_1, \dots, A_n)$. If $\mathcal{M}$ is a transversal matroid of rank r , then there are r sets $A_1, \dots, A_r$ such that $\mathcal{M} = \mathcal{M}(A_1, \dots, A_r)$. The family $(A_1, \dots, A_r)$ is called a *presentation* of $\mathcal{M}$. We recall Hall's theorem which says that the family $(A_1, \dots, A_r)$ has a transversal (thus a system of distinct representatives) if and only if

$$\left| \bigcup_{i \in K} A_i \right| \geq |K| \quad (K \subseteq \{1, \dots, r\}).$$

Now let $\mathcal{M}$ be an arbitrary matroid of rank r on the finite set E . We regard the join-semilattice $\mathcal{F}(\mathcal{M})$ as a weighted join-semilattice by letting $\omega(F) = \rho(F)$ for $F \in \mathcal{F}(\mathcal{M})$. The unique flat in $\mathcal{F}(\mathcal{M})$ of weight 0 is the closure of the empty set. In the terminology of §2 the characterization of transversal matroids given in [3] is the following: $\mathcal{M}$ is a transversal matroid if and only if $\mathcal{F}(\mathcal{M})$ has a spread $(F_1, \dots, F_r)$ where $\rho(\bigcap_{i \in K} F_i) \leq r - |K|$ ($K \subseteq \{1, \dots, r\}$). It is also proved in [3] that if $\mathcal{M}$ is a transversal matroid, then $\mathcal{M} = \mathcal{M}(E \setminus F_1, \dots, E \setminus F_r)$; indeed $(E \setminus F_1, \dots, E \setminus F_r)$ is the maximal presentation of $\mathcal{M}$. This means that we cannot enlarge any of the sets $E \setminus F_1, \dots, E \setminus F_r$ and still have a presentation of $\mathcal{M}$. It is enough to know that the sets $F_1, \dots, F_r$ have no coloops, in order to conclude that $(E \setminus F_1, \dots, E \setminus F_r)$ is the maximal presentation of $\mathcal{M}$ ([1] and [3]).

If G_1, G_2 are flats in $\mathcal{L}(\mathbf{M})$ with $G_1 \subseteq G_2$, then $\mathcal{F}(\mathbf{M})_{[G_1, G_2]}$ denotes the join-subsemilattice of $\mathcal{L}(\mathbf{M})$ consisting of all flats of $\mathcal{L}(\mathbf{M})$ with no coloops which lie in the interval $[G_1, G_2]$ of $\mathcal{L}(\mathbf{M})$. Note that $\mathcal{F}(\mathbf{M}) = \mathcal{F}(\mathbf{M})_{[\emptyset, E]}$, and that $\mathcal{F}(\mathbf{M})_{[G_1, G_2]}$ is a join-subsemilattice of $\mathcal{F}(\mathbf{M})$. We regard $\mathcal{F}(\mathbf{M})_{[G_1, G_2]}$ as weighted by rank (or we could assign $F \in \mathcal{F}(\mathbf{M})_{[G_1, G_2]}$ the weight $\rho(F) - \rho(G_2)$).

Theorem 3.1. *Let $\mathbf{M}$ be a matroid of rank r on the finite set E . Then $\mathbf{M}$ is a transversal matroid if and only if for all $G_1, G_2 \in \mathcal{L}(\mathbf{M})$ with $G_1 \subseteq G_2$ the weighted join-semilattice $\mathcal{F}(\mathbf{M})_{[G_1, G_2]}$ has a spread of at most $\rho(G_2) - \rho(G_1)$ members or, equivalently, there is a weight-preserving join-semilattice isomorphism from $\mathcal{F}(\mathbf{M})_{[G_1, G_2]}$ to a Boolean lattice on an $\rho(G_2) - \rho(G_1)$ element set.*

Proof. By Theorem 2.4 the two criteria are equivalent. Suppose first that $\mathbf{M}$ is a transversal matroid of rank r . Then $\mathcal{F}(\mathbf{M})$ has a spread $(F_1, \dots, F_r)$. Let $G \in \mathcal{L}(\mathbf{M})$ with $\rho(G) = r - k$. Then those members of $(F_1, \dots, F_r)$ which contain G are the members of a spread of $\mathcal{F}(\mathbf{M})_{[G, E]}$. Let this spread be $(F_k : k \in K)$ where $K \subseteq \{1, \dots, r\}$. Since $\rho(\bigcap_{i \in K} F_i) \leq r - |K|$ and since $G \subseteq \bigcap_{i \in K} F_i$ we have $\rho(G) = r - k \leq \rho(\bigcap_{i \in K} F_i)$, and we conclude that $|K| \leq k$. Thus $\mathcal{F}(\mathbf{M})_{[G, E]}$ has a spread of at most k members where $k = \rho(E) - \rho(G)$. Since $\mathbf{M}_{G_2}$ is a transversal matroid of rank $\rho(G_2)$ on G_2 and $\mathcal{F}(\mathbf{M})_{[G_1, G_2]}$ is isomorphic to $\mathcal{F}(\mathbf{M}_{G_2})_{[G_1, G_2]}$, we conclude that $\mathcal{F}(\mathbf{M})_{[G_1, G_2]}$ has a spread of at most $\rho(G_2) - \rho(G_1)$ members for any $G_1, G_2 \in \mathcal{L}(\mathbf{M})$ with $G_1 \subseteq G_2$.

Suppose now $\mathbf{M}$ is a matroid of rank r such that for all $G \in \mathcal{L}(\mathbf{M})$, $\mathcal{F}(\mathbf{M})_{[G, E]}$ has a spread of at most $r - \rho(G)$ members. Thus, in particular, $\mathcal{F}(\mathbf{M})$ has a spread $(F_1, \dots, F_r)$ with r members. Suppose for some $K \subseteq \{1, \dots, r\}$, $\rho(\bigcap_{i \in K} F_i) > r - |K|$. Let $G = \bigcap_{i \in K} F_i$. Then $\mathcal{F}(\mathbf{M})_{[G, E]}$ has a spread with at most $r - \rho(G)$ members. But a spread of $\mathcal{F}(\mathbf{M})_{[G, E]}$ consists of all members of the spread of $\mathcal{F}(\mathbf{M})$ which contain G ; thus F_i ($i \in K$) are members of the spread of $\mathcal{F}(\mathbf{M})_{[G, E]}$. We conclude that $|K| \leq r - \rho(G)$ or $\rho(G) \leq r - |K|$, and this is a contradiction. Hence $\rho(\bigcap_{i \in K} F_i) \leq r - |K|$ ($K \subseteq \{1, \dots, r\}$) and $\mathbf{M}$ is a transversal matroid.

We mention one application to an interesting class of matroids. Let E be a set and $\{X_i : 1 \leq i \leq k\}$ a collection of subsets of E such that (i) $|X_i| \geq r - 1$ ($1 \leq i \leq k$) and (ii) every $r - 1$ element subset of E is a subset of exactly one of $X_1, \dots, X_k$. Then [4] the set E , the X_i ($1 \leq i \leq k$), and all subsets A of E with $|A| \leq r - 2$ are the flats of a geometry (therefore matroid $\mathbf{M}$) on E of rank r . Such a geometry is called a Hartmanis geometry [4]. In this case $\mathcal{F}(\mathbf{M})$ consists of those X_i with $|X_i| \geq r$ (these are flats of rank $r - 1$), $\emptyset$, and possibly E . Thus $\mathcal{F}(\mathbf{M})$ has a

spread if and only if $|J| \leq r$ where $J = \{i: 1 \leq i \leq k, |X_i| \geq r\}$. The spread is then $(X_i: i \in J)$ along with the $\emptyset$ with the correct multiplicity to give r sets in total.

Theorem 3.2. *The Hartmanis geometry M is a transversal geometry if and only if $|\bigcap_{i \in I} X_i| \leq r - |I|$ ($I \subseteq J$, $|I| \geq 2$).*

If $I \subseteq J$, $|I| \geq 2$, then $\rho(\bigcap_{i \in I} X_i) = |\bigcap_{i \in I} X_i|$.

4. Construction of transversal matroids. Let M be a transversal matroid of rank r on a finite set E , and let $\mathcal{F}(M)$ be the join-semilattice of flats with no coloops, weighted by rank. Then we know there is a join-subsemilattice J of the Boolean lattice on an r element set such that $\mathcal{F}(M)$ and J are isomorphic as weighted join-semilattices. Since $\mathcal{F}(M)$ has a minimal element $\bar{\phi}$, $\mathcal{F}(M)$ is a lattice. (Note, however, $\mathcal{F}(M)$ is not in general a sublattice of $\mathcal{L}(M)$; it is, however, a join-subsemilattice of $\mathcal{L}(M)$.) We consider the following question. Suppose J is a join-subsemilattice with minimal element of the Boolean lattice on an r element set, weighted by cardinality. Is there a transversal matroid M of rank r such that $\mathcal{F}(M)$ and J are isomorphic as weighted join-semilattices?

Theorem 4.1. *Let J be a join-subsemilattice of a Boolean lattice on an r element set, weighted by cardinality, such that $0, 1 \in J$ with $\omega(0) = 0$, $\omega(1) = r$. Then there is a transversal matroid M of rank r on a finite set E such that $\mathcal{F}(M)$ and J are isomorphic as weighted partially ordered sets; that is, there is a bijection $\sigma: J \rightarrow \mathcal{F}(M)$ such that*

$$(4.1.1) \quad a < b \text{ if and only if } \sigma(a) < \sigma(b) \quad (a, b \in J),$$

$$(4.1.2) \quad |a| = |\omega(\sigma(a))| \quad (a \in J).$$

We shall devote the remainder of this section to proving this theorem. The proof will be divided into several parts, but first we need a construction.

Let E' be some sufficiently large set. Corresponding to each $a \in J$ we define a subset F_a of E' as follows:

$$(0) \quad F_0 = \emptyset.$$

(1) If $a \in J$ with $\omega(a) = 1$, choose distinct elements x, y of E' and set $F_a := \{x, y\}$. We do this for each $a \in J$ of weight 1 in such a way that all elements chosen are distinct: $F_a \cap F_b = \emptyset$ if $a, b \in J$, $\omega(a) = \omega(b) = 1$, $a \neq b$.

$\vdots$

(k) If $a \in J$ with $\omega(a) = k$, let $J_a = \{x \in J: x < a\}$. For each $x \in J_a$, $\omega(x) < k$. If $a = \bigvee \{x: x \in J_a\}$, set $F_a = \bigcup \{F_x: x \in J_a\}$. If $\bigvee \{x \in J_a\} = b < a$ and $\omega(b) = l < k$, then choose a subset X_a of E' with $|X_a| = k - l + 1$ where the elements of X_a are different from any chosen previously. Then set $F_a = F_b \cup X_a$. We do

this for each element of J of weight k in such a way that $X_{a_1} \cap X_{a_2} = \emptyset$ whenever $\omega(a_1) = \omega(a_2) = k$ and $a_1 \neq a_2$.

$\vdots$

The construction ends after we have gone through all elements of J . The family of sets $\mathcal{F} = (F_a : a \in J)$ obtained is partially ordered by set-theoretic inclusion. Let $E = \bigcup_{a \in J} F_a$.

$$(4.1.3) \quad a \leq b \text{ if and only if } F_a \subseteq F_b \quad (a, b \in J).$$

Thus $a \neq b$ implies $F_a \neq F_b$, and the partially ordered sets $\mathcal{F}$ and J are isomorphic.

By construction it is clear that if $a \leq b$ then $F_a \subseteq F_b$. We need to prove conversely that $F_a \subseteq F_b$ implies $a \leq b$, and we do this by induction on weight. It is certainly true by construction if a and b have weight at most 1. Let $k > 1$ and assume that $F_a \subseteq F_b$ implies $a \leq b$ if $\omega(a) < k$, $\omega(b) < k$. Now consider $a, b \in J$ with $\omega(a) \leq k$, $\omega(b) \leq k$. We may assume by the induction that one of the latter is an equality.

We first make the following observation. For $x \in E$ let $\beta(x)$ be the element of J such that $x \in X_{\beta(x)}$. Thus in the construction x makes its first appearance in the set $F_{\beta(x)}$. It then follows for $x \in E$ and $c \in J$ that $\beta(x) \leq c$ if and only if $x \in F_c$.

Now if $a \neq \bigvee_{x < a} x$, then $X_a \neq \emptyset$. Let $z \in X_a$. Since $F_a \subseteq F_b$, $z \in F_b$; hence $a = \beta(z) \leq b$. Thus we may assume $a = \bigvee_{x < a} x$. Let $p = \bigvee_{z \in F_a} \beta(z)$. Thus $F_p = F_a$. Since $z \in F_a$ also implies $z \in F_b$, $\beta(z) \leq a, b$ for $z \in F_a$ and hence $p \leq a \wedge b$. If $p = a$, then $a \leq b$. If $p < a$, then consider $x \in J$ with $x < a$. We have $F_x \subseteq F_p$. Since $\omega(x), \omega(p) < \omega(a) = k$, we have by induction that $x \leq p$. Hence $a = \bigvee_{x < a} x \leq p$. Since $p \leq a$, this implies $a = p$, a contradiction. Thus $a = p$ and $a \leq b$.

(4.1.4) The meet operation in the lattice $\mathcal{F}$ is set-theoretic intersection.

Let $a, b \in J$ and $c = a \wedge b$, so that $F_c = F_a \wedge F_b$. Then $F_c \subseteq F_a \cap F_b$. Suppose there were an $x \in (F_a \cap F_b) \setminus F_c$; thus $\beta(x) \leq a, b$ so that $\beta(x) \leq a \wedge b = c$. This means $x \in F_c$, which is a contradiction.

We let the isomorphism $\sigma: J \rightarrow \mathcal{F}$ where $\sigma(a) = F_a$ carry over the weight function of J to $\mathcal{F}$. That is, we define $\omega(F_a) = \omega(a)$ ($a \in J$). Since J is a join-subsemilattice of the Boolean lattice of an r element set with 0 and 1, weighted by cardinality, J has a spread $(a_i : 1 \leq i \leq r)$. Thus $(F_{a_i} : 1 \leq i \leq r)$ is the spread of $\mathcal{F}$. Consider the transversal matroid $M = M(E \setminus F_{a_1}, \dots, E \setminus F_{a_r})$. We have several things to verify concerning $\mathcal{F}$ and the matroid M .

(4.1.5) If $a > b$, then $|F_a \setminus F_b| \geq \omega(a) - \omega(b) + 1 \geq 2$.

To prove this we apply induction to $\omega(a)$. If $\omega(a) = 1$ then $\omega(b) = 0$ and by construction $F_b = \emptyset$, $|F_a| \geq 2$. Thus assume $\omega(a) = k$ and that the result holds when the weight is less than k . If there is $c \in J$ such that $a > c > b$, then by induction $|F_c \setminus F_b| \geq \omega(c) - \omega(b) + 1$. Thus if $|F_a \setminus F_c| \geq \omega(a) - \omega(c) + 1$ then $|F_a \setminus F_b| \geq \omega(a) - \omega(b) + 2$. Thus we might as well assume that there is no such c . If $x < a$ implies $x \leq b$, then $\bigvee \{x: x < a\} \leq b$. Thus by construction $|F_a \setminus F_b| = \omega(a) - \omega(b) + 1$. Otherwise there is an $x < a$ such that $x \not\leq b$. Then $\omega(x \wedge b) < \omega(x) < k$ so that by induction $A = F_x \setminus F_{x \wedge b} = F_x \setminus (F_x \cap F_b)$ has cardinality at least $\omega(x) - \omega(x \wedge b) + 1$. But since J is a join-subsemilattice of a Boolean lattice and weighted by cardinality, $\omega(x \vee b) + \omega(x \wedge b) \leq \omega(x) + \omega(b)$. Since $b < x \vee b \leq a$, we have $a = x \vee b$. Thus $\omega(a) - \omega(b) \leq \omega(x) - \omega(x \wedge b)$. Since $A \subseteq F_a \setminus F_b$ and $|A| \geq \omega(a) - \omega(b) + 1$, we are done.

(4.1.6) The family $(E \setminus F_{a_1}, \dots, E \setminus F_{a_r})$ has a transversal.

We need to show that the condition for the existence of a transversal is satisfied here. We calculate that for $\emptyset \neq K \subseteq \{1, \dots, r\}$

$$\left| \bigcup_{i \in K} E \setminus F_{a_i} \right| = \left| E \setminus \bigcap_{i \in K} F_{a_i} \right| = |E| - |F_z|$$

where $z = \bigwedge \{a_i: i \in K\}$. But since $(a_i: 1 \leq i \leq r)$ is a spread of J , $\omega(\bigwedge_{i \in K} a_i) \leq r - |K|$; otherwise we contradict the definition of a spread. Thus $\omega(z) \leq r - |K|$. If we apply (4.1.5) with $a = 1$ ($F_1 = E$) and $b = z$, we have

$$|E \setminus F_z| \geq r - \omega(z) + 1 \geq r - (r - |K|) + 1 = |K| + 1.$$

Thus we have a transversal.

(4.1.7) For $a \in J$, F_a is a flat of M .

Let $\omega(a) = r - k$. Then exactly k members of the spread $(a_i: 1 \leq i \leq r)$, say $a_1, \dots, a_k$, satisfy $a_i \geq a$ ($i = 1, \dots, k$) and $a = \bigwedge (a_i: 1 \leq i \leq k)$. Since $\mathcal{F}$ is lattice isomorphic to J via $\sigma(a) = F_a$, $F_a = \bigcap (F_{a_i}: 1 \leq i \leq k)$. Thus

$$F_a \cap (E \setminus F_{a_i}) = \emptyset \quad (1 \leq i \leq k).$$

Now let $x \in E \setminus F_a$. Thus $x \in \bigcup_{i=1}^k (E \setminus F_{a_i})$. If B is a maximum partial transversal contained in F_a (thus the rank of F_a in M is $|B|$), then $B \cup x$ is also a partial transversal. Thus F is closed and therefore a flat of M .

(4.1.8) If $a \in J$ has weight $r - k$, then the flat F_a of $\mathbf{M}$ has rank equal to $r - k$. Thus the rank function coincides with the weight function on $\mathcal{F}$.

Since $\omega(a) = r - k$, there are exactly k members of $(a_i: 1 \leq i \leq r)$, say $a_1, \dots, a_k$, which are greater than or equal to a . Thus $F_a \subseteq F_{a_i}$ ($1 \leq i \leq k$). Consider the family $((E \setminus F_{a_i}) \cap F_a: k+1 \leq i \leq r)$ of subsets of F_a . We show that this family has a transversal which will prove $\rho(F) = r - k$. Let $K \subseteq \{k+1, \dots, r\}$. Then

$$\left| \bigcup_{i \in K} (E \setminus F_{a_i}) \cap F_a \right| = \left| (E \setminus \bigcap_{i \in K} F_{a_i}) \cap F_a \right| = \left| F_a \setminus \bigcap_{i \in K} (F_{a_i} \cap F_a) \right|.$$

Let $\bigcap_{i \in K} F_{a_i} \cap F_a = F_b$. Since $F_a = \bigcap_{i=1}^k F_{a_i}$, at least $k + |K|$ members of the spread $(F_{a_i}: 1 \leq i \leq r)$ of $\mathcal{F}$ contain F_b . Thus $\omega(F_b) \leq r - (k + |K|)$, and so by (4.1.5)

$$|F_a \setminus F_b| \geq \omega(a) - \omega(b) + 1 \geq r - k - (r - (k + |K|)) + 1 = |K| + 1.$$

Thus the defined family has a transversal, which proves the statements made.

(4.1.9) For $a \in J$, the flat F_a of $\mathbf{M}$ has no coloops.

We prove this by induction on $\rho(F_a) = \omega(F_a)$. If $\rho(F_a) = 0$ or 1 , this is true by construction. Let $\rho(F_a) = k > 1$, and assume the result is true for rank smaller than k . Suppose first $a = \bigvee_{x < a} x$. Then $F_a = \bigvee_{x < a} F_x$ and by induction each F_x is a flat of $\mathbf{M}$ with no coloops. Since the join of flats with no coloops of a matroid has no coloops, this proves F_a has no coloops. Suppose now $\bigvee_{x < a} x = b < a$ and thus $\rho(F_b) < k$. Since by induction F_b has no coloops, no element of F_b can be a coloop of F_a . But $F_a = F_b \cup X_a$ where $|X_a| = \rho(F_a) - \rho(F_b) + 1$. Thus if B is a maximal independent set of $\mathbf{M}$ contained in F_b , then $B \cup (X_a \setminus x)$ is a maximal independent set contained in F_a . Thus no element of X_a is a coloop of F_a , and F_a has no coloops.

Finally we show

(4.1.10) If F is a flat of $\mathbf{M}$ with no coloops, then $F = F_a$ for some $a \in J$.

Consider a flat F of $\mathbf{M}$ with no coloops and let $\rho(F) = r - k$. Since $\mathbf{M} = \mathbf{M}(E \setminus F_{a_1}, \dots, E \setminus F_{a_r})$ where F_{a_i} is a flat with no coloops of $\mathbf{M}$ ($1 \leq i \leq r$), then $(E \setminus F_{a_1}, \dots, E \setminus F_{a_r})$ is the maximal presentation of $\mathbf{M}$. Thus $(F_{a_1}, \dots, F_{a_r})$ is the spread of $\mathcal{F}(\mathbf{M})$ (recall it was defined to be the spread of $\mathcal{F}$). Thus since $\rho(F) = r - k$ there are exactly k members of $(F_{a_1}, \dots, F_{a_r})$, say $F_{a_1}, \dots, F_{a_k}$

which contain F . Thus $F = \bigcap_{i=1}^k F_{a_i}$. But by (4.1.4) $\bigcap_{i=1}^k F_{a_i} = F_b$ for some $b \in J$. Thus $F = F_b$.

This now completes the proof of Theorem 4.1.

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