HEREDITARY QI-RINGS⁽¹⁾

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ABSTRACT. We consider in this paper rings in which every quasi-injective right R -module is injective. These rings will be referred to as right QI-rings. For a hereditary ring, this is equivalent to the condition that R be noetherian and a right V -ring. We also consider rings in which proper cyclic right R -modules are injective. These are right QI-rings which are either semisimple or right hereditary, right Ore domains in which indecomposable injective right R -modules are either simple or isomorphic to the injective hull of R_R .

Introduction. It has been shown that the following three conditions on a ring R are equivalent: (1) each simple right R -module is injective; (2) each right ideal is the intersection of maximal right ideals; (3) $\text{Rad } M = 0$ for all $M \in \text{Mod-}R$. Rings satisfying any one of the above conditions are called *right V-rings* after Villamayor, who proved the equivalence of (1) and (2). In finding a counterexample to the conjecture that every V -ring is regular, Cozzens [3] produced an example of a noetherian V -ring in which every cyclic right R -module is either semisimple or free. This condition on the cyclics forces every quasi-injective right R -module to be injective. We shall call a ring in which every quasi-injective right R -module is injective a *right QI-ring*. In this paper we will consider the problem of characterizing hereditary QI-rings.

The class of finitely generated torsion right R -modules plays an important role in this consideration. We find that over a hereditary, noetherian, right V -ring, the class of finitely generated torsion right R -modules, the class of finitely generated semisimple right R -modules, and the class of finitely generated injective right R -modules are all synonymous. We use this equivalence to show that over a hereditary, noetherian ring R , R is a right V -ring if and only if R is a right QI-ring.

Following the example of Cozzens more closely, we also consider rings in which proper cyclic right R -modules are injective. We call these *right PCI-rings*. Every right PCI-ring is a right QI-ring. Further we show that a right PCI-ring is either semisimple or a right hereditary, right Ore domain in which every indecomposable injective right R -module is either simple or isomorphic to the injective hull of R_R .

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1. Preliminaries. Throughout this paper a ring R is associated with an identity element. Each R -module is unitary. Unless right or left is specified, a condition will be assumed to hold on both sides of the ring. If $M \in \text{mod-}R$, then $E(M)$ denotes the injective hull of M_R .

An R -module M_R is *quasi-injective* if each homomorphism of any submodule N into M can be extended to a homomorphism of M into M . Semisimple right modules are clearly examples of quasi-injective modules. Johnson and Wong [8] have related injectives and quasi-injectives by proving that a module is quasi-injective if and only if it is closed under endomorphisms of its injective hull.

A submodule J_R is uniform if for any two submodules A_R and B_R , $A \cap B \neq 0$.

An element r of the ring R is called *regular* if $sr \neq 0$ and $rs \neq 0$ for any nonzero element s of R . For any right R -module M , an element $m \in M$ is a *torsion element* if $md = 0$ for some regular element $d \in R$. If every nonzero element of M is a torsion element, then M is said to be a *torsion module*; if no element of M is torsion, M is said to be *torsion-free*. Given a right R -module M , let tM denote the collection of torsion elements of M . In the case where R is a commutative ring, tM always forms a submodule of M . In the noncommutative case however, we can conclude that tM forms a submodule of M for any R -module M if and only if R has a classical right quotient ring (see Levy [10] and Gentile [6]).

A ring R is called a *right QI-ring* if every quasi-injective right R -module is injective.

A right R -module C is a *proper cyclic* in case $R \rightarrow C \rightarrow 0$ is exact but $0 \rightarrow R \rightarrow C \rightarrow 0$ is not. A ring R is called a *right PCI-ring* if every proper cyclic right R -module is injective.

Before proceeding to the main body of the work, we shall consider several lemmas which will prove useful later.

Lemma 1 (Kurshan [9]). *If R is a ring in which every semisimple right R -module is injective, then R is right noetherian.*

Lemma 2. *Let M be a quasi-injective right R -module. If M contains a copy of R , then M is injective.*

Proof. This is an obvious consequence of Baer's criterion for injectivity [1].

Lemma 3. *Let R be a noetherian, hereditary, semiprime ring and let $\mathfrak{F}_R$ denote the Serre class of finitely generated torsion right R -modules. Then $\text{Ext}_R^1(-, R): \mathfrak{F}_R \rightarrow {}_R\mathfrak{F}$ defines a duality.*

Proof. Let Q be the classical ring of right quotients of R . Then $\text{Hom}_R(-, Q/R)$ and $\text{Ext}_R^1(-, R)$ are naturally isomorphic on $\mathfrak{F}_R$. Using this it follows readily that the functor takes $\mathfrak{F}_R$ into ${}_R\mathfrak{F}$.

To complete the proof all we need show is that $\text{Hom}_R(-, Q/R)$ is inverse to $\text{Ext}_R^1(-, R)$ in $\mathcal{F}_R$. Let $A \in \mathcal{F}_R$. Then by Cartan and Eilenberg [2],

$$\text{Hom}_R(\text{Ext}_R^1(A, R), Q/R) \approx \text{Tor}_1^R(A, \text{Hom}_R(R, Q/R)) \approx \text{Tor}_1^R(A, Q/R)$$

and by Sandomierski [12], $\text{Tor}_1^R(A, Q/R) \approx A$, giving us the desired result.

2. Hereditary right QI-rings. In this section we will be concerned with characterizing hereditary right QI-rings.

Theorem 4. *Let R be a noetherian, hereditary right V -ring. Then finitely generated torsion right R -modules are semisimple and injective.*

Proof. Since $\text{Rad } R = 0$, R is semiprime and by Lemma 3 there exists a duality $\text{Ext}_R^1(-, R): \mathcal{F}_R \rightarrow {}_R\mathcal{F}$. Since $\mathcal{F}_R$ is a noetherian category, ${}_R\mathcal{F}$ is an artinian category, and hence a noetherian category, by this duality. Using the duality once more, since ${}_R\mathcal{F}$ is a noetherian category, $\mathcal{F}_R$ is an artinian category. Thus every finitely generated torsion right R -module contains a simple module which splits off since R is a right V -ring. Hence every finitely generated torsion module is semisimple.

Theorem 5. *Let R be a hereditary, left noetherian ring. Then R is a right QI-ring if and only if R is a right noetherian, right V -ring.*

Proof. Suppose R is a right noetherian, right V -ring. By Faith [4], R is a product of simple noetherian, right V -rings. Hence it suffices to consider a simple, noetherian hereditary right V -ring. Since R is noetherian, it contains a uniform finitely generated, right ideal U . Since R is hereditary and simple, U is a projective generator. By Mitchell [11] there exists a category equivalence $\text{Mod-}R \rightarrow \text{Mod-End } U_R$ where $\text{End } U_R$ is an Ore domain by Goldie [7]. Since quasi-injectives and injectives are Morita invariants, it suffices to work in a hereditary, noetherian, right V -ring, right Ore domain.

Let M be a quasi-injective right R -module. Since R is noetherian, $E(M)$ is a direct sum of indecomposable, injective right submodules. By intersecting M with each of these indecomposable injectives, we get a direct sum of indecomposable quasi-injectives, the direct sum of which is isomorphic to M . Hence it suffices to show that indecomposable quasi-injective right R -modules are injective.

Let M be an indecomposable quasi-injective right R -module. Let $m \in M$. If $mR \approx R$ then M contains a copy of the ring and, by Lemma 2, M is injective. If $\text{ann}_R(m) \neq 0$, then $mR \approx R/I$ for some right ideal I . Since R is a domain, I contains a regular element and hence R/I is a finitely generated torsion module. By Theorem 4, mR is injective and thus $mR = E(M)$ forcing M to be injective.

Conversely, suppose R is a right QI-ring. Since simple right R -modules are quasi-injective, R is a right V -ring. Since semisimples are also quasi-injective, R is right noetherian by Lemma 1.

We can take our knowledge of the structure of these hereditary, noetherian, right V -rings a step further by characterizing the class of finitely generated injectives.

Corollary 6. *Let R be a noetherian, hereditary, right V -ring. Then a finitely generated right R -module is injective if and only if it is semisimple.*

Proof. Since R is right noetherian and a right V -ring, every semisimple right R -module is injective.

Conversely, assume that M is a finitely generated injective right R -module. By Theorem 4, tM , the torsion submodule of M , is semisimple and injective. Thus $M \approx tM \oplus K$ where K is torsion-free. Suppose $K \neq 0$. As in the proof of Theorem 5, we have a category equivalence $T: \text{Mod-}R \rightarrow \text{Mod-}S$ where S is a hereditary, noetherian, right V -ring, right Ore domain. By Theorem 4, all finitely generated torsion modules are semisimple both in $\text{Mod-}R$ and $\text{Mod-}S$. If $T(K)$ is not a torsion module, then $T(K)$ must contain a copy of S since this is the only cyclic in $\text{Mod-}S$ which is not a torsion module. Since $T(K)$ is injective, it contains $E(S_S)$ which implies that $E(S_S)$ is finitely generated. Then by Faith and Walker [5], S is right artinian. However, every right V -ring is semiprime. So S must be semisimple by the Wedderburn-Artin theorem. R must also be semisimple in which case every right R -module is semisimple. Hence we may suppose that $K = 0$. Then $M = tM$ is semisimple.

3. Right PCI-rings. As was mentioned in the introduction, one of the motivating factors for this topic was the example of Cozzens which provided a nontrivial right QI-ring. Cozzens' example has the property that proper cyclic right R -modules are injective. We shall call rings with this property right PCI-rings.

Theorem 7. *If R is a right noetherian, right PCI-ring, then R is a right hereditary, right QI-ring.*

Proof. Let M be any right R -module with injective hull $E(M)$. Consider $E(M)/M$. If $\bar{m} \in E(M)/M$, then $\text{ann}_R(\bar{m}) \neq 0$ and is in fact a large right ideal. Thus $\bar{m}R$ is a proper cyclic and is injective by hypothesis. So $\bar{m}R$ splits off as a summand of $E(M)/M$. Take a maximal collection $\{\bar{m}_i R\}_{i \in I}$ of linearly independent cyclic summands of $E(M)/M$. Suppose $E(M) \neq \sum_{i \in I} \bar{m}_i R$. Since R is right noetherian, every direct sum of injectives is injective. Thus $\sum_{i \in I} \bar{m}_i R$ is injective and $E(M)/M \approx \sum_{i \in I} \bar{m}_i R \oplus A$ for some submodule A of $E(M)/M$. Every cyclic submodule C of A is proper and thus injective. So C splits off contradicting the maximality of $\{\bar{m}_i R\}_{i \in I}$. Hence $E(M)/M = \sum_{i \in I} \bar{m}_i R \approx \sum_{i \in I} \bar{m}_i R$ is a direct sum of proper cyclics and is therefore injective. Thus $0 \rightarrow M \rightarrow E(M) \rightarrow E(M)/M \rightarrow 0$ is an injective resolution for M . Thus $\text{inj dim } R_R \leq 1$ implying that R is right hereditary.

Since R is right noetherian, for any quasi-injective right R -module M we can get a decomposition of M into a direct sum of indecomposable quasi-injective right R -modules. Thus it suffices to show that indecomposable quasi-injectives are injective. Let M be an indecomposable quasi-injective right R -module and let $m \in M$. If $mR \approx R$, then M is injective by Lemma 2. Otherwise mR is a proper

cyclic and is injective. Since M is indecomposable, $mR = M$ and M is injective. Thus R is a right QI-ring.

It can be observed from the above proof that every indecomposable injective right R -module is either cyclic or isomorphic to $E(R)$. Using the following lemma due to Faith, we can say even more about the structure of these rings.

Lemma 8. *Let R be a noetherian right PCI-ring. Then the endomorphism ring of an indecomposable injective right R -module is a field.*

Proof. Let M be an indecomposable injective right R -module. If M has a cyclic submodule C such that $C \approx R$, then $M \approx E(R)$. Thus $E(R)$ is indecomposable. R has a semisimple classical quotient ring Q and since the singular right ideal is zero, $Q \approx E(R)$. Thus $E(R)$ is a right and left semisimple artinian ring. Since it is indecomposable over R , $E(R)$ is a field. End $M \approx \text{End } E(R) \approx E(R)$. Thus End M is a field.

Otherwise any nonzero submodule N of M contains a proper cyclic C . Since C is injective, $C = N = M$. Thus M has no proper submodules and is therefore simple. So End M is a field.

Corollary 9. *If R is a right noetherian, right PCI-ring, then either R is a semisimple ring or else R is a right hereditary, right Ore domain in which every indecomposable injective right R -module is either simple or isomorphic to the right quotient field $E(R)$.*

Proof. If R has no indecomposable injective right R -module isomorphic to $E(R)$, then as in the proof of Lemma 8, every indecomposable injective is simple, and hence given a right R -module M it is contained in a semisimple right R -module $E(M)$ and thus is semisimple.

If R is not semisimple, then there exists an indecomposable injective right R -module M such that $M \approx E(R)$. This implies that $E(R)$ is indecomposable. Since $E(R)$ is indecomposable as a ring, it is a field. Thus R is a right Ore domain and every indecomposable injective right R -module is either simple or isomorphic to $E(R)$.

Corollary 10. *Suppose R is not semisimple. Then the following conditions on a left noetherian, left hereditary ring R are equivalent:*

- (1) R is a right noetherian, right PCI-ring.
- (2) R is a right hereditary, right QI-ring, right Ore domain.
- (3) R is a right hereditary, right noetherian V -ring which is a right Ore domain.

Proof. (1) *implies* (2). This follows from Theorem 7 and Corollary 10.

(2) *implies* (3). This follows from Theorem 5.

(3) *implies* (1). We want to show that proper cyclics are injective. By Theorem 4, finitely generated torsion right R -modules are injective. Let C be a proper cyclic, $C \approx R/I$ for some right ideal I . Since R is an Ore domain, $r \in I$ implies that r is regular and $cr = 0$, $c \in C$. Thus C is a finitely generated torsion module and is injective.

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