INEQUALITIES FOR POLYNOMIALS WITH A PRESCRIBED ZERO

BY

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ABSTRACT. Inequalities for the derivative and for the maximum modulus on a larger circle of a polynomial with a given zero on the unit circle are obtained in terms of its degree and maximum modulus on the unit circle; examples are given to show that these are sharp with respect to the degree (best constants are not known). Inequalities for L^p norms, in particular L^2 norms, are also derived. Also certain functions of exponential type are considered and similar inequalities are obtained for them. Finally, the problem of estimating $P_n(r)$ (with $0 < r < 1$) given $P_n(1) = 0$ is taken up.

1. Introduction and statement of results. If $P_n(z)$ is a polynomial of degree n such that $\max_{|z|=1} |P_n(z)| = 1$ then

$$(1.1) \quad \max_{|z|=1} |P'_n(z)| \leq n,$$

$$(1.2) \quad \max_{|z|=R>1} |P_n(z)| \leq R^n.$$

Inequality (1.1) is an immediate consequence of S. Bernstein's theorem on the derivative of a trigonometric polynomial (for references see [18]). Inequality (1.2) is a simple deduction from the maximum principle (see [15, p. 346], or [12, vol. 1, p. 137, Problem III 269]).

In both (1.1), (1.2) equality holds only for $P_n(z) = e^{i\gamma} z^n$, i.e. when all the zeros of $P_n(z)$ lie at the origin. Erdős conjectured and later Lax [11] verified that if $P_n(z) \neq 0$ in $|z| < 1$ then (1.1) can be replaced by

$$(1.3) \quad \max_{|z|=1} |P'_n(z)| \leq n/2$$

and in (1.3) equality holds if all the zeros of $P_n(z)$ lie on $|z| = 1$. Ankeny and Rivlin [2] used (1.3) to prove that if $|P_n(z)| \leq 1$ for $|z| = 1$ and $P_n(z) \neq 0$ in $|z| < 1$ then

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$$(1.4) \quad \max_{|z|=R>1} |P_n(z)| \leq (R^n + 1)/2$$

which is much better than (1.2). Besides, in (1.4) equality is possible and in fact holds for $P_n(z) = \alpha + \beta z^n$ where $|\alpha| = |\beta| = 1/2$.

Several years ago Professor R. P. Boas, Jr. asked one of us, namely Rahman, as to what can be said about $\max_{|z|=1} |P'_n(z)|$, $\max_{|z|=R>1} |P_n(z)|$ if we assume $P_n(z)$ to have precisely k zeros in $|z| \geq 1$ instead of all the zeros as Erdős did. In particular, what happens if $P_n(1) = 0$ or (more generally) if $|P_n(1)| = a$? We are thus led to consider the class $\mathcal{P}_{n,a}$ of all polynomials $P_n(z)$ of degree n such that $\max_{|z|=1} |P_n(z)| = 1$, $\min_{|z|=1} |P_n(z)| \leq a$ where $0 \leq a \leq 1$. It is clear that every polynomial $P_n(z)$ with $\max_{|z|=1} |P_n(z)| = 1$ belongs to $\mathcal{P}_{n,a}$ for some a .

In view of (1.3), one might expect that if $P_n(z) \in \mathcal{P}_{n,0}$ then $\max_{|z|=1} |P'_n(z)| \leq n - c$ where c is a constant, possibly equal to $1/2$. But this is far from the truth as the next two theorems show.

Theorem 1. *If $P_n(z) \in \mathcal{P}_{n,a}$ then for $|z| \leq 1$*

$$(1.5) \quad |P'_n(z)| \leq n - (1-a)\{1 - a - \sin(1-a)\}/4\pi n.$$

Theorem 2. *There exists an absolute constant $c_1 > 0$ such that*

$$(1.6) \quad \max_{P_n(z) \in \mathcal{P}_{n,a}} \left(\max_{|z|=1} |P'_n(z)| \right) \geq n - (1-a)c_1/n.$$

We also prove

Theorem 3. *If $P_n(z) \in \mathcal{P}_{n,a}$ then for $R > 1$*

$$(1.7) \quad \max_{|z|=R} |P_n(z)| \leq R^n \left\{ 1 - \frac{1}{([\pi/(1-a)] + 1)^2 n} (1 - e^{-\pi/2})(1 - R^{-1})^2 \right\}.$$

Theorem 4. *There exists an absolute constant $c_2 > 0$ such that for $R > n^2$*

$$(1.8) \quad \max_{P_n(z) \in \mathcal{P}_{n,a}} \left(\max_{|z|=R} |P_n(z)| \right) > R^n \{1 - c_2(1-a)/n\}.$$

Theorem 1 says in particular that if $P_n(z) \in \mathcal{P}_{n,0}$ then

$$(1.5') \quad \max_{|z|=1} |P'_n(z)| \leq n - (1 - \sin 1)/4\pi n.$$

But in this special case namely for polynomials $P_n(z) \in \mathcal{P}_{n,0}$ we obtain the better estimate:

$$(1.5'') \quad \max_{|z|=1} |P'_n(z)| \leq n - (2 - \sqrt{2})/4n.$$

We are also able to replace

$$(1.7') \quad \max_{P_n(z) \in \mathcal{P}_{n,0}} \left(\max_{|z|=R>1} |P_n(z)| \right) \leq R^n \left\{ 1 - \frac{1}{16n} (1 - e^{-\pi/2})(1 - R^{-1})^2 \right\}$$

obtainable from Theorem 3 by

$$(1.7'') \quad \max_{P_n(z) \in \mathcal{P}_{n,0}} \left(\max_{|z|=R>1} |P_n(z)| \right) \leq R^n \left\{ 1 - \frac{2 - \sqrt{2}}{2n} (1 - R^{-1})^2 \right\}.$$

Coming back to the original question of Boas, namely what happens if $p_n(z)$ has k zeros in $|z| \geq 1$, we can prove the following theorem:

Theorem 5. *Corresponding to every $\epsilon > 0$, there exist a $\delta > 0$ and an integer n_0 such that for all $n > n_0$ there is a polynomial $P_n(z) \in \mathcal{P}_{n,0}$ which has at least $\delta\sqrt{n}$ zeros in $|z| \geq 1$ and is such that $\max_{|z|=1} |P'_n(z)| > n - \epsilon$.*

Theorem 2 gives us an idea as to how large $\max_{|z|=1} |P'_n(z)|$ can be if $P_n(z)$ is a polynomial of degree n such that $\max_{|z|=1} |P_n(z)| = 1$ and $|P_n(1)| = a$. It is natural to ask how small $\max_{|z|=1} |P'_n(z)|$ can be under these conditions. This question turns out to be easy. Indeed, from

$$P_n(z) = P_n(1) + \int_1^z P'_n(\zeta) d\zeta$$

we obtain, for $|z| \leq 1$,

$$|P_n(z)| \leq a + |z - 1| \max_{|z| \leq 1} |P'_n(z)| \leq a + 2 \max_{|z| \leq 1} |P'_n(z)|$$

that is

$$(1.9) \quad \max_{|z| \leq 1} |P'_n(z)| \geq (1 - a)/2.$$

We may consider the polynomial

$$(1.10) \quad P_n(z) = a + \frac{1 - a}{2(k + 1)} (1 - z)\{k + (-z)^{n-1}\}$$

with sufficiently large positive k to see that the bound $(1 - a)/2$ is best possible.

In fact, $P_n(1) = a$, $\max_{|z|=1} |P_n(z)| = 1$ and, for every given $\epsilon > 0$,

$$\max_{|z|=1} |P'_n(z)| = (1 - a) \left(\frac{1}{2} + \frac{n - 1}{k + 1} \right) < \frac{1 - a}{2} + \epsilon$$

if $k > (1 - a)(n - 1)/\epsilon - 1$.

Applying Theorem 3 to the polynomial $z^n P_n(1/z)$ we deduce

$$(1.11) \quad \max_{P_n(z) \in \mathcal{P}_{n,a}} \left(\max_{|z|=\rho < 1} |P_n(z)| \right) \leq 1 - \frac{1}{([\pi/(1-a)] + 1)^2 n} (1 - e^{-\pi/2})(1 - \rho)^2.$$

For polynomials $P_n(z) \in \mathcal{P}_{n,0}$ we may use (1.7'') to get

$$(1.11') \quad \max_{|z|=\rho < 1} |P_n(z)| \leq 1 - \frac{2 - \sqrt{2}}{2n} (1 - \rho)^2.$$

Recently, but in quite a different context Halász asked how large $\min_{|z|=1-(\omega/n)} |P_n(z)|$ can be for a given ω in $(0, n]$ if $P_n(z) \in \mathcal{P}_{n,0}$. It has been shown by Rahman and Stenger [14] that for given

$$\lambda > \frac{1}{\pi} \int_{-\infty}^{\infty} |\log(1 - \sin^2 u/u^2)| du$$

there exists a positive number $A(\lambda)$ depending on λ such that

$$(1.12) \quad \mu(\omega, n) = \max_{P_n(z) \in \mathcal{P}_{n,0}} \left\{ \max_{|z|=1-(\omega/n)} |P_n(z)| \right\} > 1 - \frac{\lambda}{\omega}$$

provided $\omega > A(\lambda)$. On the other hand they showed that if ω is large then for $n \geq \omega$

$$(1.13) \quad \mu(\omega, n) \leq 1 - 1/(e\omega) + o(1/\omega).$$

Here we shall prove the following theorem which gives a better estimate for $\mu(\omega, n)$ than (1.13). Besides, in lieu of requiring $\max_{|z|=1} |P_n(z)| \leq 1$ we only assume $|P_n(\exp(ij\pi/n))| \leq 1$ for $j = 1, 2, 3, \dots, 2n-1$ if 1 is the point on $|z| = 1$ where $P_n(z)$ vanishes.

Theorem 6. *If $P_n(z)$ is a polynomial of degree n such that $P_n(1) = 0$, $|P_n(\exp(ij\pi/n))| \leq 1$ for $j = 1, 2, 3, \dots, 2n-1$ then, for $0 < \omega \leq n$,*

$$(1.14) \quad \left| P_n \left(1 - \frac{\omega}{n} \right) \right| \leq 1 - \left(\frac{1}{\omega} - \frac{1}{2n} \right) + \left(\frac{1}{\omega} - \frac{1}{2n} \right) \left(1 - \frac{\omega}{n} \right)^n.$$

Inequalities (1.1), (1.2) can be obtained by letting $p \rightarrow \infty$ in the inequalities

$$(1.15) \quad \left(\frac{1}{2\pi} \int_0^{2\pi} |P'_n(e^{i\theta})|^p d\theta \right)^{1/p} \leq n \left(\frac{1}{2\pi} \int_0^{2\pi} |P_n(e^{i\theta})|^p d\theta \right)^{1/p}, \quad p \geq 1,$$

and

$$(1.16) \quad \left(\frac{1}{2\pi} \int_0^{2\pi} |P_n(Re^{i\theta})|^p d\theta \right)^{1/p} \leq R^n \left(\frac{1}{2\pi} \int_0^{2\pi} |P_n(e^{i\theta})|^p d\theta \right)^{1/p}, \quad p > 0, R > 1,$$

respectively. Inequality (1.15) is due to Zygmund [20] who proved it for all trigonometric polynomials of degree n and not only for those which are of the form $P_n(e^{i\theta})$. As for (1.16) it is difficult to trace its origin. We can deduce it from a well-known result of G. H. Hardy [10] according to which for every function $f(z)$ analytic in $|z| < \rho_0$ and, for every $p > 0$,

$$\left(\frac{1}{2\pi} \int_0^{2\pi} |f(\rho e^{i\theta})|^p d\theta \right)^{1/p}$$

is a nondecreasing function of ρ for $0 < \rho < \rho_0$. If $P_n(z)$ is a polynomial of degree n , then $f(z) = z^n P_n(1/\bar{z})$ is again a polynomial, i.e. an entire function and by Hardy's result

$$\left(\frac{1}{2\pi} \int_0^{2\pi} |f(\rho e^{i\theta})|^p d\theta \right)^{1/p} \leq \left(\frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^p d\theta \right)^{1/p}, \quad p > 0,$$

for $\rho = R^{-1} < 1$. This is equivalent to (1.16).

In both (1.15), (1.16) equality holds only if $P_n(z)$ is a constant multiple of z^n .

It was shown by de Bruijn [8] that if $P_n(z) \neq 0$ in $|z| < 1$ then (1.15) can be replaced by

$$(1.17) \quad \left(\frac{1}{2\pi} \int_0^{2\pi} |P'_n(e^{i\theta})|^p d\theta \right)^{1/p} \leq n C_p \left(\frac{1}{2\pi} \int_0^{2\pi} |P_n(e^{i\theta})|^p d\theta \right)^{1/p}, \quad p \geq 1,$$

where

$$C_p = \left(\frac{1}{2\pi} \int_0^{2\pi} |1 + e^{i\phi}|^p d\phi \right)^{-1/p}.$$

The corresponding refinement of (1.16) namely

$$(1.18) \quad \left(\frac{1}{2\pi} \int_0^{2\pi} |P_n(R e^{i\theta})|^p d\theta \right)^{1/p} \leq K_p \left(\frac{1}{2\pi} \int_0^{2\pi} |P_n(e^{i\theta})|^p d\theta \right)^{1/p}, \quad p \geq 1, R > 1,$$

where

$$K_p = \left(\frac{1}{2\pi} \int_0^{2\pi} |1 + R^n e^{in\phi}|^p d\phi \right)^{1/p} / \left(\frac{1}{2\pi} \int_0^{2\pi} |1 + e^{in\phi}|^p d\phi \right)^{1/p}$$

if $P_n(z) \neq 0$ in $|z| < 1$ was proved by Boas and Rahman [7].

Both inequalities (1.17), (1.18) are sharp. Equality holds for all polynomials of the form $\lambda + \mu z^n$ with $|\lambda| = |\mu|$.

If we let p tend to infinity in (1.17), (1.18) we get (1.3), (1.4) respectively.

In the special case $p = 2$ inequalities (1.17), (1.18) reduce to

$$(1.17') \quad \frac{1}{2\pi} \int_0^{2\pi} |P'_n(e^{i\theta})|^2 d\theta \leq \left(\frac{n^2}{2} \right) \frac{1}{2\pi} \int_0^{2\pi} |P_n(e^{i\theta})|^2 d\theta,$$

$$(1.18') \quad \frac{1}{2\pi} \int_0^{2\pi} |P_n(R e^{i\theta})|^2 d\theta \leq \frac{R^{2n} + 1}{2} \frac{1}{2\pi} \int_0^{2\pi} |P_n(e^{i\theta})|^2 d\theta$$

respectively, i.e. the hypothesis $P_n(z) \neq 0$ in $|z| < 1$ allows us to put $n^2/2$ in-

stead of n^2 on the right-hand side of (1.17') and $(R^{2n} + 1)/2$ instead of R^{2n} on the right-hand side of (1.18'). It may be asked what happens if we simply assume $P_n(z)$ to have a zero on $|z| = 1$. The answer is provided by Corollaries 1 and 2 of the following theorem.

Theorem 7. *If $P_n(z) = \sum_{k=0}^n a_k z^k$ is a polynomial of degree n with $P_n(e^{i\theta_0}) = 0$ for some θ_0 in $[0, 2\pi)$ and $\lambda_0, \lambda_1, \dots, \lambda_n$ are nonnegative numbers such that $\lambda_j > \lambda_k$ for $k = 0, 1, \dots, j-1, j+1, \dots, n$, then*

$$(1.19) \quad \sum_{k=0}^n \lambda_k |a_k|^2 \leq (\lambda_j - \lambda) \sum_{k=0}^n |a_k|^2$$

where λ is the unique root of the equation

$$(1.20) \quad \sum_{k=0}^n \frac{1}{\lambda_j - \lambda_k - x} = 0$$

in the interval $(0, \Lambda = \min_{0 \leq k \leq n; k \neq j} (\lambda_j - \lambda_k))$.

That equation (1.20) has one and only one root in the interval $(0, \Lambda)$ follows on writing it as $f'(x)/f(x) = 0$ where

$$f(x) = \prod_{k=0}^n \{x - (\lambda_j - \lambda_k)\}.$$

Since all the zeros of the polynomial $f(x)$ are real and $0, \Lambda$ are consecutive zeros, Rolle's theorem shows that $f'(x)$ vanishes once and only once in $(0, \Lambda)$.

Inequality (1.19) is sharp and becomes an equality if (and only if) $P_n(z)$ is a constant multiple of $\sum_{k=0}^n (e^{-i\theta_0 z})^k / (\lambda_j - \lambda_k - \lambda)$.

The following two corollaries of Theorem 7 are obtained on setting $\lambda_k = k^2$ ($0 \leq k \leq n$), $\lambda_k = R^{2k}$ ($0 \leq k \leq n$) respectively.

Corollary 1. *If $P_n(z)$ is a polynomial of degree n with $P_n(e^{i\theta_0}) = 0$ for some θ_0 in $[0, 2\pi)$, then*

$$(1.21) \quad \frac{1}{2\pi} \int_0^{2\pi} |P'_n(e^{i\theta})|^2 d\theta \leq (n^2 - \alpha_n) \frac{1}{2\pi} \int_0^{2\pi} |P_n(e^{i\theta})|^2 d\theta$$

where α_n is the unique root of the equation

$$(1.22) \quad \sum_{k=0}^n \frac{1}{n^2 - k^2 - x} = 0$$

in the interval $(0, 2n - 1)$.

Corollary 2. *If $P_n(z)$ is a polynomial of degree n with $P_n(e^{i\theta_0}) = 0$ for some θ_0 in $[0, 2\pi)$, then for $R > 1$*

$$(1.23) \quad \frac{1}{2\pi} \int_0^{2\pi} |P_n(Re^{i\theta})|^2 d\theta \leq (R^{2n} - \beta_n) \frac{1}{2\pi} \int_0^{2\pi} |P_n(e^{i\theta})|^2 d\theta$$

where β_n is the unique root of the equation

$$(1.24) \quad \sum_{k=0}^n \frac{1}{R^{2n} - R^{2k} - x} = 0$$

in the interval $(0, R^{2n} - R^{2n-2})$.

Although we do not know the explicit values of α_n, β_n we can show that (i) $\alpha_n \sim 2n/(\log n)$ as $n \rightarrow \infty$, (ii) $\beta_n \sim R^{2n}/(n+1)$ as $R \rightarrow \infty$ and n is fixed.

(i) That $\alpha_n \sim 2n/(\log n)$ as $n \rightarrow \infty$ may be deduced from the fact that

$$(1.25) \quad \sum_{k=0}^{n-1} \frac{1}{n^2 - k^2 - 2n/(\log n)} \sim \frac{\log n}{2n} \text{ as } n \rightarrow \infty.$$

Indeed, if $F_n(x)$ denotes the sum $\sum_{k=0}^{n-1} [1/(n^2 - k^2 - x)]$ then by virtue of (1.25) we have for every $\epsilon > 0$

$$F_n\left(\frac{(2+\epsilon)n}{\log n}\right) > F_n\left(\frac{2n}{\log n}\right) > \frac{\log n}{(2+\epsilon)n}$$

and

$$F_n\left(\frac{(2-\epsilon)n}{\log n}\right) < F_n\left(\frac{2n}{\log n}\right) < \frac{\log n}{(2-\epsilon)n}$$

provided n is sufficiently large. Thus $F_n(x) - 1/x$ changes sign between $(2-\epsilon)n/\log n$ and $(2+\epsilon)n/\log n$, i.e. $(2-\epsilon)n/\log n < \alpha_n < (2+\epsilon)n/\log n$. In other words, $\alpha_n \sim 2n/(\log n)$ as $n \rightarrow \infty$.

In order to prove (1.25) we write $\sum_{k=0}^{n-1} [1/(n^2 - k^2 - 2n/\log n)]$ as

$$\frac{1}{2(n^2 - 2n/\log n)^{1/2}} \sum_{k=0}^{n-1} \left\{ \frac{1}{(n^2 - 2n/\log n)^{1/2} - k} + \frac{1}{(n^2 - 2n/\log n)^{1/2} + k} \right\}$$

which is equal to

$$\frac{1}{2n(1 - 2/n \log n)^{1/2}} \left\{ \sum_{k=0}^{n-1} \left(\frac{1}{n-k} + \frac{1}{n+k} \right) + r_n \right\}$$

where the positive number r_n remains bounded as $n \rightarrow \infty$. In fact, the inequalities

$$\log(N+1) < \sum_{l=1}^N \frac{1}{l} < 1 + \log N$$

show that (for sufficiently large n)

$$\begin{aligned}
0 &< \sum_{k=0}^{n-1} \frac{1}{(n^2 - 2n/\log n)^{1/2} - k} - \sum_{k=0}^{n-1} \frac{1}{n-k} \\
&\leq \frac{n - n(1 - 2/n \log n)^{1/2}}{1 - n + n(1 - 2/n \log n)^{1/2}} \sum_{k=0}^{n-1} \frac{1}{n-k} \\
&< \frac{2}{\log n} \frac{1 + \log n}{1 - n + n(1 - 2/n \log n)^{1/2}} = O(1)
\end{aligned}$$

and that

$$\begin{aligned}
0 &< \sum_{k=0}^{n-1} \frac{1}{(n^2 - 2n/\log n)^{1/2} + k} - \sum_{k=0}^{n-1} \frac{1}{n+k} \\
&\leq \frac{n - n(1 - 2/n \log n)^{1/2}}{n(1 - 2/n \log n)^{1/2}} \sum_{k=0}^{n-1} \frac{1}{n+k} \\
&< \frac{2}{\log n} \frac{1 + \log(2n-1)}{n(1 - 2/n \log n)^{1/2}} = o(1).
\end{aligned}$$

Thus

$$\begin{aligned}
\sum_{k=0}^{n-1} \frac{1}{n^2 - k^2 - 2n/\log n} &= \frac{1}{2n(1 - 2/n \log n)^{1/2}} \left(\sum_{l=0}^{2n-1} \frac{1}{l} + \frac{1}{n} + r_n \right) \\
&\sim \frac{\log n}{2n} \text{ as } n \rightarrow \infty.
\end{aligned}$$

This proves (1.25), and the verification of the claim that " $\alpha_n \sim 2n/(\log n)$ as $n \rightarrow \infty$ " is complete.

(ii) That, for fixed n , $\beta_n \sim R^{2n}/(n+1)$ as $R \rightarrow \infty$ can be deduced from the trivial fact:

$$\sum_{k=0}^{n-1} \frac{1}{R^{2n} - R^{2k} - R^{2n}/(n+1)} \sim \frac{n+1}{R^{2n}} \text{ as } R \rightarrow \infty.$$

Remark 1. Theorem 7 is easily seen to be equivalent to the following:

Theorem 7'. If $f(s) = \sum_{k=0}^n a_k e^{s\sqrt{\lambda_k}}$ ($s = \sigma + it$) where $0 \leq \lambda_0 < \lambda_1 < \dots < \lambda_{n-1} < \lambda_n$ and $f(\sigma_0) = 0$, then

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |f'(\sigma_0 + it)|^2 dt \leq (\lambda_n - \lambda) \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |f(\sigma_0 + it)|^2 dt$$

where λ is the smallest root of the equation

$$\sum_{k=0}^n \frac{1}{\lambda_n - \lambda_k - x} = 0.$$

Remark 2. Instead of assuming $P_n(e^{i\theta_0}) = 0$ we may assume $P_n(\rho e^{i\theta_0}) = 0$ for some $\rho > 0$ and ask what becomes of (1.19).

Theorem 7". If $\lambda_j > \lambda_k \geq 0$ for $k = 0, 1, 2, \dots, j-1, j+1, \dots, n$ and $P_n(z) = \sum_{k=0}^n a_k z^k$ vanishes at $z = \rho e^{i\theta_0}$ then

$$(1.26) \quad \sum_{k=0}^n \lambda_k |a_k|^2 \leq (\lambda_j - \lambda^{(\rho)}) \sum_{k=0}^n |a_k|^2$$

where $\lambda^{(\rho)}$ is the unique root of the equation

$$(1.27) \quad \sum_{k=0}^n \frac{\rho^{2k}}{\lambda_j - \lambda_k - x} = 0$$

in $(0, \Lambda = \min_{0 \leq k \leq n; k \neq j} (\lambda_j - \lambda_k))$.

Equality holds in (1.26) for

$$P_n(z) = a \sum_{k=0}^n \frac{\rho^k}{\lambda_j - \lambda_k - \lambda^{(\rho)}} (e^{-i\theta_0} z)^k.$$

Since $f(x) = \prod_{k=0}^n (\lambda_j - \lambda_k - x)^{\rho^{2k}}$ vanishes at $x = 0, x = \Lambda$ the equation

$$\sum_{k=0}^n \frac{\rho^{2k}}{\lambda_j - \lambda_k - x} \equiv - \frac{f'(x)}{f(x)} = 0$$

must have at least one root in $(0, \Lambda)$ by the mean value theorem. That it has only one becomes obvious on writing it as

$$\frac{\rho^{2j}}{x} = \sum_{k=0; k \neq j}^n \frac{\rho^{2k}}{\lambda_j - \lambda_k - x}$$

and noting that the left-hand side decreases whereas the right-hand side increases as x increases from 0 to Λ .

On setting $\lambda_k = k^2$ ($0 \leq k \leq n$), $\lambda_k = R^{2k}$ ($0 \leq k \leq n$) in Theorem 7" we obtain sharp estimates for

$$\left(\frac{1}{2\pi} \int_0^{2\pi} |P'_n(e^{i\theta})|^2 d\theta \right) / \left(\frac{1}{2\pi} \int_0^{2\pi} |P_n(e^{i\theta})|^2 d\theta \right),$$

$$\left(\frac{1}{2\pi} \int_0^{2\pi} |P'_n(R e^{i\theta})|^2 d\theta \right) / \left(\frac{1}{2\pi} \int_0^{2\pi} |P_n(e^{i\theta})|^2 d\theta \right)$$

under the hypothesis that $P_n(z)$ has a zero on $|z| = \rho$. These estimates constitute generalizations of Corollaries 1, 2 respectively which deal with the case $\rho = 1$.

The following corollary is obtained on setting $\lambda_j = 1$ and $\lambda_k = 0$ for $k \neq j$ in Theorem 7".

Corollary 3. *If $P_n(z) = \sum_{k=0}^n a_k z^k$ has a zero on $|z| = \rho$ then for $0 \leq j \leq n$*

$$|a_j| \leq \left\{ \left(\sum_{k=0; k \neq j}^n \rho^{2k} \right) / \left(\sum_{k=0}^n \rho^{2k} \right) \right\}^{1/2} \left(\frac{1}{2\pi} \int_0^{2\pi} |P_n(e^{i\theta})|^2 d\theta \right)^{1/2}.$$

If $S_n(\theta) = \sum_{k=-n}^n c_k e^{ik\theta}$ is a trigonometric polynomial of degree n with $S_n(\theta_0) = 0$ for some θ_0 in $[0, 2\pi)$ then $e^{in\theta} S_n(\theta) = P_{2n}(e^{i\theta})$ where $P_{2n}(z) = \sum_{k=0}^{2n} a_k z^k = \sum_{k=0}^{2n} c_{k-n} z^k$ is a polynomial of degree $2n$ with $P_{2n}(e^{i\theta_0}) = 0$. Corollary 3 is applicable with $2n$ in place of n giving sharp upper bound for $|a_{k+n}| \equiv |c_k|$ in terms of $((1/2\pi) \int_0^{2\pi} |S_n(\theta)|^2 d\theta)^{1/2}$. That is how we get

Theorem 8. *If $S_n(\theta) = \sum_{k=-n}^n c_k e^{ik\theta}$ is a trigonometric polynomial of degree n with $S_n(\theta_0) = 0$ for some θ_0 in $[0, 2\pi)$ then for $-n \leq k \leq n$*

$$(1.28) \quad |c_k| \leq \left(\frac{2n}{2n+1} \right)^{1/2} \left(\frac{1}{2\pi} \int_0^{2\pi} |S_n(\theta)|^2 d\theta \right)^{1/2}$$

where equality holds if and only if $S_n(\theta)$ is a constant multiple of

$$\sum_{l=-n; l \neq k}^n (e^{-i\theta_0} e^{i\theta})^l - 2n(e^{-i\theta_0} e^{i\theta})^k.$$

Theorem 8 is the L^2 analogue of a theorem of Boas [6] according to which

$$(1.29) \quad |c_0| \leq \frac{n}{n+1} \max_{0 \leq \theta < 2\pi} |S_n(\theta)|.$$

He, in fact, showed that the maximum on the right-hand side of (1.29) may be taken only over the points $\theta = \theta_0 + 2k\pi/(n+1)$, $k = 1, 2, \dots, n$. It follows from his argument that

$$(1.30) \quad |c_0| \leq \frac{1}{n+1} \left\{ a + n \max_{1 \leq k \leq n} \left| S_n \left(\theta_0 + \frac{2k\pi}{n+1} \right) \right| \right\}$$

if $|S_n(\theta_0)| = a$ where a is any nonnegative number. Applying (1.30) to the trigonometric polynomial $|P_n(e^{i\theta_0})|^2$ as Boas [6] did we obtain

$$(1.31) \quad \frac{1}{2\pi} \int_0^{2\pi} |P_n(e^{i\theta})|^2 d\theta \leq \frac{1}{n+1} \left\{ a^2 + n \max_{1 \leq k \leq n} \left| P_n \left(e^{i(\theta_0 + 2k\pi/(n+1))} \right) \right|^2 \right\}$$

if $P_n(z)$ is a polynomial of degree n with $|P_n(e^{i\theta_0})| = a$. In (1.31) equality holds for all polynomials $P_n(z)$ for which

$$|P_n(e^{i\theta})|^2 = a^2 + (\mu^2 - a^2) \left\{ 1 - \left(\frac{1}{n+1} \frac{\sin(n+1)(\theta - \theta_0)/2}{\sin \frac{1}{2}(\theta - \theta_0)} \right)^2 \right\}$$

for $0 \leq \theta < 2\pi$. For all such polynomials $|P_n(e^{i\theta_0})| = a$,

$$\max_{1 \leq k \leq n} |P_n(e^{i(\theta_0 + 2k\pi/(n+1))})| = \mu^2 = \max_{|z|=1} |P_n(z)| \quad \text{if } a \leq \mu.$$

Hence if $a \leq \mu$ we will not get any improvement in (1.31) if maximum of $|P_n(z)|$ is taken on $|z| = 1$ rather than at the points

$$z = e^{i(\theta_0 + 2k\pi/(n+1))} \quad (k = 1, 2, \dots, n).$$

In general

$$(1.32) \quad \|P_n(e^{i\theta})\|_{p_1} \leq \|P_n(e^{i\theta})\|_{p_2}$$

if $0 < p_1 < p_2 \leq \infty$. We know from above that

$$\|P_n(e^{i\theta})\|_2 \leq \{(a^2 + n\|P_n(e^{i\theta})\|_\infty^2)/(n+1)\}^{1/2}$$

if $|P_n(e^{i\theta_0})| = a$ for some θ_0 in $[0, 2\pi)$. It is natural to ask what improvement results in (1.32) if $|P_n(e^{i\theta_0})| = a < \|P_n(e^{i\theta})\|_\infty$ for some θ_0 in $[0, 2\pi)$. Although we cannot answer this general question we note that by applying Theorem 8 to the trigonometric polynomial $|P_n(e^{i\theta})|^2$ we get

Corollary 4. *If $P_n(e^{i\theta_0}) = 0$ for some θ_0 in $[0, 2\pi)$, then*

$$(1.33) \quad \|P_n(e^{i\theta})\|_2 \leq (2n/(2n+1))^{1/4} \|P_n(e^{i\theta})\|_4.$$

In (1.33) equality holds for all polynomials $P_n(z)$ for which

$$|P_n(e^{i\theta})|^2 = n - \sum_{k=1}^n \cos k(\theta - \theta_0)$$

for $0 \leq \theta < 2\pi$.

Entire functions of exponential type. If $P_n(z)$ is a polynomial of degree n such that $|P_n(z)| \leq 1$ for $|z| = 1$ then $f(z) = P_n(e^{iz})$ is an entire function of exponential type n and $|f(x)| \leq 1$ for $-\infty < x < \infty$. From (1.1) we know that $|f'(x)| \leq n$ for $-\infty < x < \infty$ whereas according to (1.2) $|f(x+iy)| \leq e^{n|y|}$ for $y > 0$. It was shown by S. Bernstein (see [3, Chapter 11]) that for every entire function $f(z)$ of exponential type τ satisfying $|f(x)| \leq 1$ for $-\infty < x < \infty$ we have $|f'(x)| \leq \tau$ for $-\infty < x < \infty$. Besides, it is a simple consequence of the Phragmén-Lindelöf principle (for references see [3, p. 82]) that for all y : $|f(x+iy)| \leq e^{\tau|y|}$ for an arbitrary entire function $f(z)$ of exponential type τ satisfying $|f(x)| \leq 1$ for $-\infty < x < \infty$.

If $P_n(z) \neq 0$ in $|z| < 1$ then $P_n(e^{iz})$ is an entire function $f(z)$ of exponential type of a special kind: if $b(\theta)$ is its indicator, we have $b(-\pi/2) = n$ and (since $P_n(0) \neq 0$) $b(\pi/2) = 0$. Thus Boas [5] extended (1.3), (1.4) to entire functions of exponential type by proving that for all real x ,

$$(1.34) \quad |f'(x)| \leq r/2$$

and for $y < 0$,

$$(1.35) \quad |f(x + iy)| \leq \frac{1}{2}(e^{r|y|} + 1)$$

if $f(z)$ is an entire function of exponential type r with $|f(x)| \leq 1$ for real x , $b(\pi/2) = 0$ and $f(z) \neq 0$ for $\text{Im } z > 0$.

Here we shall prove

Theorem 9. *Let $f(z)$ be an entire function of exponential type r with $|f(x)| \leq 1$ for real x ,*

$$b(\pi/2) = \limsup_{y \rightarrow \infty} y^{-1} \log |f(iy)| \leq 0$$

and $f(0) = 0$. Then for all real x we have

$$(1.36) \quad |f'(x)| \leq r\{1 - (4 - \pi)/2(r|x| + 2)^2\}$$

and for $y < 0$ we have

$$(1.37) \quad |f(x + iy)| \leq e^{r|y|} \left\{ 1 - \frac{1}{2} r|y| \frac{(1 - e^{-r|y|})(4 - \pi)}{(ry)^2 + (r|x| + 2)^2} \right\}.$$

We shall show that (1.36) is "essentially" best possible.

Theorem 6 can be reformulated as follows:

Theorem 6'. *If the entire function $f(z)$ of exponential type n is periodic on the real axis with period 2π (and hence bounded for real x), such that $b(\pi/2) \leq 0$, $f(0) = 0$ and $|f(j\pi/n)| \leq 1$ for $j = 1, 2, \dots, 2n - 1$, then, for $0 < \omega < n$,*

$$(1.14') \quad |f(-i \log(1 - \omega/n))| \leq 1 - (1/\omega - 1/2n) + (1/\omega - 1/2n)(1 - \omega/n)^n.$$

Indeed, an entire function $f(z)$ of exponential type r is periodic on the real axis with period 2π if and only if $f(z) = \sum_{k=-n}^n a_k e^{ikz}$ ($n \leq r$). If, in addition, $b(\pi/2) \leq 0$, then

$$f(z) = \sum_{k=0}^n a_k e^{ikz} = P_n(e^{iz})$$

where $P_n(z)$ satisfies the hypotheses of Theorem 6. Hence (1.14') holds.

We observe that the requirement of periodicity in Theorem 6' can be dropped with little change in the conclusion if $f(z)$ is bounded on the real axis.

Theorem 10. If $f(z)$ is an entire function of exponential type τ such that $|f(x)|$ is bounded on the real axis, $b(\pi/2) \leq 0$, $f(0) = 0$ and $|f(j\pi/\tau)| \leq 1$ for $j = \pm 1, \pm 2, \dots$, then, for $0 < \omega < \tau$, we have

$$(1.38) \quad \begin{aligned} |f(-i \log(1 - \omega/\tau))| &\leq 1 + \frac{1 - (1 - \omega/\tau)^\tau}{\log(1 - \omega/\tau)^\tau} \\ &< 1 - (1/\omega - 1/2\tau) + (1/\omega - 1/2\tau)(1 - \omega/\tau)^\tau + \omega\{1 - (1 - \omega/\tau)^\tau\}/2\tau^2 \end{aligned}$$

2. Lemmas. We now prove or simply quote certain results which we shall need later. First, an interpolation formula:

Lemma 1. If $P_n(z)$ is a polynomial of degree n then for $R > 1$

$$(2.1) \quad P_n(Re^{i\phi}) = P_n(e^{i\phi}) + \frac{1}{2n} \sum_{k=1}^{2n} (-1)^k A_k P_n(e^{i(\phi + k\pi/n)})$$

where

$$A_k = (R^n - 1) + 2 \sum_{j=1}^{n-1} (R^{n-j} - 1) \cos j \frac{k\pi}{n}.$$

The coefficients A_k are positive and

$$(2.2) \quad \frac{1}{2n} \sum_{k=1}^{2n} A_k = R^n - 1.$$

Proof. Let $t(\theta) = P_n(e^{i\theta}) = \sum_{\nu=0}^n (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta)$. As

$$a_\nu = \frac{1}{\pi} \int_{-\pi}^{\pi} t(\theta) \cos \nu\theta \, d\theta, \quad b_\nu = \frac{1}{\pi} \int_{-\pi}^{\pi} t(\theta) \sin \nu\theta \, d\theta$$

for $\nu = 1, 2, \dots, n$, we have

$$\begin{aligned} P_n(Re^{i\phi}) - P_n(e^{i\phi}) &= \sum_{\nu=1}^n (R^\nu - 1)(a_\nu \cos \nu\phi + b_\nu \sin \nu\phi) \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} t(\theta + \phi) \left(\sum_{\nu=1}^n (R^\nu - 1) \cos \nu\theta \right) d\theta. \end{aligned}$$

Since $t(\theta + \phi)$ is a trigonometric polynomial of degree n in θ we may add to the sum terms in $e^{i\nu\theta}$ ($|\nu| > n$) without changing the value of the integral. Thus, we can write

$$\begin{aligned} P_n(Re^{i\phi}) - P_n(e^{i\phi}) &= \frac{1}{\pi} \int_{-\pi}^{\pi} t(\theta + \phi) \left[(R^n - 1) \cos n\theta + \sum_{\nu=1}^{n-1} (R^\nu - 1) \{\cos \nu\theta - \cos(2n - \nu)\theta\} \right] d\theta \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} t(\theta + \phi) (\cos n\theta) \left\{ R^n - 1 + 2 \sum_{j=1}^{n-1} (R^{n-j} - 1) \cos j\theta \right\} d\theta. \end{aligned}$$

Once again, we can replace $\cos n\theta$ by $b(n\theta)$, where $b(\theta)$ is simply required to be

continuous, periodic (with period 2π) and have a Fourier series of the form $b(\theta) \sim \cos \theta + c_2 \cos 2\theta + d_2 \sin 2\theta + \dots$. If $0 < \rho < 1$, we may choose

$$b(\theta) = \frac{1}{4\rho} \left(\frac{1 - \rho^2}{1 - 2\rho \cos \theta + \rho^2} - \frac{1 - \rho^2}{1 + 2\rho \cos \theta + \rho^2} \right) \\ = \cos \theta + \rho^2 \cos 3\theta + \rho^4 \cos 5\theta + \dots,$$

the series being uniformly convergent. We will then have

$$P_n(Re^{i\phi}) - P_n(e^{i\phi}) = \lim_{\rho \rightarrow 1} \frac{1}{4\pi\rho} \int_{-\pi}^{\pi} t(\theta + \phi) \left\{ R^n - 1 + 2 \sum_{j=1}^{n-1} (R^{n-j} - 1) \cos j\theta \right\} \\ \times \left(\frac{1 - \rho^2}{1 - 2\rho \cos n\theta + \rho^2} - \frac{1 - \rho^2}{1 + 2\rho \cos n\theta + \rho^2} \right) d\theta.$$

Now, by a well-known property of Poisson's kernel, we have, for every continuous periodic function $F(\theta)$ with period 2π ,

$$\lim_{\rho \rightarrow 1} \frac{1}{4\pi\rho} \int_{-\pi}^{\pi} F(\theta) \left(\frac{1 - \rho^2}{1 - 2\rho \cos n\theta + \rho^2} - \frac{1 - \rho^2}{1 + 2\rho \cos n\theta + \rho^2} \right) d\theta \\ = \frac{1}{2n} \sum_{k=1}^{2n} (-1)^k F(k\pi/n).$$

Applying this to the function

$$F(\theta) = t(\theta + \phi) \left\{ R^n - 1 + 2 \sum_{j=1}^{n-1} (R^{n-j} - 1) \cos j\theta \right\}$$

we get the interpolation formula (2.1). The relation (2.2) follows from (2.1) if we set $P_n(z) = z^n$ and $\phi = 0$.

The idea of the above proof comes from [19].

The fact that the coefficients A_k are positive is clear from Lemma 2 below.

Lemma 2. For $R > 1$

$$(2.3) \quad A_k = (R^n - 1) + 2 \sum_{j=1}^{n-1} (R^{n-j} - 1) \cos j \frac{k\pi}{n} \geq R^{n-2} (R - 1)^2.$$

Proof. If $\lambda_n \geq 0$, $\lambda_{n-1} - 2\lambda_n \geq 0$ and $\lambda_{j-1} - 2\lambda_j + \lambda_{j+1} \geq 0$ for $j = 1, 2, \dots, n-1$ then (see [17, p. 75])

$$\lambda_0 + 2 \sum_{j=1}^n \lambda_j \cos j\theta \geq 0$$

for all real θ . This result may be applied with $\lambda_j = R^{n-j} - 1$ for $1 \leq j \leq n$ and $\lambda_0 = 2R^{n-1} - R^{n-2} - 1$. Thus

$$\chi(\theta) = (2R^{n-1} - R^{n-2} - 1) + 2 \sum_{j=1}^{n-1} (R^{n-j} - 1) \cos j\theta \geq 0$$

for all real θ and $A_k \equiv R^{n-2}(R-1)^2 + \chi(k\pi/n) \geq R^{n-2}(R-1)^2$.

For the proof of Theorems 1 and 3 we shall need the following

Lemma 3. Let $P_n(z)$ be a polynomial of degree n . If $\max_{|z|=1} |P_n(z)| = 1$ and $|P_n(1)| = a$, then, for $|\theta| \leq (1-a)/n$,

$$(2.4) \quad |P_n(e^{i\theta})| \leq (1+a)/2.$$

Proof. Without loss of generality we may suppose that $P_n(z) \neq 0$ in $|z| < 1$. In fact, if $z_1, z_2, \dots, z_k$ are the zeros of $P_n(z)$ in $|z| < 1$ then

$$P_n^*(z) = P_n(z) \prod_{\nu=1}^k \frac{1 - \bar{z}_\nu z}{z - z_\nu}$$

is a polynomial of degree n which does not vanish in $|z| < 1$ and

$|P_n^*(e^{i\theta})| = |P_n(e^{i\theta})|$ for $0 \leq \theta < 2\pi$. Since

$$P_n(e^{i\theta}) = P_n(1) + \int_1^{e^{i\theta}} P_n'(z) dz$$

we have

$$|P_n(e^{i\theta})| \leq a + |e^{i\theta} - 1| \max_{|z|=1} |P_n'(z)|.$$

Hence, if $\max_{|z|=1} |P_n(z)| \leq 1$ and $P_n(z) \neq 0$ in $|z| < 1$ (as we may assume), (1.3) implies

$$|P_n(e^{i\theta})| \leq a + n|\sin(\theta/2)| \leq a + n|\theta|/2 \leq (1+a)/2$$

provided that $|\theta| \leq (1-a)/n$. This concludes the proof of Lemma 3.

Whereas the bound in (2.4) is not attained for any $\theta \neq 0$ (unless $a = 1$) the following lemma gives sharp estimate for $|P_n(e^{i\theta})|$ for every θ in $[-\pi/n, \pi/n]$.

Lemma 4. If $P_n(z)$ is a polynomial of degree n such that $\max_{0 \leq k \leq n-1} |P_n(e^{i((2k+1)/n)\pi})| \leq 1$ and $P_n(1) = 0$, then for $|\theta| \leq \pi/n$

$$(2.5) \quad |P_n(e^{i\theta})| \leq |\sin(n\theta/2)|.$$

Proof. This result is, essentially, due to Boas. Consider the trigonometric polynomial of degree n

$$S_n(\theta) = e^{-in\theta} P_n(e^{2i\theta}).$$

Since $\max_{0 \leq k \leq 2n-1} |S_n[(2k+1)\pi/(2n)]| \leq 1$ and $S_n(0) = 0$ it follows from a theorem of Boas (see [6, p. 43]) that $|S_n(\theta)| \leq |\sin n\theta|$ for $|\theta| \leq \pi/(2n)$. In terms of $P_n(z)$ this says that, for $|\theta| \leq \pi/n$, $|P_n(e^{i\theta})| \leq |\sin(n\theta/2)|$ which is the desired estimate.

Lemma 5 below will be used in the proofs of Theorems 2 and 4.

Lemma 5. *The integral*

$$I = \int_0^{\pi/2} -n \log \left\{ 1 - \frac{1}{(n+1)^2} \frac{\sin^2(n+1)\theta}{\sin^2 \theta} \right\} d\theta$$

remains bounded as n tends to infinity over the positive integers.

Proof. Break the range of integration $[0, \pi/2]$ into two parts, namely $[0, \pi/(2n)]$, $[\pi/(2n), \pi/2]$. Let the integral over $[0, \pi/(2n)]$ be denoted by I_1 and that over $[\pi/(2n), \pi/2]$ by I_2 .

In the range $\pi/(2n) \leq t \leq \pi/2$ we have

$$0 < |\sin(n+1)t|/|(n+1) \sin t| \leq \pi/\{2(n+1)t\} < 1$$

so that

$$\begin{aligned} 0 &\leq -n \int_{\pi/(2n)}^{\pi/2} \log \left(1 - \frac{1}{(n+1)^2} \frac{\sin^2(n+1)t}{\sin^2 t} \right) dt \\ &\leq -n \int_{\pi/(2n)}^{\pi/2} \log \left(1 - \frac{\pi^2}{2(n+1)^2 t^2} \right) dt \\ &= n \int_{\pi/(2n)}^{\pi/2} \left\{ \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{\pi}{2(n+1)} \right)^{2k} t^{-2k} \right\} dt \\ &= n \frac{\pi}{2} \sum_{k=1}^{\infty} \left(\frac{1}{n+1} \right)^{2k} \frac{1}{k(1-2k)} (1 - n^{2k-1}) < \frac{\pi}{2} \sum_{k=1}^{\infty} \frac{1}{k(2k-1)} \left(\frac{n}{n+1} \right)^{2k}, \end{aligned}$$

i.e., for all positive n , $|I_2| < (\pi/2) \sum_{k=1}^{\infty} (1/k(2k-1))$.

On the other hand,

$$\frac{\sin(n+1)t}{\sin t} = e^{-int} + e^{-i(n-2)t} + \dots + e^{i(n-2)t} + e^{int}$$

so that

$$\frac{\sin(n+1)t}{\sin t} = 1 + 2 \sum_{k=1}^{n/2} \cos 2kt$$

if n is even and

$$\frac{\sin(n+1)t}{\sin t} = 2 \sum_{k=0}^{(n-1)/2} \cos(2k+1)t$$

in case n is odd. For $0 \leq x \leq 1$,

$$\cos x = 1 - x^2/2 + x^4/24 - x^6/720 + \dots \leq 1 - x^2/2 + x^4/24 \leq 1 - 11x^2/24.$$

Hence, for $0 \leq t \leq \pi/(2n)$, we have

$$\cos 2kt \leq 1 - 11 k^2 t^2 / 6 \quad (1 \leq k \leq j < n/\pi)$$

and

$$\cos (2k+1)t \leq 1 - 11 (2k+1)^2 t^2 / 24 \quad (0 \leq k \leq j < (2n-\pi)/2\pi).$$

Thus, for even n and $j < n/\pi$,

$$\frac{\sin (n+1)t}{\sin t} \leq 1 + 2 \left(j - \frac{11}{6} t^2 \sum_{k=1}^j k^2 \right) + 2(n/2 - j) = (n+1) - \frac{11}{8} t^2 j(j+1)(2j+1),$$

whereas, for odd n and $j < (2n-\pi)/2\pi$,

$$\begin{aligned} \frac{\sin (n+1)t}{\sin t} &\leq 2 \left\{ j + 1 - \frac{11}{24} t^2 \sum_{k=0}^j (2k+1)^2 \right\} + 2((n-1)/2 - j) \\ &= (n+1) - \frac{11}{12} t^2 (2j/3 + 1)(j+1)(2j+1). \end{aligned}$$

In case n is even we may choose $j = [n/\pi] > (n/\pi) - 1$ to conclude that for $0 < t \leq \pi/(2n)$:

$$(2.6) \quad 0 < \frac{\sin (n+1)t}{(n+1)\sin t} \leq 1 - \frac{11}{9} t^2 \frac{(n-\pi)n(n-\pi/2)}{(n+1)\pi^3} < 1 \quad \text{if } n \geq 4.$$

Therefore for even $n \geq 4$

$$\begin{aligned} 0 \leq I_1 &\leq -n \int_0^{\pi/(2n)} \log \left\{ 1 - \left(1 - \frac{11}{9} t^2 \frac{(n-\pi)n(n-\pi/2)}{(n+1)\pi^3} \right)^2 \right\} dt \\ &= - \int_0^{\pi/2} \log \left\{ 1 - \left(\frac{11}{9} s^2 \frac{(n-\pi)(n-\pi/2)}{n(n+1)\pi^3} \right)^2 \right\} ds = O(1). \end{aligned}$$

For odd n , we may choose $j = [(n-\pi)/\pi]$ to obtain, corresponding to (2.6),

$$(2.7) \quad 0 < \frac{\sin (n+1)t}{(n+1)\sin t} \leq 1 - \frac{11}{36} t^2 \frac{(2n-\pi)(n-\pi)(2n-3\pi)}{(n+1)\pi^3} < 1 \quad \text{if } n \geq 5,$$

and then argue as before.

For the proof of Theorem 2 we shall also need

Lemma 6. *The integral*

$$-n \int_{\pi/4}^{\pi/2} \log \left(1 - \frac{1}{(n+1)^2} \frac{\sin^2(n+1)t}{\sin^2 t} \right) \frac{dt}{1 + \cos 2t}$$

remains bounded as n tends to infinity over the odd positive integers.

Proof. Write the integral as

$$\left(\int_{\pi/4}^{\pi/2 - \pi/(2n)} + \int_{\pi/2 - \pi/(2n)}^{\pi/2} \right) - n \log \left(1 - \frac{1}{(n+1)^2} \frac{\sin^2(n+1)t}{\sin^2 t} \right) \frac{dt}{1 + \cos 2t}.$$

For the second integral, we have, since n is odd

$$\begin{aligned} 0 &\leq \int_{\pi/2 - \pi/(2n)}^{\pi/2} - n \log \left(1 - \frac{1}{(n+1)^2} \frac{\sin^2(n+1)t}{\sin^2 t} \right) \frac{dt}{1 + \cos 2t} \\ &= \int_0^{\pi/(2n)} - n \log \left(1 - \frac{1}{(n+1)^2} \frac{\sin^2(n+1)t}{\cos^2 t} \right) \frac{dt}{2 \sin^2 t} \\ &\leq \frac{\pi^2}{8} \int_0^{\pi/(2n)} - n \log \left(1 - \frac{t^2}{\cos^2 t} \right) \frac{dt}{t^2} \\ &= \frac{\pi^2}{8} n \int_0^{\pi/(2n)} \sum_{k=1}^{\infty} \frac{1}{k} \frac{t^{2(k-1)}}{\cos^{2k} t} dt = O(1). \end{aligned}$$

The first integral can be estimated as follows.

$$\begin{aligned} 0 &\leq \int_{\pi/4}^{\pi/2 - \pi/(2n)} - n \log \left(1 - \frac{1}{(n+1)^2} \frac{\sin^2(n+1)t}{\sin^2 t} \right) \frac{dt}{1 + \cos 2t} \\ &\leq \int_{\pi/4}^{\pi/2 - \pi/(2n)} - n \log \left(1 - \frac{\pi^2}{(n+1)^2 4t^2} \right) \frac{dt}{2 \sin^2(t - \pi/2)} \\ &\leq \frac{\pi^2}{8} \int_{-\pi/4}^{-\pi/(2n)} - n \log \left(1 - \frac{\pi^2}{(n+1)^2 4(t + \pi/2)^2} \right) \frac{dt}{t^2} \\ &= \frac{\pi^2}{8} \int_{-\pi/4}^{-\pi/(2n)} n \left\{ \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{\pi}{2(n+1)} \right)^{2k} \frac{1}{t^{2(t + \pi/2)^{2k}}} \right\} dt \\ &\leq \frac{\pi^2}{8} n \sum_{k=1}^{\infty} \left\{ \frac{1}{k} \left(\frac{\pi}{2(n+1)} \right)^{2k} \int_{-\pi/4}^{-\pi/(2n)} \frac{dt}{(\pi/4)^{2k} t^2} \right\} \\ &< \frac{\pi n^2}{(n+1)^2} \sum_{k=0}^{\infty} \frac{1}{(k+1)} \left(\frac{2}{n+1} \right)^{2k} = O(1). \end{aligned}$$

The next lemma is, in fact, the well-known interpolation formula of M. Riesz [16] expressing the derivative of a trigonometric polynomial in terms of the values of the polynomial at $2n$ different points.

Lemma 7. *If $S(\theta)$ is a trigonometric polynomial of degree n then*

$$(2.8) \quad S'(\theta) = \frac{1}{2n} \sum_{k=1}^{2n} \frac{(-1)^k}{1 - \cos \{(2k+1)/2n\}\pi} S\left(\theta + \frac{2k+1}{2n} \pi\right).$$

Besides,

$$(2.9) \quad \frac{1}{2n} \sum_{k=1}^{2n} \frac{1}{1 - \cos\{(2k+1)/2n\}\pi} = n.$$

There is a corresponding interpolation formula for entire functions of exponential type. We shall need it for the proof of the first inequality appearing in the statement of Theorem 9 and it reads as follows (see [3, p. 210]).

Lemma 8. *If $f(z)$ is an entire function of exponential type τ bounded on the real axis, then for all real x*

$$(2.10) \quad f'(x) = \tau(4/\pi^2) \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(2n+1)^2} f\left(x + \frac{2n+1}{2\tau}\pi\right).$$

Further,

$$(2.11) \quad (4/\pi^2) \sum_{n=-\infty}^{\infty} \frac{1}{(2n+1)^2} = 1.$$

The next interpolation formula will be needed for the proof of the second inequality appearing in the statement of Theorem 9 as well as for the proof of Theorem 10.

Lemma 9. *If $f(z)$ is an entire function of exponential type τ bounded on the real axis, such that $b(\pi/2) \leq 0$, then*

$$(2.12) \quad f(x+iy) = \tau y \sum_{k=-\infty}^{\infty} \frac{1 - (-1)^k e^{-\tau y}}{(ry)^2 + (k\pi)^2} f\left(x + \frac{k\pi}{\tau}\right).$$

Proof. Suppose first that $f(z) = \int_0^{\tau} e^{izt} \phi(t) dt$ with $\phi \in L^2(0, \tau)$. Setting

$$g(t) = \begin{cases} e^{-yt} & \text{if } 0 \leq t \leq \tau, \\ e^{yt} & \text{if } -\tau \leq t \leq 0, \end{cases} \quad \text{and} \quad \psi(t) = \begin{cases} e^{ixt} \phi(t) & \text{if } 0 \leq t \leq \tau, \\ 0 & \text{if } -\tau \leq t \leq 0, \end{cases}$$

we can write

$$(2.13) \quad f(x+iy) = \int_{-\tau}^{\tau} g(t) \psi(t) dt.$$

Now let $\sum_{k=-\infty}^{\infty} c_k e^{ik(\pi/\tau)t}$ be the Fourier series of the continuous function of bounded variation $g(t)$ on the interval $[-\tau, \tau]$. Then

$$c_k = \frac{1}{2\pi} \int_{-\tau}^{\tau} g(t) e^{-ik(\pi/\tau)t} dt = \tau y \frac{1 - (-1)^k e^{-\tau y}}{(ry)^2 + (k\pi)^2}$$

and

$$g(t) = \tau y \sum_{k=-\infty}^{\infty} \frac{1 - (-1)^k e^{-\tau y}}{(ry)^2 + (k\pi)^2} e^{ik(\pi/\tau)t}$$

for $-\tau \leq t \leq \tau$. Since the series is uniformly convergent, we may substitute it for $g(t)$ in (2.13) and integrate term by term to obtain

$$\begin{aligned}
 f(x + iy) &= ry \sum_{k=-\infty}^{\infty} \frac{1 - (-1)^k e^{-\tau y}}{(ry)^2 + (k\pi)^2} \int_0^{\tau} e^{i(x+k\pi/\tau)t} \phi(t) dt \\
 &= ry \sum_{k=-\infty}^{\infty} \frac{1 - (-1)^k e^{-\tau y}}{(ry)^2 + (k\pi)^2} f(x + k\pi/\tau),
 \end{aligned}$$

which proves the desired interpolation formula in the special case under consideration. In the general case, consider the function

$$f_{\delta}(z) = f(z) e^{i\delta z} \frac{\sin \delta z}{\delta z}$$

with $\delta > 0$. It is clear that $f_{\delta}(z)$ is an entire function of exponential type $\tau + 2\delta$ such that

$$\limsup_{y \rightarrow \infty} y^{-1} \log |f_{\delta}(iy)| \leq 0.$$

Besides, it belongs to L^2 on the real axis. Hence by the Paley-Wiener theorem (see [3, p. 103]), it has the form $\int_0^{\tau+2\delta} e^{izt} \phi_{\delta}(t) dt$ with $\phi_{\delta} \in L^2[0, \tau + 2\delta]$. By the above argument,

$$f_{\delta}(x + iy) = (\tau + 2\delta)y \sum_{k=-\infty}^{\infty} \frac{1 - (-1)^k e^{-(\tau+2\delta)y}}{((\tau + 2\delta)y)^2 + (k\pi)^2} f_{\delta}\left(x + k \frac{\pi}{\tau + 2\delta}\right)$$

and we obtain (2.12) on letting δ tend to zero.

This kind of reasoning has been used before for proving certain other interpolation formulas (see for instance [4], [13]).

For the proof of Theorem 9 we shall also need:

Lemma 10. *Let $f(z)$ be an entire function of exponential type τ . If $f(0) = 0$,*

$$\sup_{-\infty < x < \infty} |f(x)| \leq 1 \quad \text{and} \quad b_f(\pi/2) = \limsup_{y \rightarrow \infty} y^{-1} \log |f(iy)| \leq 0$$

then for all real x

$$(2.14) \quad |f(x)| \leq (\tau/2)|x|.$$

Proof. An entire function of order less than 1 which is bounded on the real axis is necessarily a constant. If $f(0) = 0$ then it is identically zero and (2.14) is trivially true.

So let $f(z)$ be an entire function of order 1 type τ . If $b_f(\pi/2) = c$ let

$$F(z) = e^{-i(\tau-c)/2 z} z^{-1} f(z).$$

The function $F(z)$ is an entire function of order 1 type $\frac{1}{2}(\tau + c)$ and $b_F(\pi/2) = b_F(-\pi/2) = \frac{1}{2}(\tau + c)$. Let x_0 be a point of the real axis where $|F(x_0)| = \max_{-\infty < x < \infty} |F(x)|$. Such a point exists since $|F(x)|$ tends to zero as $|x| \rightarrow \infty$.

Choose γ such that $e^{i\gamma}F(x_0)$ is positive. If $e^{i\gamma}F(z) = \sum_{n=0}^{\infty} a_n z^n$ then the function

$$G(z) = \sum_{n=0}^{\infty} (\operatorname{Re} a_n) z^n$$

is an entire function of exponential type $\frac{1}{2}(r+c)$. Besides it is real for real x and

$$|G(x)| = |\operatorname{Re} \{e^{i\gamma}F(x)\}| \leq |x|^{-1}.$$

The function $H(z) = zG(z)$ is therefore an entire function of exponential type $\frac{1}{2}(r+c)$ which is real for real x and whose modulus is bounded by 1 on the real axis. According to a theorem of Duffin and Schaeffer (see [9, p. 555]) we have for all real x

$$((r+c)/2)^2 H^2(x) + \{H'(x)\}^2 \leq ((r+c)/2)^2,$$

i.e.

$$((r+c)/2)^2 x^2 G^2(x) + \{G(x) + xG'(x)\}^2 \leq ((r+c)/2)^2.$$

At the point x_0 where $G(x)$ attains its maximum we have $G'(x_0) = 0$ and therefore

$$((r+c)/2)^2 x_0^2 G^2(x_0) + G^2(x_0) \leq ((r+c)/2)^2$$

or

$$G(x_0) \leq ((r+c)/2) / \{1 + ((r+c)/2)^2 x_0^2\}^{1/2} \leq (r+c)/2 \leq r/2.$$

Since

$$G(x_0) = \max_{-\infty < x < \infty} |G(x)| = \max_{-\infty < x < \infty} |e^{i\gamma}F(x)| = \max_{-\infty < x < \infty} |x|^{-1} |f(x)|$$

we get $|f(x)| \leq (r/2)|x|$ for all real x . This proves Lemma 10.

3. Proofs of results announced in §1.

Proof of Theorem 1. Let $P_n(z) \in \mathcal{P}_{n,a}$. There is no loss of generality in supposing that $|P_n(1)| = a$. If $P_n(z) = a_0 + a_1 z + \dots + a_n z^n$ then

$$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_n(e^{i\phi}) e^{ik\phi} d\phi \quad (0 \leq k \leq n).$$

Hence

$$\begin{aligned} e^{i\theta} P'_n(e^{i\theta}) &= \sum_{k=1}^n k a_k e^{ik\theta} = \sum_{k=1}^n \frac{k}{2\pi} \int_{-\pi}^{\pi} P_n(e^{i\phi}) e^{ik(\theta-\phi)} d\phi \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} P_n(e^{i(\theta+t)}) e^{-it} \sum_{k=1}^n k e^{-i(k-1)t} dt. \end{aligned}$$

Since $P_n(e^{i(\theta+t)}) e^{-it} = a_0 e^{-it} + a_1 e^{i\theta} + a_2 e^{2i\theta} e^{it} + \dots + a_n e^{in\theta} e^{i(n-1)t}$, we may

add to the sum $\sum_{k=1}^n k e^{-i(k-1)t}$ terms in $e^{-ik t}$ with $k \geq n$ without changing the value of the integral provided $n > 1$. In particular, noting the identity

$$\left(\sum_{k=1}^n z^{k-1} \right)^2 = \sum_{k=1}^n k z^{k-1} + \text{higher powers of } z$$

we can write (also if $n = 1$)

$$e^{i\theta} P'_n(e^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_n(e^{i(\theta+t)}) e^{-it} \{1 + e^{-it} + e^{-i2t} + \dots + e^{-i(n-1)t}\}^2 dt.$$

Consequently,

$$|P'_n(e^{i\theta})| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |P_n(e^{i(\theta+t)})| \{(\sin nt/2)/(\sin t/2)\}^2 dt.$$

Since

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{\sin nt/2}{\sin t/2} \right)^2 dt = n$$

we find, using (2.4) together with $|P_n(z)| \leq 1$ ($|z| \leq 1$) that

$$\begin{aligned} |P'_n(e^{i\theta})| &\leq \frac{1}{2\pi} \left(\int_{(1-a)/n \leq |\theta+t| \leq \pi} + \int_{|\theta+t| \leq (1-a)/n} \right) |P_n(e^{i(\theta+t)})| \left(\frac{\sin nt/2}{\sin t/2} \right)^2 dt \\ &\leq n - \frac{1}{2\pi} \int_{|\theta+t| \leq (1-a)/n} \{1 - |P_n(e^{i(\theta+t)})|\} \left(\frac{\sin nt/2}{\sin t/2} \right)^2 dt \\ &\leq n - \frac{1-a}{4\pi} \int_{|\theta+t| \leq (1-a)/n} \left(\frac{\sin nt/2}{\sin t/2} \right)^2 dt \\ &\leq n - \frac{1-a}{4\pi} \int_{-\theta-(1-a)/n}^{-\theta+(1-a)/n} \sin^2 \frac{nt}{2} dt \\ &= n - \frac{1-a}{4\pi n} \{(1-a) - \cos n\theta \sin(1-a)\} \\ &\leq n - \frac{1-a}{4\pi n} \{(1-a) - \sin(1-a)\}. \end{aligned}$$

This proves Theorem 1.

Proof of Theorem 2. Since the trigonometric polynomial

$$1 - \frac{1}{(n+1)^2} \left(\sum_{k=0}^n e^{ik\theta} \right) \left(\sum_{l=0}^n e^{-il\theta} \right) = 1 - \frac{1}{(n+1)^2} \frac{\sin^2(n+1)\theta/2}{\sin^2\theta/2}$$

is nonnegative for all real θ , it follows from a classical result of Fejér and Riesz (see [1, p. 152]) that the equation

$$|P_n(e^{i\theta})|^2 + \frac{1}{(n+1)^2} \frac{\sin^2(n+1)\theta/2}{\sin^2\theta/2} = 1$$

defines a family of polynomials belonging to $\mathcal{P}_{n,0}$. Let $\hat{P}_n(z)$ be the one which does not vanish in $|z| < 1$ and assumes a positive value at the origin. For $|z| < 1$

$$(3.1) \quad \log \dot{P}_n(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log |\dot{P}_n(e^{it})| dt.$$

In fact, given z_0 ($|z_0| < 1$)

$$\log \dot{P}_n(z_0) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\rho e^{it} + z}{\rho e^{it} - z} \log |\dot{P}_n(\rho e^{it})| dt$$

for $|z_0| < \rho < 1$ by Poisson-Schwarz formula. Since

$$1 \geq |\rho e^{it} - 1| \geq 2\sqrt{\rho} |\sin t/2| \geq 2\sqrt{\rho} |t| \geq \sqrt{2} |t|/\pi$$

for $\rho \geq 1/2$ and hence

$$|\log |\rho e^{it} - 1|| \leq \log(\pi/\sqrt{2}) + |\log |t||,$$

Lebesgue's dominated convergence theorem allows us to let ρ tend to 1 under the integral sign giving (3.1). Besides, the function $\log |\dot{P}_n(e^{it})|$ being integrable over $[0, 2\pi]$, we can differentiate (3.1) under the integral sign and get

$$(3.2) \quad \dot{P}'_n(z) = \dot{P}_n(z) \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it}}{(e^{it} - z)^2} \log |\dot{P}_n(e^{it})|^2 dt$$

for $|z| < 1$. But we are interested in $\dot{P}'_n(-1)$ and will like (3.2) to hold for $z = -1$ as well. We observe that for *odd* n it is indeed the case. For odd n , the function

$$\log \{\dot{P}_n(e^{iz})\}^2 = \log \left\{ 1 - \frac{1}{(n+1)^2} \frac{\sin^2(n+1)z/2}{\sin^2 z/2} \right\}$$

is analytic in $|z - \pi| < \rho_0$ for some $\rho_0 > 0$ and has a double zero at $z = \pi$. Let $\delta = 1/2 \min(\rho_0, 1)$. Then for $|t - \pi| \leq \delta$:

$$\log |\dot{P}_n(e^{it})|^2 = (t - \pi)^2 \alpha(t)$$

where $\alpha(t)$ is continuous on $|t - \pi| \leq \delta$. Hence, if

$$f_\rho(t) = \frac{e^{it}}{(e^{it} + \rho)^2} \log \left\{ 1 - \frac{1}{(n+1)^2} \frac{\sin^2(n+1)t/2}{\sin^2 t/2} \right\}$$

then, for $0 < \rho < 1$, $|t - \pi| \leq \delta$,

$$|f_\rho(t)| = \frac{(t - \pi)^2 \alpha(t)}{|e^{it} + \rho|^2} \leq \frac{(t - \pi)^2 \alpha(t)}{4\rho \sin^2((t - \pi)/2)} = \frac{1}{\rho} \beta(t)$$

where the function $\beta(t)$ is integrable over $|t - \pi| \leq \delta$. For $0 < \rho < 1$, $\delta < |t - \pi| \leq \pi$, we have

$$|f_\rho(t)| \leq \frac{-1}{1 + 2\rho \cos(\pi - \delta) + \rho^2} \log \left\{ 1 - \frac{1}{(n+1)^2} \frac{\sin^2(n+1)t/2}{\sin^2 t/2} \right\};$$

Once again, Lebesgue's dominated convergence theorem shows that

$$\lim_{\rho \rightarrow 1} \frac{1}{2\pi} \int_0^{2\pi} f_\rho(t) dt = \frac{1}{2\pi} \int_0^{2\pi} f_1(t) dt,$$

i.e.

$$(3.3) \quad \dot{P}'_n(-1) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it}}{(e^{it} + 1)^2} \log \left\{ 1 - \frac{1}{(n+1)^2} \frac{\sin^2(n+1)t/2}{\sin^2 t/2} \right\} dt.$$

Formula (3.3) can be rewritten as

$$|\dot{P}'_n(-1)| = -\dot{P}'_n(-1) = -\frac{1}{\pi} \int_0^{\pi/2} \log \left\{ 1 - \frac{1}{(n+1)^2} \frac{\sin^2(n+1)t}{\sin^2 t} \right\} \frac{dt}{1 + \cos 2t}.$$

Therefore, Lemmas 5 and 6 show that there exists a positive constant c such that $|\dot{P}'_n(-1)| \leq c/n$. If n is even and $m = n/2$ is odd, $g_n(z) = (\dot{P}_m(z))^2$ is a polynomial of degree n such that $g_n(1) = 0$ and $\max_{|z|=1} |g_n(z)| = g_n(-1) = 1$. Besides,

$$|g'_n(-1)| = 2|\dot{P}'_m(-1)| \leq 4c/n.$$

If n is even and $m = n/2$ is also even, the polynomial $b_n(z) = \dot{P}_{m-1}(z)\dot{P}_{m+1}(z) \in \mathcal{P}_{n,0}$ (with $b_n(-1) = 1$) and

$$|b'_n(-1)| = -b'_n(-1) \leq 4cn/(n^2 - 4).$$

Thus, there exist a positive constant c_1 and, for each n , a polynomial $P_n(z) \in \mathcal{P}_{n,0}$ with $P_n(1) = 0$, $P_n(-1) = 1$, $P'_n(-1) < 0$ and

$$(3.4) \quad |P'_n(-1)| \leq c_1/n.$$

Now, let $G_n(z) = (-1)^n z^n P_n(1/z)$ which belongs to $\mathcal{P}_{n,0}$ with $G_n(1) = 0$, $G_n(-1) = 1$. Besides, using (3.4) we see that

$$|G'_n(-1)| = -G'_n(-1) \geq n - c_1/n.$$

Finally, given $a \in [0, 1]$, set $H_n(z) = (-1)^n a z^n + (1-a)G_n(z)$. Then $H_n(z) \in \mathcal{P}_{n,a}$ (with $|H_n(1)| = a$ and $H_n(-1) = 1$) and

$$|H'_n(-1)| = |na - (1-a)G'_n(-1)| = na + (1-a)|G'_n(-1)| \geq n - c_1(1-a)/n.$$

This completes the proof of Theorem 2.

Proof of Theorem 3. Let $P_n(z) \in \mathcal{P}_{n,a}$. Without loss of generality we may suppose $P_n(1) = a$. Let $m = [\pi/(1-a)] + 1$; and consider the polynomial $P^*(z) = (P_n(z))^m$. By Lemma 1, we have for $R > 1$

$$P^*(Re^{i\phi}) = P^*(e^{i\phi}) + \frac{1}{2mn} \sum_{k=1}^{2mn} (-1)^k A_k P^*(e^{i(\phi + k\pi/mn)})$$

where

$$\frac{1}{2mn} \sum_{k=1}^{2mn} A_k = R^{mn} - 1,$$

and according to Lemma 2 $A_k \geq R^{mn}(1 - R^{-1})^2$. By Lemma 3

$$|P^*(e^{i\theta})| \leq ((1+a)/2)^m < ((1+a)/2)^{\pi/(1-a)} < e^{-\pi/2} \quad \text{for } |\theta| < (1-a)/n.$$

At least two of the points $\phi_k = \phi + k\pi/mn$, $k = 1, 2, \dots, 2mn$ (say ϕ_{j_1}, ϕ_{j_2}) lie in the angle $|\theta| \leq \pi/mn < (1-a)/n$. Hence from above

$$\begin{aligned} |P^*(Re^{i\phi})| &\leq 1 + \frac{1}{2mn} \sum_{k=1}^{2mn} A_k |P^*(e^{i(\phi+k\pi/mn)})| \\ &\leq 1 + \frac{1}{2mn} \sum_{k=1}^{2mn} A_k - \frac{1}{2mn} (1 - e^{-\pi/2})(A_{j_1} + A_{j_2}) \\ &\leq R^{mn} \{1 - (mn)^{-1}(1 - e^{-\pi/2})(1 - R^{-1})^2\} \end{aligned}$$

and

$$\begin{aligned} |P_n(Re^{i\phi})| &= |P^*(Re^{i\phi})|^{1/m} \leq R^n \{1 - (mn)^{-1}(1 - e^{-\pi/2})(1 - R^{-1})^2\}^{1/m} \\ &\leq R^n \{1 - (m^2n)^{-1}(1 - e^{-\pi/2})(1 - R^{-1})^2\} \end{aligned}$$

by virtue of the inequality $(1-x)^a < 1 - ax$ valid for $0 < a < 1$ and $0 < x < 1$.

Since $m = [\pi/(1-a)] + 1$ we have, in fact, proved (1.7).

Proof of Theorem 4. We again consider the polynomial

$$\dot{P}_n(z) = \sum_{k=0}^n \dot{a}_k z^k \in \mathcal{P}_{n,0}$$

defined by the relation

$$|P_n(e^{i\theta})|^2 = 1 - \frac{1}{(n+1)^2} \frac{\sin^2(n+1)\theta/2}{\sin^2\theta/2}$$

not vanishing in $|z| < 1$ and assuming a positive value at the origin. From (3.1)

$$0 < -\log \dot{a}_0 = \frac{1}{\pi} \int_0^{\pi/2} -\log \left\{ 1 - \frac{1}{(n+1)^2} \frac{\sin^2(n+1)\theta}{\sin^2\theta} \right\} d\theta.$$

Hence according to Lemma 5 there exists a constant c_2 such that $-\log \dot{a}_0 < (c_2 - 1)/n$, i.e. $\dot{a}_0 > \exp \{-(c_2 - 1)/n\} > 1 - (c_2 - 1)/n$. Now if

$$\dot{Q}_n(z) = z^n \dot{P}_n(1/z) = \sum_{k=0}^n \dot{a}_k z^{n-k}$$

then

$$\begin{aligned} |\dot{Q}_n(-R)| &\geq \dot{a}_0 R^n - \sum_{k=1}^n |a_k| R^{n-k} \\ &\geq \dot{a}_0 R^n - nR^{n-1} \quad (\text{since } |\dot{Q}_n(z)| \leq 1 \text{ for } |z| = 1) \\ &> R^n \{1 - (c_2 - 1)/n - n/R\} > R^n (1 - c_2/n) \end{aligned}$$

if $R > n^2$. The polynomial $\dot{Q}_n^+(z)$ is real on the real axis and for odd n , $\dot{Q}_n^+(-R) < 0$ if $R > 1$ (note that $\dot{Q}_n^+(z)$ does not vanish for $|z| > 1$ and $Q_n^+(-R)$ tends to $-\infty$ as $R \rightarrow \infty$). Hence, if n is odd and

$$V_n(z) = (1-a)\{[a/(1-a)]z^n + \dot{Q}_n^+(z)\}$$

then $|V_n(z)| \leq 1$ for $|z| \leq 1$, $V_n(-1) = -1$ and $V_n(1) = a$. Thus $V_n(z) \in \mathcal{P}_{n,a}$ and for $R > n^2$:

$$\begin{aligned} |V_n(-R)| &= (1-a)\left\{\frac{a}{1-a}R^n + |\dot{Q}_n^+(-R)|\right\} \\ &> aR^n + (1-a)R^n(1 - c_2/n) = R^n\{1 - c_2(1-a)/n\}. \end{aligned}$$

If n is even, we may consider

$$W_n(z) = (1-a)\{(a/(1-a))z^n + z\dot{Q}_{n-1}^+(z)\}$$

instead of $V_n(z)$. Clearly $W_n(z) \in \mathcal{P}_{n,a}$ and for $R > n^2$

$$|W_n(-R)| > R^n\{1 - c_2(1-a)/n\}.$$

Proof of (1.5"). Let $P_n(z) \in \mathcal{P}_{n,0}$, where we may suppose $P_n(1) = 0$. Since $P_n(e^{i\theta})$ is a trigonometric polynomial of degree n we have by Lemma 7

$$|P'_n(e^{i\theta})| \leq \frac{1}{2n} \sum_{k=1}^{2n} \frac{k}{1 - \cos(2k+1)\pi/2n} |P_n(e^{i(\theta+(2k+1)\pi/2n)})|.$$

One of the points $e^{i(\theta+(2k+1)\pi/2n)}$ ($1 \leq k \leq 2n$) is 1 if θ is an odd multiple of $\pi/(2n)$ and hence for such values of θ

$$\begin{aligned} |P'_n(e^{i\theta})| &\leq n - \frac{1}{2n} \frac{1}{1 - \cos(2j+1)\pi/2n} \quad (\text{for some integer } j) \\ &\leq n - \frac{1}{2n} \frac{1}{1 + \cos(\pi/2n)} < n - \frac{1}{4n}. \end{aligned}$$

If θ is not an odd multiple of $\pi/(2n)$ then precisely two of the points $\theta + (2k+1)\pi/2n$ (say $\theta + \pi(2j_1+1)/2n$, $\theta + \pi(2j_2+1)/2n$) lie in the interval $(-\pi/n, \pi/n)$ and by Lemma 4

$$\begin{aligned} |P'_n(e^{i\theta})| &\leq n - \frac{1}{2n} \left\{ \frac{1 - |\sin n(\theta + (2j_1+1)\pi/2n)/2|}{1 - \cos(2j_1+1)\pi/2n} + \frac{1 - |\sin n(\theta + (2j_2+1)\pi/2n)/2|}{1 - \cos(2j_2+1)\pi/2n} \right\} \\ &= n - \frac{1}{2n} \left\{ \frac{1 - |\sin n(\theta + (2j_1+1)\pi/2n)/2|}{1 - \cos(2j_1+1)\pi/2n} + \frac{1 - |\cos n(\theta + (2j_1+1)\pi/2n)/2|}{1 - \cos(2j_2+1)\pi/2n} \right\} \\ &\leq n - \frac{1}{2n} \left\{ \frac{2 - |\sin n(\theta + (2j_1+1)\pi/2n)/2| - |\cos n(\theta + (2j_1+1)\pi/2n)/2|}{1 + \cos(\pi/2n)} \right\} \\ &\leq n - \frac{2 - \sqrt{2}}{4n} \end{aligned}$$

which completes the proof of (1.5").

Proof of (1.7"). Let $P_n(z) \in \mathcal{P}_{n,0}$. Clearly, we may suppose $P_n(1) = 0$. In this case two and possibly three of the interpolation points in the formula

$$P_n(Re^{i\phi}) = P_n(e^{i\phi}) + \frac{1}{2n} \sum_{k=1}^{2n} (-1)^k A_k P_n e^{i(\phi + k\pi/n)}$$

lie in $|\theta| \leq \pi/n$ where by Lemma 4 $|P_n(e^{i\theta})| \leq |\sin(n\theta/2)|$. If three of the points (say $\phi + j_1\pi/n$, $\phi + j_2\pi/n$, $\phi + j_3\pi/n$) fall in the interval $[-\pi/n, \pi/n]$ then they have to be $-\pi/n$, 0 , π/n and

$$\begin{aligned} |P_n(Re^{i\phi})| &\leq R^n - \frac{1}{2n} \{A_{j_1}(1 - |\sin -\pi/2|) + A_{j_2}(1 - |\sin 0|) + A_{j_3}(1 - |\sin \pi/2|)\} \\ &\leq R^n \{1 - (1 - R^{-1})^2/2n\}. \end{aligned}$$

If there are only two points (say $\phi + j_1\pi/n$, $\phi + j_2\pi/n$) in $[-\pi/n, \pi/n]$, then

$$|\sin n(\phi + j_2\pi/n)/2| \equiv |\cos n(\phi + j_1\pi/n)/2|$$

and hence

$$\begin{aligned} |P_n(Re^{i\phi})| &\leq R^n - \frac{1}{2n} \{A_{j_1}(1 - |\sin n(\phi + j_1\pi/n)/2|) + A_{j_2}(1 - |\cos n(\phi + j_1\pi/n)/2|)\} \\ &\leq R^n \{1 - (2 - \sqrt{2})(1 - R^{-1})^2/2n\} \end{aligned}$$

which completes the proof of (1.7").

Proof of Theorem 5. In the proof of Theorem 2 it was shown that for each positive integer n there exists a polynomial $G_n(z) \in \mathcal{P}_{n,0}$ with $G_n(1) = 0$, $G_n(-1) = 1$ and $|G'_n(-1)| = -G'_n(-1) \geq n - c_1/n$ where c_1 is an absolute constant. Given $\epsilon > 0$, and a positive integer n let $m = [\sqrt{n}]$, $k = [\epsilon\sqrt{n}/2c_1]$, $j = n - [\sqrt{n}][\epsilon\sqrt{n}/2c_1]$. Then $P_n(z) = \{G_m(z)\}^k G_j(z)$ has a zero of multiplicity $k+1 > \epsilon\sqrt{n}/2c_1 = \delta\sqrt{n}$ at $z = 1$. Moreover

$$\begin{aligned} |P'_n(-1)| &= |kG'_m(-1) + G'_j(-1)| \\ &\geq k(m - c_1/m) + (j - c_1/j) = n - c_1(k/m + 1/j) > n - \epsilon \end{aligned}$$

if n is large enough, say $n > n_0$.

Proof of Theorem 6. If $Q_n(z) = z^n \overline{P_n(1/\bar{z})}$ then by the hypothesis of Theorem 6

$$\begin{aligned} Q_n(1) = P_n(1) = 0, \quad |Q_n(\exp ij\pi/n)| &= |P_n(\exp ij\pi/n)| \leq 1 \\ &\text{for } j = 1, 2, \dots, 2n-1. \end{aligned}$$

Hence applying Lemma 1 to $Q_n(z)$ we get for $R > 1$

$$\begin{aligned}
 |Q_n(R)| &\leq \frac{1}{2n} \sum_{k=1}^{2n} A_k - \frac{1}{2n} A_{2n} = R^n - 1 - \frac{1}{2n} A_{2n} \\
 &= R^n - 1 - \frac{1}{2n} \left\{ R^n - 1 + 2 \sum_{j=1}^{n-1} (R^{n-j} - 1) \right\} \\
 &= R^n - 1 - \frac{1}{2n} \left\{ (R^n - 1) \frac{R+1}{R-1} - 2n \right\}.
 \end{aligned}$$

Thus

$$|P_n(1/R)| \leq \left\{ (1 - R^{-n}) \left(1 - \frac{1}{2n} \frac{1 + R^{-1}}{1 - R^{-1}} \right) + R^{-n} \right\}.$$

If $0 < \omega \leq n$ then $(1 - \omega/n)^{-1} > 1$ and hence, in particular,

$$|P_n(1 - \omega/n)| \leq 1 - (1/\omega - 1/2n) + (1/\omega - 1/2n)(1 - \omega/n)^n.$$

The above reasoning also shows that if $|P_n(1)| = a < 1$ and $|P_n(\exp i j \pi/n)| \leq 1$ for $j = 1, 2, \dots, 2n-1$ then for $0 < \omega \leq n$

$$(3.5) \quad |P_n(1 - \omega/n)| \leq 1 - (1-a)(1/\omega - 1/2n) + (1-a)(1/\omega - 1/2n)(1 - \omega/n)^n.$$

Besides, the formula obtained by equating the real parts on the two sides of (2.1) may be applied to the polynomial $Q_n(z) = z^n \overline{P_n(1/\bar{z})}$ to prove in exactly the same way as above that:

$$\begin{aligned}
 &\text{(i) if } P_n(z) \text{ is a polynomial of degree } n \text{ such that } |\operatorname{Re} P_n(1)| = a < 1, \\
 &|\operatorname{Re} P_n(\exp i j \pi/n)| \leq 1 \text{ for } j = 1, 2, \dots, 2n-1 \text{ then for } 0 < \omega \leq n \\
 (3.6) \quad &|\operatorname{Re} P_n(1 - \omega/n)| \leq 1 - (1-a)(1/\omega - 1/2n) + (1-a)(1/\omega - 1/2n)(1 - \omega/n)^n;
 \end{aligned}$$

$$\begin{aligned}
 &\text{(ii) if } P_n(z) \text{ is a polynomial of degree } n \text{ such that } \operatorname{Re} P_n(1) = a < 1, \\
 &\operatorname{Re} P_n(\exp i j \pi/n) \leq 1 \text{ for } j = 1, 2, \dots, 2n-1 \text{ then for } 0 < \omega \leq n
 \end{aligned}$$

$$(3.7) \quad \operatorname{Re} P_n(1 - \omega/n) \leq 1 - (1-a)(1/\omega - 1/2n) + (1-a)(1/\omega - 1/2n)(1 - \omega/n)^n.$$

Proof of Theorem 7. There is no loss of generality in supposing that the point of the unit circle where the polynomial vanishes is 1, i.e.

$$(3.8) \quad \sum_{k=0}^n a_k = 0.$$

We write the left-hand side of (1.19) as

$$\begin{aligned}
 \sum_{k=0}^n \lambda_k |a_k|^2 &= \lambda_j \sum_{k=0}^n |a_k|^2 - \sum_{k=0; k \neq j}^n (\lambda_j - \lambda_k) |a_k|^2 \\
 &= \lambda_j \sum_{k=0}^n |a_k|^2 - \sum_{k=0; k \neq j}^n (\lambda_j - \lambda_k - \lambda) |a_k|^2 - \lambda \sum_{k=0; k \neq j}^n |a_k|^2,
 \end{aligned}$$

where for the moment λ is a constant such that

$$0 < \lambda < \Lambda = \min_{0 \leq k \leq n; k \neq j} (\lambda_j - \lambda_k).$$

From (3.8) and Schwarz's inequality we obtain

$$\begin{aligned} |a_j|^2 &= \left| \sum_{k=0; k \neq j}^n a_k \right|^2 \leq \left(\sum_{k=0; k \neq j}^n |a_k| \right)^2 \\ &= \left\{ \sum_{k=0; k \neq j}^n (\lambda_j - \lambda_k - \lambda)^{1/2} |a_k| (\lambda_j - \lambda_k - \lambda)^{-1/2} \right\}^2 \\ &\leq \sum_{k=0; k \neq j}^n (\lambda_j - \lambda_k - \lambda) |a_k|^2 \sum_{k=0; k \neq j}^n (\lambda_j - \lambda_k - \lambda)^{-1}, \end{aligned}$$

so that

$$- \sum_{k=0; k \neq j}^n (\lambda_j - \lambda_k - \lambda) |a_k|^2 \leq -|a_j|^2 \left\{ \sum_{k=0; k \neq j}^n (\lambda_j - \lambda_k - \lambda)^{-1} \right\}^{-1}.$$

Now if λ happens to be the root of the equation (1.20) lying in $(0, \Lambda)$, then

$$\left\{ \sum_{k=0; k \neq j}^n (\lambda_j - \lambda_k - \lambda)^{-1} \right\}^{-1} = \lambda$$

and we get

$$\sum_{k=0}^n \lambda_k |a_k|^2 \leq \lambda_j \sum_{k=0}^n |a_k|^2 - \lambda |a_j|^2 - \lambda \sum_{k=0; k \neq j}^n |a_k|^2 = (\lambda_j - \lambda) \sum_{k=0}^n |a_k|^2$$

which is the desired inequality.

Observe that in (1.19) equality is possible only if

$$\left| \sum_{k=0; k \neq j}^n a_k \right| = \sum_{k=0; k \neq j}^n |a_k|$$

and equality holds in Schwarz's inequality, i.e. $a_k = a(\lambda_j - \lambda_k - \lambda)^{-1}$ if $k \neq j$ whereas $a_j = -a/\lambda$. Thus the extremal polynomial has the form

$$P_n(z) = a \sum_{k=0}^n \frac{1}{\lambda_j - \lambda_k - \lambda} (e^{-i\theta_0} z)^k.$$

The proof of Theorem 7'' is analogous and we omit it.

Proof of Theorem 9. For an entire function $f(z)$ satisfying the hypotheses of the theorem we have by virtue of Lemma 8

$$|f'(x)| \leq r(4/\pi^2) \sum_{k=-\infty}^{\infty} \frac{1}{(2n+1)^2} |f(x + (2n+1)\pi/2r)|$$

for all real x . According to (2.14) $|f(x)| < (r/2)|x|$ in $(-2/r, 2/r)$. If only one of the interpolation points (say $x + (2m+1)\pi/2r$) falls in this interval it must, in fact, lie in the subinterval $[-(\pi-2)/r, (\pi-2)/r]$ and hence by (2.14) in conjunction with (2.11)

$$|f'(x)| \leq r \left\{ 1 - \frac{4}{\pi^2} \frac{1}{(2m+1)^2} \left(1 - \frac{r}{2} \left| x + \frac{2m+1}{2r} \pi \right| \right) \right\} \leq r \left\{ 1 - \frac{4}{\pi^2} \frac{2-\pi/2}{(2m+1)^2} \right\}.$$

Since

$$|(2m+1)\pi/2r - |x|| \leq |x + (2m+1)\pi/2r| \leq (\pi-2)/r$$

we have

$$1/(2m+1)^2 \geq \pi^2/4(r|x| + \pi - 2)^2$$

and hence

$$|f'(x)| \leq r \{ 1 - (4-\pi)/2(r|x| + \pi - 2)^2 \}.$$

If, on the other hand, two points $x + (2m+1)\pi/2r$, $x + (2m+3)\pi/2r$ lie in $(-2/r, 2/r)$, then

$$|f'(x)| \leq r \left\{ 1 - \frac{4}{\pi^2} \left(\frac{1-r|x + (2m+1)\pi/2r|/2}{(2m+1)^2} + \frac{1-r|x + (2m+3)\pi/2r|/2}{(2m+3)^2} \right) \right\}.$$

Since $|x + (2m+1)\pi/2r| < 2/r$ and $|x + (2m+3)\pi/2r| < 2/r$, we have

$$1/(2m+1)^2 > \pi^2/4(r|x| + 2)^2, \quad 1/(2m+3)^2 > \pi^2/4(r|x| + 2)^2$$

and hence

$$|f'(x)| < r \{ 1 - (4-\pi)/2(r|x| + 2)^2 \}$$

which completes the proof of (1.36).

To prove inequality (1.37), we use formula (2.12) which we may apply to the constant function 1 to first conclude that

$$(3.9) \quad ry \sum_{k=-\infty}^{\infty} \frac{1 - (-1)^k e^{-ry}}{(ry)^2 + (k\pi)^2} = 1.$$

Now let $y > 0$. If only one of the interpolation points, say $x + n\pi/r$, belongs to $[-2/r, 2/r]$, we have in fact $|x + n\pi/r| \leq (\pi-2)/r$ so that $(n\pi)^2 \leq (\pi-2+r|x|)^2$ and

$$\begin{aligned} |f(x + iy)| &\leq ry \sum_{k=-\infty; k \neq n}^{\infty} \frac{1 - (-1)^k e^{-ry}}{(ry)^2 + (k\pi)^2} + ry \frac{1 - (-1)^n e^{-ry}}{(ry)^2 + (n\pi)^2} \cdot r|x + n\pi/r|/2 \\ &= 1 - \frac{ry(1 - (-1)^n e^{-ry})}{(ry)^2 + (n\pi)^2} (1 - r|x + n\pi/r|/2) \\ &\leq 1 - \frac{ry(1 - (-1)^n e^{-ry})}{(ry)^2 + (n\pi)^2} (2 - \pi/2) \leq 1 - ry \frac{(2 - \pi/2)(1 - e^{-ry})}{(ry)^2 + (\pi - 2 + r|x|)^2}. \end{aligned}$$

If two interpolation points $x + n\pi/r$ and $x + (n+1)\pi/r$ are in $[-2/r, 2/r]$, we similarly obtain

$$|f(x + iy)| \leq 1 - ry \frac{(2 - \pi/2)(1 - e^{-ry})}{(ry)^2 + (2 + r|x|)^2}$$

for $y > 0$. Thus, in any case

$$|f(x + iy)| \leq 1 - ry \frac{(2 - \pi/2)(1 - e^{-ry})}{(ry)^2 + (2 + r|x|)^2}$$

if $y > 0$.

Finally, applying this inequality to the function $g(z) = \overline{f(\bar{z})} e^{i\tau z}$ which satisfies the same hypothesis as the function $f(z)$ we get

$$|f(x + iy)| \leq e^{r|y|} \left\{ 1 - r|y| \frac{(1 - e^{-r|y|})(2 - \pi/2)}{(ry)^2 + (2 + r|x|)^2} \right\}$$

for $y < 0$.

We observe that inequality (1.36) is best possible in the sense that there exists an absolute constant A such that given r and a positive number B , we can find a function f satisfying the hypothesis of Theorem 9 and a point $x > B$ where

$$(3.10) \quad |f'(x)| > r - A/\tau x^2.$$

In the proof of Theorem 2 it was shown that for each positive integer n there exists a polynomial $P_n(z) \in \mathcal{P}_{n,0}$ such that $P_n(1) = 0$, $P_n(-1) = 1$, $P'_n(-1) < 0$ and $|P'_n(-1)| > n - c_1/n$ where c_1 is an absolute constant. If $P_m(z)$ is such a polynomial of degree $m = [rB/\pi] + 1$ then

$$f(z) = P_m(\exp(irz/m))$$

satisfies all the hypotheses of Theorem 9 and

$$|f'(m\pi/r)| = \frac{r}{m} |P'_m(-1)| > r - \frac{c_1\pi^2}{r(m\pi/r)^2} = r - \frac{A}{r(m\pi/r)^2}$$

where, clearly, $x = m\pi/r > B$.

We do not make such a remark concerning inequality (1.37).

Proof of Theorem 10. By Lemma 9 we have

$$f(iy) = ry \sum_{k=-\infty}^{\infty} \frac{1 - (-1)^k e^{-ry}}{(ry)^2 + (k\pi)^2} f(k\pi/r),$$

which in conjunction with (3.9) gives us $|f(iy)| \leq 1 - (1 - e^{-ry})/ry$. If $0 < \omega < r$ we may set $y = -\log(1 - \omega/r)$ to get the desired inequality

$$|f(-i \log(1 - \omega/r))| \leq 1 + \frac{1 - (1 - \omega/r)^r}{\log(1 - \omega/r)^r}.$$

In very much the same way, inequalities (3.5), (3.6) and (3.7) can also be extended to entire functions of exponential type.

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