CONVEX HULLS AND EXTREME POINTS OF SOME FAMILIES OF UNIVALENT FUNCTIONS⁽¹⁾

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ABSTRACT. The closed convex hull and extreme points are obtained for the functions which are convex, starlike, and close-to-convex and in addition are real on $(-1, 1)$. We also obtain this result for the functions which are convex in the direction of the imaginary axis and real on $(-1, 1)$. Integral representations are given for the hulls of these families in terms of probability measures on suitable sets. We also obtain such a representation for the functions $f(z)$ analytic in the unit disk, normalized and satisfying $\operatorname{Re} f'(z) > \alpha$ for $\alpha < 1$. These results are used to solve extremal problems. For example, the upper bounds are determined for the coefficients of a function subordinate to some function satisfying $\operatorname{Re} f'(z) > \alpha$.

Introduction. We shall determine the closed convex hulls and extreme points of some families of univalent functions. We utilize these results to solve specific extremal problems over certain of the families.

Let Δ denote the unit disk $\{z: |z| < 1\}$ and let A denote the set of functions analytic in Δ . Then A is known to be a locally convex linear topological space where the topology is given by uniform convergence on compact subsets of Δ [14, p. 150]. We let S denote the subset of A consisting of the functions f that are univalent in Δ and normalized so as to satisfy $f(0) = 0$ and $f'(0) = 1$. Let K , St and C denote the well-known subfamilies of S which are respectively convex, starlike and close-to-convex. We will consider the subfamilies K_R , St_R and C_R where for any class of functions $\mathfrak{G}$ we let $\mathfrak{G}_R = \{f: f \in \mathfrak{G} \text{ and } f \text{ is real on } (-1, 1)\}$. We also consider the family of functions denoted by F_R which are convex in the direction of the imaginary axis and real on $(-1, 1)$. The functions in F_R were studied by M. S. Robertson in [10]. Further we let $P(\alpha)$ denote the subfamily of S consisting of those functions satisfying $\operatorname{Re} f'(z) > \alpha$ where $0 \leq \alpha < 1$.

The study of the convex hulls and extreme points of various families of univalent functions was initiated by L. Brickman, T. H. MacGregor, and D. R. Wilken in [2]. It was continued by the above authors and the present author in [1]. We shall use some of the basic results contained in [2]. $\mathcal{HF}$ shall denote the

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closed convex hull of a family of functions F . $\mathcal{E}\mathcal{H}F$ shall denote the set of extreme points of $\mathcal{H}F$. We recall that whenever F is compact, $\mathcal{E}\mathcal{H}F \subset F$.

We find that $\mathcal{H}K_R$ consists of the functions represented as

$$f(z) = \int_X \frac{1}{x - \bar{x}} \log \frac{1 - \bar{x}z}{1 - xz} d\mu(x)$$

where μ varies over the probability measures on $X = \{x: |x| = 1 \text{ and } \operatorname{Im} x \geq 0\}$. We show that $\mathcal{H}K_R = \mathcal{H}F_R$. We find further that $\mathcal{H}St_R = \mathcal{H}C_R = \mathcal{H}S_R$ where $\mathcal{H}S_R$ was determined in [2, p. 95]. Finally we prove that $\mathcal{H}P(\alpha)$ consists of the functions represented as

$$f(z) = (2\alpha - 1)z + (2\alpha - 2) \int_X \bar{x} \log(1 - xz) d\mu(x)$$

where μ varies over the probability measures on the unit circle X .

We recall the definition of subordination between two functions f and g which are both analytic in Δ . We say that f is subordinate to g , denoted by $f < g$, if there exists an analytic function $\phi(z)$ so that $\phi(0) = 0$, $|\phi(z)| < 1$ and $f(z) = g(\phi(z))$ for z in Δ . We prove that if $f < F$ where $F \in P(\alpha)$ and if $f(z) = \sum_{n=1}^{\infty} a_n z^n$ then $|a_n| \leq 1$ for $n = 1, 2, 3, \dots$. We also show that if $f \in P(\alpha)$ then

$$f(z)/z < (2\alpha - 1) + (2\alpha - 2)\log(1 - z)/z.$$

As a consequence we show that if $f \in P(\alpha)$ then

$$\operatorname{Re}(f(z)/z) \geq (2 - 2\alpha)(1/|z|)\log(1 + |z|) + (2\alpha - 1)$$

for all z in Δ . We also determine the radius of convexity of $P(1/2)$ to be $1/\sqrt{2}$. This result has also recently been announced by D. Shaffer [13].

1. The convex hulls and extreme points of K_R , St_R , C_R and F_R .

Theorem 1. Let X be $\{z: |z| = 1, \operatorname{Im} z \geq 0\}$, let $\mathcal{P}$ be the set of probability measures on X , and let $\mathfrak{D}$ be the set of functions f_μ on Δ defined by

$$f_\mu(z) = \int_X \frac{z}{(1 - xz)(1 - \bar{x}z)} d\mu(x), \quad \mu \in \mathcal{P}.$$

Then $\mathcal{H}St_R = \mathcal{H}C_R = \mathfrak{D}$, the map $\mu \rightarrow f_\mu$ is one-to-one, and the extreme points of $\mathcal{H}St_R$ and $\mathcal{H}C_R$ are precisely the functions $z/(1 - xz)(1 - \bar{x}z)$, $x \in X$.

Proof. Since $z/(1 - xz)(1 - \bar{x}z)$ is in St_R and $St_R \subset C_R$ we have as a consequence of [2, Theorem 1, p. 93] that $\mathfrak{D} \subset \mathcal{H}St_R \subset \mathcal{H}C_R$. If $f \in St_R$ then $f \in S_R$. In [2] it was proven that $S_R \subset \mathfrak{D}$. Hence, $St_R \subset C_R \subset \mathfrak{D}$ which implies that $\mathcal{H}St_R \subset \mathcal{H}C_R \subset \mathfrak{D}$ since $\mathfrak{D}$ is convex and compact.

Hence $\mathcal{H}St_R = \mathcal{H}C_R = \mathfrak{D}$. The map $\mu \rightarrow f_\mu$ is known [2] to be one-to-one and consequently the remark concerning the extreme points follows from [2, Theorem 1].

Remarks. It is easy to show that $St_R \subseteq C_R \subseteq S_R$. In [2, Theorem 4] it was shown that $\mathcal{H}S_R = \mathfrak{V}$ where $\mathfrak{V}$ is defined as above. Hence we may conclude that $\mathcal{H}St_R = \mathcal{H}C_R = \mathcal{H}S_R$. Of course it follows that each of these hulls have the same extreme points. We remark that $\mathfrak{V} = T$, the set of typically real functions [2, p. 95].

Theorem 2. Let X be $\{z: |z| = 1, \operatorname{Im} z \geq 0\}$, let $\mathcal{P}$ be the set of probability measures on X and let $\mathfrak{V}$ be the set of all functions f_μ defined on Δ by

$$f_\mu(z) = \int_X \frac{1}{x - \bar{x}} \log \frac{1 - \bar{x}z}{1 - xz} d\mu(x), \quad \mu \in \mathcal{P}.$$

Then $\mathcal{H}K_R = \mathfrak{V}$, the map $\mu \rightarrow f_\mu$ is one-to-one, and the extreme points of $\mathcal{H}K_R$ are precisely the functions $(x - \bar{x})^{-1} \log((1 - \bar{x}z)/(1 - xz))$ for $|x| = 1$.

Proof. The operator L defined by $(Lf)(z) = \int_0^z (f(\omega)/\omega) d\omega$ is known to be a linear homeomorphism of the space of analytic functions on Δ that vanish at 0 and further that $L(St) = K$. Since the operator L also preserves real coefficients, it follows that $L(St_R) = K_R$. We note that since

$$L\left(\frac{z}{(1 - xz)(1 - \bar{x}z)}\right) = \frac{1}{x - \bar{x}} \log \frac{1 - \bar{x}z}{1 - xz},$$

it follows by Theorem 1 that $L(\mathcal{H}St_R) = \mathcal{H}K_R = \mathfrak{V}$. The fact that the map $\mu \rightarrow f_\mu$ is one-to-one follows from the previous theorem since the operator does not alter this condition. Hence the conclusion concerning the extreme points follows from [2, Theorem 1].

Remarks. The kernel function $K(z, x) = (x - \bar{x})^{-1} \log((1 - \bar{x}z)/(1 - xz))$ can be defined continuously at $x = 1, -1$ and indeed $K(z, 1) = z/(1 - z)$ and $K(z, -1) = z/(1 + z)$.

Theorem 3. Let X be $\{z: |z| = 1, \operatorname{Im} z \geq 0\}$, let $\mathcal{P}$ be the set of probability measures on X and let $\mathfrak{V}$ be the set of all functions f_μ defined on Δ by

$$f_\mu(z) = \int_X \frac{1}{x - \bar{x}} \log \frac{1 - \bar{x}z}{1 - xz} d\mu(x), \quad \mu \in \mathcal{P}.$$

Then $\mathcal{H}F_R = \mathfrak{V}$, the map $\mu \rightarrow f_\mu$ is one-to-one, and the extreme points of $\mathcal{H}F_R$ are precisely the functions $(x - \bar{x})^{-1} \log((1 - \bar{x}z)/(1 - xz))$ for $|x| = 1$.

Proof. It is an equivalent form of a result of M. S. Robertson [11] that if $f \in F_R$ then

$$f(z) = \int_X \frac{1}{x - \bar{x}} \log \frac{1 - \bar{x}z}{1 - xz} d\mu(x) \quad \text{where } \mu \in \mathcal{P}.$$

Hence $F_R \subset \mathfrak{V}$ and consequently $\mathcal{H}F_R \subset \mathfrak{V}$ since $\mathfrak{V}$ is closed and convex. However, each function $(x - \bar{x})^{-1} \log((1 - \bar{x}z)/(1 - xz))$ is real on $(-1, 1)$ and convex by the previous theorem, since $\mathcal{E}\mathcal{H}K_R \subset K_R$. These functions can also be

seen to be convex by a simple direct computation. Hence each kernel function $K(z, x) = (x - \bar{x})^{-1} \log((1 - \bar{x}z)/(1 - xz))$ is in F_R . Consequently by [2, Theorem 1], we conclude that $\mathfrak{D} \subset \mathcal{H}F_R$. Therefore $\mathcal{H}F_R = \mathfrak{D}$. By Theorem 2 the mapping $\mu \rightarrow f_\mu$ is known to be one-to-one and so the conclusion about the extreme points follows from Theorem 1 of [2].

Remarks. It is easy to show that $K_R \subseteq F_R$. However, by the results of Theorems 2 and 3, we see that $\mathcal{H}K_R = \mathcal{H}F_R$ and these hulls have the same extreme points.

2. The convex hulls and extreme points of the class $P(\alpha)$. We recall that $P(\alpha)$ consists of those functions satisfying $f(0) = 0$, $f'(0) = 1$, and $\operatorname{Re} f'(z) > \alpha$ for z in Δ where $0 \leq \alpha < 1$. We remark that the third condition is known [8] to imply that $f(z)$ is one-to-one and so given our normalizations we have $P(\alpha)$ a compact subfamily of S . The class $P(0)$ was investigated by T. H. MacGregor in [4]. We remark that every theorem dealing with $P(\alpha)$ contained in this paper is valid for $\alpha < 1$ with the single exception of Theorem 5. We note that if $\alpha < 0$ then the derivative of functions in $P(\alpha)$ can have a zero in Δ , so that $P(\alpha) \not\subset S$.

Theorem 4. Let X be $\{z: |z| = 1\}$, let $\mathcal{P}$ be the set of probability measures on X and let $\mathfrak{D}$ be the set of functions f_μ on Δ defined by

$$f_\mu(z) = (2\alpha - 1)z + (2\alpha - 2) \int_X \bar{x} \log(1 - xz) d\mu(x), \quad \mu \in \mathcal{P}.$$

Then $\mathcal{H}P(\alpha) = P(\alpha) = \mathfrak{D}$, the map $\mu \rightarrow f_\mu$ is one-to-one, and the extreme points of $\mathcal{H}P(\alpha)$ are precisely the functions $(2\alpha - 1)z + (2\alpha - 2)\bar{x} \log(1 - xz)$, $x \in X$.

Proof. Since $\operatorname{Re} f'(z) > \alpha$ for z in Δ and $f'(0) = 1$, it follows by applying the Herglotz formula to $(f'(z) - \alpha)/(1 - \alpha)$ that

$$f'(z) = \int_X \frac{1 + (1 - 2\alpha)xz}{1 - xz} d\mu(x) \quad \text{for some } \mu \in \mathcal{P}.$$

Hence by using $f(0) = 0$ we conclude through integration that

$$f(z) = (2\alpha - 1)z + (2\alpha - 2) \int_X \bar{x} \log(1 - xz) d\mu(x).$$

Hence $P(\alpha) \subset \mathfrak{D}$ and so $\mathcal{H}P(\alpha) \subset \mathcal{H}\mathfrak{D} = \mathfrak{D}$. The statement $\mathcal{H}P(\alpha) = P(\alpha)$ is clear since $P(\alpha)$ is a closed convex set. It is easy to verify that the kernel function

$$K(z, x) = (2\alpha - 1)z + (2\alpha - 2)\bar{x} \log(1 - xz)$$

is an element of $P(\alpha)$ for each $x \in X$. Hence by [2, Theorem 1] it follows that $\mathfrak{D} \subset \mathcal{H}P(\alpha)$. Therefore $\mathcal{H}P(\alpha) = P(\alpha) = \mathfrak{D}$. The mapping $\mu \rightarrow f_\mu$ is easily seen to be one-to-one since this property is known to hold for the Herglotz representation [9, p. 30].

The assertion about the extreme points follows in the usual way from [2, Theorem 1].

Remarks. We recall the definition of the class of functions which are convex of order α which we denote by $K(\alpha)$. A function f is in $K(\alpha)$ if and only if $\operatorname{Re}(1 + zf''(z)/f'(z)) > \alpha$ for z in Δ where $0 \leq \alpha < 1$ and f has the same normalizations as S . These compact subfamilies of S were introduced by M. S. Robertson in [10]. In [1] the set $\mathcal{HK}(\alpha)$ was determined exactly for all $\alpha < 1$. When $\alpha = 1/2$ it was shown that

$$\mathcal{HK}(1/2) = \left\{ \int_X -\bar{x} \log(1 - xz) d\mu(x) : \mu \in \mathcal{P} \right\}$$

where X is the unit circle. We see from Theorem 4 above that $\mathcal{HK}(1/2) = \mathcal{HP}(1/2) = P(1/2)$. This is of some interest since $K(1/2) \subseteq P(1/2)$ is a direct implication of our Theorem 7 and the foregoing remark given that $K(1/2) \subset \mathcal{HK}(1/2)$.

Corollary 4.1. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in P(\alpha)$, then $|a_n| \leq (2 - 2\alpha)/n$ for $n = 2, 3, \dots$.*

Proof. By Theorem 4 the extreme points of $P(\alpha)$ are given by $(2\alpha - 1)z + (2\alpha - 2)\bar{x} \log(1 - xz)$, $x \in X$. It is easy to verify that the n th coefficient in the power series expansion for the kernel function is bounded in modulus by $(2 - 2\alpha)/n$. Therefore, the same bound will hold for each function in $P(\alpha)$.

Remarks. This corollary could easily have been proven without knowledge of the extreme points by using the known bound on the modulus of the n th coefficient in the power series expansion of a function p satisfying $\operatorname{Re} p(z) > \alpha$ for z in Δ . We include it because of its relevance to the next theorem.

The next theorem depends upon a fact proven by T. H. MacGregor in [6]. Suppose G is a compact subset of A and F is the class of all functions subordinate to each function in G . Let F_0 be the subset of F of functions subordinate to some function in $\mathcal{EHG}$. If J is a complex-valued, continuous, linear functional on A , then

$$\max_{f \in F} |J(f)| = \max_{f \in F_0} |J(f)|.$$

We also need a result of W. Rogozinski [12, p. 64] which we state now. Suppose that $f(z) = \sum_{n=1}^{\infty} a_n z^n$ is subordinate to $F(z) = \sum_{n=1}^{\infty} A_n z^n$ in Δ . If, for $1 \leq k \leq n$, the numbers A_k are nonnegative, nonincreasing, and convex, then $|a_n| \leq A_1$, for $k = 1, 2, \dots, n$.

Theorem 5. *Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$ be analytic in Δ and be subordinate to some function in $P(\alpha)$ where $0 \leq \alpha < 1$. Then $|a_n| \leq 1$ for $n = 1, 2, \dots$.*

Proof. Since the family $P(\alpha)$ is compact the arguments referred to above by T. H. MacGregor given in [6] show that in order to maximize $|a_n|$ we need only consider the functions f which are subordinate to a function in $\mathcal{HP}(\alpha)$. Hence, by Theorem 4, $f(z)$ has the form

$$f(z) = (2\alpha - 1)\phi(z) + (2\alpha - 2)\bar{x} \log(1 - x\phi(z)),$$

where $|x| = 1$, ϕ is analytic for $|z| < 1$, $|\phi(z)| < 1$, and $\phi(0) = 0$. The function $\phi(z)/x$ has the same properties as ϕ and a function f and xf have n th coefficients with the same modulus. Hence we may assume $x = 1$, that is, f is subordinate to $F(z) = (2\alpha - 1)z + (2\alpha - 2)\log(1 - z)$. Let $F(z) = \sum_{n=1}^{\infty} A_n z^n$. Then $A_1 = 1$ and $A_n = (2 - 2\alpha)/n$ for $n = 2, 3, \dots$. Since $0 \leq \alpha < 1$ the sequence $\{A_n\}$ is seen to consist of nonnegative real numbers. Since

$$A_{n+1} - A_n = -(2 - 2\alpha)/n(n + 1)$$

the sequence is seen to be nonincreasing. Finally, since we have

$$A_n - 2A_{n+1} + A_{n+2} = (2 - 2\alpha) \frac{2}{n(n + 1)(n + 2)}$$

we conclude that the sequence $\{A_n\}$ is also convex. Hence by a direct application of Rogozinski's result mentioned above we conclude $|a_n| \leq 1$ for $n = 2, 3, \dots$.

Remarks. This result can be seen to be sharp for each n by choosing $\phi(z) = z^n$.

Before we prove our next theorem it is appropriate to introduce a result proven by R. J. Libera in [3] which we need for our proof.

Lemma 1. Let f be analytic and univalent for $|z| < 1$. Suppose $f(0) = 0$ and $f(\Delta)$ is convex. If $g(z) = (1/z) \int_0^z f(\omega) d\omega$ then g is also univalent and convex for $|z| < 1$.

Using Libera's result above, we prove the following simple lemma.

Lemma 2. The function $\log(1 - z)/z$ is univalent and convex for z in Δ .

Proof. We may write

$$1 + \frac{\log(1 - z)}{z} = \frac{1}{z} \int_0^z \left(1 + \frac{-1}{1 - \omega}\right) d\omega.$$

The function $f(z) = 1 + 1/(1 - z)$ is univalent and convex for z in Δ . It also satisfies $f(0) = 0$. Consequently, by Lemma 1 above we conclude that $1 + \log(1 - z)/z$ is univalent and convex for z in Δ and hence $\log(1 - z)/z$ also has these same properties.

Theorem 6. If $f(z) \in P(\alpha)$ then

$$f(z)/z < (2\alpha - 1) + (2\alpha - 2)\log(1 - z)/z \quad \text{for } z \text{ in } \Delta.$$

Proof. We know by Theorem 4 that

$$\frac{f(z)}{z} = (2\alpha - 1) + (2\alpha - 2) \int_X \frac{\log(1 - xz)}{xz} d\mu(x)$$

where μ is a probability measure on X , the unit circle. By the previous lemma we know that $\log(1 - z)/z$ is univalent and convex for $|z| < 1$. The result follows directly.

Corollary 6.1. *If $f(z) \in P(\alpha)$, then*

$$\operatorname{Re}(f(z)/z) \geq (2\alpha - 1) + (2 - 2\alpha)(1/|z|)\log(1 + |z|) \quad \text{for } |z| < 1.$$

Proof. By Theorem 6 we know that

$$f(z)/z \prec g(z) = (2\alpha - 1) + (2\alpha - 2)\log(1 - z)/z$$

for $|z| < 1$. The function $g(z)$ is univalent and convex by Lemma 2. It follows by the convexity, conformality, and the realness of g that the image of the disk $|z| \leq r$ for any $0 \leq r < 1$ has a support line at $(2\alpha - 1) + (2 - 2\alpha)(1/|z|) \cdot \log(1 + |z|)$ which is perpendicular to the real axis at that point. It is clear that the image must lie wholly to the right of this line. Since the image of $|z| \leq r$ under $f(z)/z$ is contained in the image of $|z| \leq r$ under $g(z)$ the result follows.

Remarks. 1. This result in the case $\alpha = 0$ was proven by T. H. MacGregor in [4, p. 533].

2. An easy argument shows that equality holds only for the kernel functions $(2\alpha - 1)z + (2 - 2\alpha)\bar{x} \log(1 - xz)$, $x \in X$.

3. It is a problem in calculus to show that $(1/|z|)\log(1 + |z|) > \log 2$ for z in Δ . Hence if $f \in P(\alpha)$ we see that $\operatorname{Re}(f(z)/z) > (2\alpha - 1) + (2 - 2\alpha)\log 2$.

4. A proof of the corollary independent of knowledge of extreme points can be fashioned by appropriately integrating the known lower bound on the real part of the derivative of a function in $P(\alpha)$.

The following result as mentioned earlier has also been announced by D. Shaffer [13].

Theorem 7. *The radius of convexity of $P(1/2)$ is $1/\sqrt{2}$.*

Proof. If $g(z) = zf'(z)$, then $g(z)$ is starlike if and only if $f(z)$ is convex. Note that $\operatorname{Re}(g(z)/z) = \operatorname{Re} f'(z) > 1/2$ for z in Δ . The radius of starlikeness of the class of functions satisfying the $g(z)/z > 1/2$ was found by T. H. MacGregor in [5, p. 75] to be $1/\sqrt{2}$. Hence the radius for $P(1/2)$ is $1/\sqrt{2}$.

Remarks. The radius of convexity of $P(0)$ was determined by T. H. MacGregor in [4, p. 33] to be $\sqrt{2} - 1$. The problem of determining the radius of convexity for arbitrary α and the problem of determining the radius of starlikeness are, as far as we can determine, open.

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