TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 194, 1974

ON THE TENSOR PRODUCT OF W^* ALGEBRAS

BY
BRUCE B. RENSHAW

ABSTRACT. We develop the algebra underlying the reduction theory of von Neumann in the language and spirit of Sakai's abstract W^* algebras, and using the maximum spectrum of an abelian von Neumann algebra rather than a measure-theoretic surrogate. We are thus enabled to obtain the basic fact of the von Neumann theory as a special case of a weaker general decomposition theorem, valid without separability or type restrictions, and adapted to comparison with Wright's theory in the finite case.

Introduction. The study of von Neumann algebras (weakly closed rings of operators on Hilbert space) has always been largely algebraic. Many of their properties follow from the spectral theorem, and can consequently be developed for the abstract AW^* algebras of Kaplansky [6], [7]. However there are some more delicate consequences of weak closure which do not survive in a general AW^* algebra. For questions of this kind we have the striking abstract characterization due to Sakai [9]: a C^* algebra is* isomorphic to a von Neumann algebra if and only if it is a dual Banach space. This is taken as the definition of a W^* algebra in [11], where much of the standard theory of von Neumann algebras is developed in a space-free manner, including the so-called "reduction theory" of von Neumann [16]. The central algebraic fact is found to be a representation of the W^* tensor product $Z \overline{\otimes} A$ of an abelian W^* algebra $Z = L^\infty(\Gamma, \mu)$ with another W^* algebra A, as the algebra $Z = L^\infty(\Gamma, \mu; A)$ of all essentially bounded weakly* measurable A-valued functions on Γ, with pointwise operations. This result is restricted to the case when A can be faithfully represented on a separable Hilbert space because of measure-theoretic difficulties. In this paper we prove an analogous theorem, without separability hypotheses, by working with the continuous Gelfand representation $Z = C(\Omega)$ of the abelian algebra, rather than a measure-theoretic one, the role of null sets being played by meager sets (i.e., sets of the first category) in Ω, as follows:

Theorem. *Let $Z = C(\Omega)$ be an abelian W^* algebra, and let A be any W^* algebra, with predual A_*, and let $B = Z \overline{\otimes} A$ be the W^* tensor product. Let $C^*(\Omega, A)$ be the Banach space of all w^* continuous functions from Ω to A, with the supremum norm. Then there is a natural isometry*

$$B \simeq C^*(\Omega, A): b \leftrightarrow \hat{b}$$

Received by the editors February 22, 1973.

AMS (MOS) subject classifications (1970). Primary 46L10; Secondary 16A48, 14D99.

Key words and phrases. Reduction theory, W^* algebra, normed modules, W^* topology, von Neumann algebra, hyperstonean space, primary decomposition.

with the following properties

(1) $(b+c)\hat{}\,(t) = \hat{b}(t) + \hat{c}(t)$, *and* $(b^*)\hat{}\,(t) = (\hat{b}(t))^*$, *for all* $t \in \Omega$.

(2) *If* $b \in Z \otimes_\lambda A$, *the* C^* *tensor product, and* $c \in Z \mathbin{\overline{\otimes}} A$ *is arbitrary, then* $(bc)\hat{}\,(t) = \hat{b}(t)\hat{c}(t)$, *for all* $t \in \Omega$.

(3) *If* $b, c \in B$ *are arbitrary, then* bc *is the unique element of* B *so that for all* $\theta \in A_*$, *we have*

$$\theta((bc)\hat{}\,(t)) = \theta(\hat{b}(t)\hat{c}(t)) \quad a.e.$$

(4) *Let* $\{b^\alpha\}$ *be a bounded monotone net of selfadjoint elements of* B, *and* $b = w^* \lim_\alpha b^\alpha$. *Then* b *is the unique element of* B *so that for all* $\theta \in A_*$, *we have*

$$\theta(\hat{b}(t)) = \lim_\alpha \theta(\hat{b}^\alpha(t)) \quad a.e.$$

In (3) and (4), "a.e." means "off a meager set in Ω", and the meager set is understood to depend on θ. In the separable case we can eliminate this dependence (see 2.7 and 3.8), and we obtain a new and more canonical version of Sakai's theorem, on which a reduction theory can be based, as in [11, §3.2].

In the general case, the multiplicative description (3) is too weak to accomodate such a direct approach to decomposition theory. However, I believe our success in generalizing the structure theorem of the tensor product suggests strongly that the correct approach in the general case is to avoid measure theory and work instead with the more intrinsic continuous representation of an abelian algebra.

The plan of the paper is as follows. In §1 we recall the basic facts about tensor products of W^* algebras. In §2 we set up the isometry of the theorem and prove all of the theorem except for assertion (3). §3 contains the proof of (3) using normed module techniques. In §4 we compare the continuous and measure-theoretic results and show that in the inseparable case the space $L^\infty(\Gamma, \mu; A)$ is definitely the wrong object. In an appendix §5 we prove an extended version of Halpern's theorem ([5, Theorem 3]) which is required in §3.

I am indebted to Professor S. Sakai and to Professor J. Feldman for encouragement and helpful criticisms in the course of this research.

1. **W^* algebras and their tensor products.** We recall some results of the space-free theory. For our purposes, a *W^* algebra* is a C^* algebra which is also a dual Banach space. If A is such an algebra, then its w^* topology is actually determined by its algebraic structure, as follows: the predual can be identified with the subspace of the dual spanned by those positive functionals which are *normal* in the sense that they preserve least upper bounds (Dixmier [2]).

If A is any C^* algebra, then its second dual A^{**} is naturally a C^* algebra, and hence a W^* algebra, in such a way that the canonical map $A \to A^{**}$ is a $*$ isomorphism of A onto a W^* dense subalgebra (Sherman [13]). The multiplication and *operation of A induce operations on A^* defined by

$$L_a\theta(b) = R_b\theta(a) = \theta(ab), \qquad \theta^*(a) = \overline{\theta(a^*)} \quad \text{for } a, b \in A \text{ and } \theta \in A^*.$$

If $L \subset A^*$ is a subspace of the dual invariant under these operations, then its annihilator $L^\perp$ in A^{**} is a W^* closed ideal, so that $L^* = A^{**}/L^\perp$ is a W^* algebra and the map $A \to L^*$ is a $*$ homorphism of A onto a w^* dense subalgebra of L^*. This will be an isomorphism iff it is one-to-one, and hence iff L is faithful on A in the sense that for all $a \in A$ there is a $\theta \in L$ with $\theta(a) \neq 0$.

Let A and B be C^* algebras. Then there is a unique smallest norm α on the algebraic tensor product $A \otimes_{\mathbf{C}} B$ making it into a pre-C^* algebra, and so that the inclusions $A, B \to A \otimes B$ are isometric (Takesaki [14]). It is in fact a *cross-norm*, that is, we have $\alpha(a \otimes b) = \|a\|\,\|b\|$ for $a \in A$, $b \in B$. The completion $A \otimes_\alpha B$ in this norm is by definition the C^* *tensor product* of A and B. If A and B are W^* algebras, with preduals A_* and B_* respectively, then $A_* \otimes B_*$ is naturally identified with a faithful invariant subspace of $(A \otimes_\alpha B)^*$, and as such inherits a cross-norm α^*. By the reasoning of the preceding paragraph, its dual $(A_* \otimes_{\alpha^*} B_*)^*$ is a W^* algebra containing $A \otimes_\alpha B$ as a W^* dense subalgebra. This algebra $A \mathbin{\overline{\otimes}} B = (A_* \otimes_{\alpha^*} B_*)^*$ is by definition the W^* *tensor* product of A and B.

When one of the algebras is abelian, we can be more specific. Thus let Z be an abelian W^* algebra and let A be any W^* algebra. Then the C^* cross-norm α is actually the *least cross norm* λ in the sense of Schatten [12], and the dual cross-norm α^* is the *greatest cross norm* γ. (See Sakai [11, §1.22] for a full account of these facts.)

2. **Proof of the theorem, part 1.** Throughout this section, A is a W^* algebra with predual A_*, $Z = C(\Omega)$ is an abelian W^* algebra with predual Z_*, and $B = Z \mathbin{\overline{\otimes}} A = (Z_* \otimes_\gamma A_*)^*$ is the W^* tensor product. We denote by $C^*(\Omega, A)$ the Banach space of all bounded w^* continuous functions from Ω to A, with the supremum norm.

2.1. **Proposition.** *There is a linear isometry $B \simeq C^*(\Omega, A)$, $b \to \hat{b}$ characterized by*

$$(*) \qquad b(\omega \otimes \theta) = \omega(\theta \circ \hat{b}) \quad \textit{for all } b \in B,\ \omega \in Z_*,\ \theta \in A_*.$$

Proof. Let $\hat{b}\colon \Omega \to A$ be a w^* continuous bounded function. Then $\theta \circ \hat{b} \in Z$ for all $\theta \in A_*$, and so $(*)$ serves to define a complex-valued function b on the generators of the tensor product $Z_* \otimes A_*$. We verify easily that $\|b\| = \sup\|\hat{b}(t)\| = \sup\{|b(\omega \otimes \theta)|\colon \|\omega\|, \|\theta\| \leq 1\}$, and it follows that b extends uniquely to a functional $b \in (Z_* \otimes_\gamma A_*)^* = B$ of norm exactly $\|\hat{b}\|$. The correspondence $\hat{b} \to b$ is clearly linear, and so we have defined a linear isometry of $C^*(\Omega, A)$ into $B = (Z_* \otimes_\gamma A_*)^*$. It remains to see that every $b \in B$ arises in this way. So let $b \in (Z_* \otimes_\gamma A_*)^*$. Then for fixed θ, the map $\omega \to b(\omega \otimes \theta)$ is a functional on Z_* of norm $\leq \|b\|\,\|\theta\|$, and so for some $z_\theta \in Z$ we have $b(\omega \otimes \theta) = \omega(z_\theta)$, for all $\omega \in Z_*$, and $\|z_\theta\| \leq \|b\|\,\|\theta\|$. Now fixing $t \in \Omega$, the map $\theta \to z_\theta(t)$ is a linear functional on A_*, of norm $\leq \|b\|$, so for some $\hat{b}(t) \in A$ we have $z_\theta(t) = \theta(\hat{b}(t))$, for all $\theta \in A_*$, and $\|\hat{b}(t)\| \leq \|b\|$. Then $t \to \hat{b}(t)$ is w^* continuous and bounded by $\|b\|$, and we have $b(\omega \otimes \theta) = \omega(z_\theta) = \omega(\theta \circ \hat{b})$, as desired.

Remark. We note that by the compactness of Ω and the uniform boundedness principle, every w^* continuous function $\Omega \to A$ is in fact bounded. (I am indebted to Professor Feldman for this observation.) In the remainder of this section, we proceed to verify assertions (1), (3) and (4) of the theorem.

2.2. **Proposition.** *Let $b \in B$ correspond to $\hat{b}$: $\Omega \to A$ under the isometry of* 2.1. *Then $b^*(t) = (\hat{b}(t))^*$ for all $t \in \Omega$. (That is,* (1) *of the theorem holds.)*

Proof. Let $\theta \in A_*$. Then for all $\omega \in Z_*$, we have

$$\omega(\theta \circ (b^*)\hat{}) = b^*(\omega \otimes \theta) = \overline{b(\omega^* \otimes \theta^*)} = \overline{\omega^*(\theta^* \circ \hat{b})} = \omega((\theta^* \circ \hat{b})^*).$$

Hence $\theta \circ (b^*)\hat{} = (\theta^* \circ \hat{b})^*$. Then evaluating at a point $t \in \Omega$, we have

$$\theta((b^*)\hat{}(t)) = (\theta^* \circ \hat{b})^*(t) = \overline{\theta^*(\hat{b}(t))} = \theta((\hat{b}(t))^*).$$

This holds for all $\theta \in A_*$, so $(b^*)\hat{}(t) = (\hat{b}(t))^*$. Q.E.D.

To verify (2) of the theorem, we first calculate the transform of a generator $z \otimes a$ of the tensor product:

2.3. **Lemma.** *Let $b = z \otimes a$ be a generator of the tensor product. Then $\hat{b}(t) = z(t)a$ for all $t \in \Omega$.*

Proof. If $b = z \otimes a$, then we have $\omega(\theta \circ \hat{b}) = b(\omega \otimes \theta) = \omega(z)\theta(a) = \omega(z\theta(a))$ for all $\omega \in Z_*$, $\theta \in A_*$. Hence $\theta \circ \hat{b} = z\theta(a)$ for each θ. Then evaluating at $t \in \Omega$ we have $\theta(\hat{b}(t)) = z(t)\theta(a) = \theta(z(t)a)$. This holds for all $\theta \in A_*$, and so $\hat{b}(t) = z(t)a$. Q.E.D.

Remark. The lemma shows that the C^* tensor product $Z \otimes_\lambda A$ is mapped by our isometry onto the uniform span of the functions $t \to z(t)a$, for $z \in Z$, $a \in A$. Since Ω is a compact Hausdorff space, this is just the subalgebra $C(\Omega, A)$ of uniformly continuous A-valued functions on Ω.

2.4. **Proposition.** *Let $b \in Z \otimes_\lambda A$ and let $c \in Z \mathbin{\overline{\otimes}} A$ be arbitrary. Then $(bc)\hat{}(t) = \hat{b}(t)\hat{c}(t)$ for all $t \in \Omega$. (That is,* (2) *of the theorem holds.)*

Proof. By linearity and uniform continuity, it is enough to prove the proposition for the generators $b = z \otimes a$ of the tensor product. But if b is of this form then $\omega(\theta \circ (bc))\hat{x} = (bc)(\omega \otimes \theta) = c(L_b(\omega \otimes \theta)) = c(L_z\omega \otimes L_a\theta) = L_z\omega(L_a\theta \circ \hat{c}) = \omega(z(L_a\theta \circ \hat{c}))$. We have this for all $\omega \in Z_*$, so $\theta \circ (bc)\hat{} = z(L_a\theta \circ \hat{c})$. That is, for each $t \in \Omega$, $\theta((bc)\hat{}(t)) = z(t)L_a\theta(\hat{c}(t)) = \theta(z(t)a\hat{c}(t)) = \theta(\hat{b}(t)\hat{c}(t))$. We have this for all $\theta \in A_*$, and so $(bc)\hat{}(t) = \hat{b}(t)\hat{c}(t)$ as desired.

Assertion (4) of the theorem will follow from the special case $A = \mathbf{C}$. We say that a property holds *almost everywhere* (a.e.) in Ω if it holds on the complement of a meager set (and hence on a dense set, by the Baire theorem).

2.5. **Lemma (cf. Dixmier [1, p. 153]).** *Let $\{z_\alpha\}$ be a bounded monotone net of selfadjoint elements of Z, and $z = w^* \lim_\alpha z_\alpha$. Then $z(t) = \lim_\alpha z_\alpha(t)$ a.e.*

Proof. We may assume $\{z_\alpha\}$ monotone increasing, so that $z = \sup_\alpha z_\alpha$. Then easily $z(t) \geq \sup_\alpha z_\alpha(t)$ for all t. On the other hand put $U_n = \{t \in \Omega \mid z(t) < \sup_\alpha z_\alpha(t) + 1/n\}$ for each positive integer n. If $t \in U_n$, then for some α we have $z(t) < z_\alpha(t) + 1/n$, and hence by continuity $z(s) < z_\alpha(s) + 1/n$ on a neighborhood of t. Thus U_n is open. It is also dense, for otherwise it would miss the support of a nonzero projection p, so that $z - (1/n)p$ would be an upper bound, a contradiction. Thus $M = \bigcup_n(\Omega - U_n)$ is meager, and $z(t) = \sup_\alpha z_\alpha(t) = \lim_\alpha z_\alpha(t)$ for $t \notin M$.

2.6. **Proposition.** *Let $\{b^\alpha\}$ be a bounded monotone net of selfadjoint elements of B, and $b = w^* \lim b^\alpha$. Then for all $\theta \in A_*$, we have*

$$\theta(\hat{b}(t)) = \lim_\alpha \theta(b^\alpha(t)) \quad a.e.$$

That is, (4) *of the theorem holds.*

Proof. It is enough to prove the assertion in the case when $\theta \in A_*$ is positive, since an arbitrary functional can be written as a **C**-linear combination of four positive ones. So let θ be positive. For definiteness assume $\{b^\alpha\}$ is monotone increasing. Then for each positive $\omega \in Z_*$, the net $\{\omega(\theta \circ \hat{b}^\alpha)\} = \{\hat{b}^\alpha(\omega \otimes \theta)\}$ is monotone increasing, with limit $\omega(\theta \circ \hat{b}) = b(\omega \otimes \theta)$. Thus $\theta \circ \hat{b}$ is the monotone limit of $\theta \circ \hat{b}^\alpha$ as elements of Z, and so by the lemma we have $\theta(\hat{b}(t)) = \lim \theta(b^\alpha(t))$ a.e.

In case the predual A_* of A is separable, we can strengthen this point:

2.7. **Corollary.** *Let Z, A, B, Ω be as in the theorem and suppose that A_* is uniformly separable. Let $\{b^\alpha\}$ be a bounded monotone net of selfadjoint elements of B, and $b = w^* \lim b^\alpha$. Then*

$$\hat{b}(t) = w^* \lim_\alpha \hat{b}^\alpha(t) \quad a.e.$$

Proof. Let $\{\theta_n\}$ be a dense sequence in A_*. Then, for each n, 2.6 gives a meager set M_n in Ω so that $\theta_n(\hat{b}(t)) = \lim_\alpha \theta_n(\hat{b}^\alpha(t))$ when $t \notin M_n$. Put $M = \bigcup_n M_n$. Then M is meager. Since the θ_n are dense and the b^α bounded, we have $\theta(\hat{b}(t)) = \lim_\alpha \theta(\hat{b}^\alpha(t))$ for *all* $\theta \in A_*$, so long as $t \notin M$. Thus $\hat{b}(t) = w^* \lim_\alpha \hat{b}^\alpha(t)$ for $t \notin M$.

3. **Proof of the theorem, part 2.** In order to establish (3) of the theorem in the general case we make use of the theory of normed modules. The module analog of Hilbert space was introduced by Kaplansky in [8] to show that any derivation of a Type I AW^* algebra is inner, and to settle some questions about homogeneous AW^* algebras. The idea was further developed by Widom [17], and has been taken up recently in a series of papers of Halpern [4], [5].

Let Z be an abelian W^* algebra. A *normed Z-module* is a Banach space X which is also a Z-module in such a way that $\|z\| \, \|x\| \leq \|z\| \, \|x\|$ for $z \in Z$,

$x \in X$. By a *Z-valued norm* on X we mean a function $|\ |: X \to Z^*$ satisfying $|zx| = |z|\,|x|$, $|x+y| \leq |x| + |y|$, and $\|\,|x|\,\| = \|x\|$.

3.1. **Proposition.** *A normed Z-module X carries a Z-valued norm iff*

$$(*) \qquad \|x\| = \sup_i \|p_i x\|$$

whenever $x \in X$ and $\{p_i\}$ is a set of projections in Z with $\mathrm{lub}_i p_i = 1$.

Proof. If X has a Z-valued norm, then $\|x\| = \|\,|x|\,\| = \sup_i \|p_i |x|\,\| = \sup_i \|\,|p_i x|\,\| = \sup_i \|p_i x\|$, so (∗) holds. On the other hand, if X satisfies (∗), and $x \in X$, define a function $|x|$ on Ω by $|x|(t) = \inf\{\|px\| \,|\, p$ a projection, $p(t) = 1\}$. Then (∗) shows that $p_c = \mathrm{lub}\{p \,|\, \|px\| \leq c\}$ is the largest projection satisfying $\|p_c x\| \leq c$. It follows that $|x|$ is continuous because $c \leq |x|(t) \leq c'$ on the open set where $p_{c'}(t) = 1$ and $p_c(t) = 0$. If $c < \|x\|$ then $p_c \neq 1$ so there is a point t where $p_c(t) = 0$. There $|x|(t) \geq c$. This shows that $\|\,|x|\,\| = \|x\|$. The triangle inequality is easy, and the relation $|zx| = |z|\,|x|$ follows from the case when z is a projection, which is clear.

Let X be a normed module with Z-valued norm. We say that X is *Z-complete* if whenever $\{p_i\}$ is a family of orthogonal projections in Z with $\mathrm{lub}_i p_i = 1$, and $\{x_i\}$ is a bounded family of elements of X, there is an $x \in X$ such that $p_i x = p_i x_i$ for all i. If X is Z-complete, and Y is a submodule of X, we denote by $\tilde{Y}$ the smallest uniformly closed and Z-complete submodule containing it. Explicitly we have $x \in \tilde{Y}$ iff there is a sequence $\{x_n\}$ converging uniformly to x and for each n, a family $\{p_{i,n}\}$ of projections in Z, such that $\mathrm{lub}_i p_{i,n} = 1$ and $p_{i,n} x_n \in Y$ (cf. Widom [17]).

3.2. **Proposition.** *If Y is a Z-complete submodule of a Z-normed module X, then X/Y, the quotient module with quotient norm, carries a Z-valued norm.*

Proof. We verify the condition (∗) of 3.1. The quotient norm is defined by $\|x + Y\| = d(x, Y) = \inf\{\|x - y\|: y \in Y\}$. Let $\{p_i\}$ be a family of orthogonal projections with $\mathrm{lub}_i p_i = 1$, and suppose $c > \mathrm{lub}_i d(p_i x, Y)$. Then for each i we can find a $y_i \in Y$ with $\|p_i x - y_i\| < c$. The family $\{y_i\}$ is bounded because $\|y_i\| \leq c + \|p_i x\| \leq c + \|x\|$. Hence by the completeness condition there is a $y \in Y$ with $p_i y = p_i y_i$. Then $\|x - y\| = \mathrm{lub}_i \|p_i x - p_i y\| = \mathrm{lub}_i \|p_i x - y_i\| \leq c$. Thus $d(x, Y) \leq c$ whenever $\mathrm{lub}_i d(p_i x, Y) < c$, so $d(x, Y) \leq \mathrm{lub}_i d(p_i x, Y)$. The opposite inequality is automatic in any normed module. Q.E.D.

By a *Z-functional* on a normed module X we mean a bounded module homomorphism $X \to Z$. The space $X^\#$ of all Z-functionals on X is again a normed Z-module in a natural way. As such it always carries a Z-valued norm and is Z-complete. In analogy with the case $Z = \mathbf{C}$, we have the following separation result:

3.3. **Proposition.** *Let X be a normed module with Z-valued norm, and let Y be a Z-complete submodule. If $Y \neq X$, then there is a nonzero Z-functional on X vanishing on Y.*

Proof. Let $|\ |$ be the Z-valued norm on X/Y guaranteed by 3.2, and take $x \in X - Y$. Then $zx + y \to z|x + Y|$ is seen to be a Z-functional on the submodule $Zx + Y$. By the Hahn-Banach theorem for normed modules over a W^* algebra [15], it extends to a Z-functional f on X, with $f(Y) = 0$ and $f(x) = |x + Y| \neq 0$.

We return now to the algebra $B = Z \overline{\otimes} A$ of the theorem. Then B is naturally a normed Z-module. Let $B_\# \subset B^\#$ be the space of Z-functionals on B which are w^* continuous. We easily check that $B_\#$ is a Z-complete submodule of $B^\#$. Moreover $B_\#$ is closed under the operations L_a, R_b defined by

$$L_a\phi(b) = R_b\phi(a) = \phi(ab) \quad \text{for } a, b \in B, \qquad \phi \in B_\#.$$

Finally for each $\theta \in A_*$, the map $\omega \to \omega \otimes \theta$ on the preduals induces a Z-functional $\tilde{\theta} \in B_\#$ defined by $\tilde{\theta}(b) = \theta \circ \hat{b}$, for $b \in B$, and $\|\tilde{\theta}\| = \|\theta\|$.

3.4. **Proposition.** *The natural map $B \to (B_\#)^\#: b \to b^{\#\#}$ defined by $b^{\#\#}(\phi) = \phi(b)$ is an isometry onto.*

Note. Thus B is a dual Z-module as well as a dual Banach space. Halpern ([5, Theorem 3]) has proved this result whenever B is a W^* algebra with center Z. In the present case Z is only a sub-W^*-algebra of the center; it will coincide with the center of B only if A is centerless, that is, a factor. Thus Halpern's theorem does not strictly apply. However, the result and the main ideas of its proof go through in the more general case of a W^* algebra B and an arbitrary sub-W^*-algebra Z of its center. For completeness we include a proof of this extended version of Halpern's theorem in an appendix §5.

3.5. **Corollary.** *$B_\#$ is generated as a Z-complete module by the elements $\tilde{\theta}$, for $\theta \in A_*$.*

Proof. By 3.3 it is enough to show that the only Z-functional on $B_\#$ which vanishes on every $\tilde{\theta}$ is the zero functional. But by 3.4 every Z-functional on $B_\#$ is given by evaluating at an element of B. If $b^{\#\#}(\tilde{\theta}) = \theta \circ \hat{b} = 0$ for all $\theta \in A_*$, then $\hat{b} = 0$ and hence $b = 0$ by the isometry of the theorem. Q.E.D.

We can give a functional representation for the elements of $B_\#$. If $\phi \in B_\#$ and $t \in \Omega$, we define a functional $\phi(t) \in A^*$ by $(\phi(t))(a) = (\phi(1 \otimes a))(t)$. The map $\phi \to \phi(t)$ is linear and norm-decreasing, and for the special elements $\tilde{\theta}$ we have $\tilde{\theta}(t) = \theta$.

3.6. **Proposition.** *Let $\phi \in B_\#$. Then there is a meager set $M = M(\phi)$ in Ω such that for $t \notin M$, we have*

(a) $\phi(t) \in A_*$.

(b) *The function $s \to \phi(s)$ is uniformly continuous at t.*

(c) *For all $b \in B$, we have*

$$(\phi(b))(t) = (\phi(t))(\hat{b}(t)).$$

Proof. Let $Y \subset B_\#$ be the submodule of elements of the form $\sum_{i=1}^k z_i\tilde{\theta}_i$ with $z_i \in Z$, $\theta_i \in A_*$. We check easily that (a)–(c) are satisfied for all t when $\phi \in Y$. If $\phi \in B_\#$ is arbitrary, then by the explicit description of the Z-completion given before 3.2, there is a sequence $\{\phi_n\}$ converging uniformly to ϕ, and for each n a family $\{p_{i,n}\}$ of projections in Z, such that $\mathrm{lub}_i p_{i,n} = 1$ and $p_{i,n}\phi_n \in Y$. Then for each n the set $F_n = \{t\colon p_{i,n}(t) = 0 \text{ for all } i\}$ is closed and nowhere dense, so that $M = \bigcup_n F_n$ is meager. If $t \notin M$, then for each n choose an i so that $p_{i,n}(t) = 1$. Then $(\phi_n)(t) = (p_{i,n}\phi_n)(t) \in A_*$ since $p_{i,n}\phi_n \in Y$, and therefore $\phi(t) = \lim_n \phi_n(t) \in A_*$, proving (a). Similarly $(\phi(b))(t) = \lim_n(\phi_n(b))(t) = \lim_n(p_{i,n}\phi_n(b))(t) = \lim_n(\phi_n(t))(\hat{b}(t)) = (\phi(t))(\hat{b}(t))$, proving (c). To prove (b) for ϕ, let $\varepsilon > 0$ be given and take n so that $\|\phi_n - \phi\| < \varepsilon/3$. Then take i so that $p_{i,n}(t) = 1$. Then the function $s \to (p_{i,n}\phi_n)(s)$ is continuous at t, so we can take an open set U contained in the support of $p_{i,n}$, with $t \in U$, and such that $\|(p_{i,n}\phi_n)(s) - (p_{i,n}\phi_n)(t)\| < \varepsilon/3$ for $s \in U$. Then $\|\phi(s) - \phi(t)\| < \varepsilon$ for $s \in U$, showing that $s \to \phi(s)$ is uniformly continuous at t. Q.E.D.

Finally we can establish the multiplicative assertion (3) of the theorem.

3.7. **Proposition.** *Let $b, c \in B$ and $\theta \in A_*$. Then*

$$\theta((bc)\hat{}\,(t)) = \theta(\hat{b}(t)\hat{c}(t)) \quad a.e.$$

(*That is,* (3) *of the theorem holds.*)

Proof. Consider the Z-functional $\phi = L_b\tilde{\theta} \in B_\#$. Using (2) of the theorem we verify that $\phi(t) = L_{\hat{b}(t)}\theta$. Now using 3.6(c) we have

$$\begin{aligned}\theta((bc)\hat{}\,(t)) &= \tilde{\theta}(t)((bc)\hat{}\,(t))\\ &= (\tilde{\theta}(bc))(t) = (\phi(c))(t)\\ &= (\phi(t))(\hat{c}(t)) = L_{\hat{b}(t)}\theta(\hat{c}(t))\\ &= \theta(\hat{b}(t)\hat{c}(t)),\end{aligned}$$

so long as $t \notin M = M(\phi)$, a meager set. Q.E.D.

As in the case of the topological statement (4), we can strengthen this result in the separable case.

3.8. **Corollary.** *Let A, B, Z, Ω be as in the theorem, and suppose A_* is separable. Then for all $b, c \in B$ we have*

$$(bc)\hat{}\,(t) = \hat{b}(t)\hat{c}(t) \quad a.e. \tag{3'}$$

Proof. As in 2.7, we take a dense sequence $\{\theta_n\}$ in A_* and meager $M_n \subset \Omega$ so that $\theta_n((bc)\hat{}\,(t)) = \theta_n(\hat{b}(t)\hat{c}(t))$ for $t \notin M_n$. Then by density we have $\theta((bc)\hat{}\,(t)) = \theta(\hat{b}(t)\hat{c}(t))$ for all $\theta \in A_*$, so long as $t \notin M = \bigcup_n M_n$, so that $(bc)\hat{}\,(t) = \hat{b}(t)\hat{c}(t)$ for $t \notin M$.

4. **Comparison with the measure-theoretic result.** Let (Γ, μ) be a *localizable* measure space, that is a direct sum of finite measure spaces. Let $Z = L^\infty(\Gamma, \mu)$ be the space of essentially bounded locally measurable functions $\Gamma \to \mathbf{C}$, and let $Z_* = L^1(\Gamma, \mu)$ be the space of all μ-integrable functions $\Gamma \to \mathbf{C}$. Then Z is an abelian W^* algebra under the pointwise operations, with predual Z_*.

If A is a W^* algebra, with predual A_*, we denote by $L^\infty(\Gamma, \mu; A)$ the space of essentially bounded funtions f: $\Gamma \to A$ which are w^* measurable in the sense that for each $\theta \in A_*$ we have $\theta \circ f \in L^\infty(\Gamma, \mu)$. There is a canonical norm-decreasing map, K: $L^\infty(\Gamma, \mu; A) \to L^\infty(\Gamma, \mu) \overline{\otimes} A$, defined as follows: If $f \in L^\infty(\Gamma, \mu; A)$, then $K(f)$ acts as a functional on $(L^\infty(\Gamma, \mu) \overline{\otimes} A)_* = L'(\Gamma, \mu) \otimes_\gamma A_*$ by the rule $\langle K(f), g \otimes \theta \rangle = \int_\Gamma (\theta \circ f) g \, d\mu$. If A_* is separable, the Dunford-Pettis theorem [3] shows that K is an isometry onto. In [10], Sakai goes on to show that in this case the multiplication in the tensor product corresponds under K to pointwise multiplication in $L^\infty(\Gamma, \mu; A)$.

We can give a simple example to show that K may have a nontrivial kernel when A_* is not separable. In fact let $\Gamma = [0, 1]$, $\mu =$ Lebesgue measure and $\nu =$ counting measure. Take $A = L^\infty(\Gamma, \nu)$ and $Z = L^\infty(\Gamma, \mu)$, and consider the function f: $\Omega \to A$ defined by $f(t) = \delta_t$. Then $\|f\| = 1$ as an element of $L^\infty(\Gamma, \mu; A)$, but $K(f) = 0$. Thus $L^\infty(\Gamma, \mu; A)$ is too big in this case.

For certain special measure spaces we can say more. In fact, if $Z = C(\Omega)$ is a given abelian W^* algebra then by a well-known procedure [11, §1.18] we can represent $Z \simeq L^\infty(\Gamma, \mu)$ as a measure algebra where Γ is an open dense subset of the spectrum Ω of Z, and μ is a perfect Borel measure. If $b \in Z \overline{\otimes} A$ is an element of the W^* tensor product, let $\hat{b}$: $\Omega \to A$ be the corresponding w^* continuous function given by our theorem. Then $\hat{b} \mid \Gamma$ is in $L^\infty(\Gamma, \mu; A)$ and $K(\hat{b} \mid \Gamma) = b$, as is easily checked. The mappings K and $b \to \hat{b} \mid \Gamma$ thus exhibit $Z \overline{\otimes} A$ as a *retract* of $L^\infty(\Gamma, \mu; A)$ as Banach spaces.

5. **Appendix: Halpern's theorem.** Throughout this section let Z be an abelian W^* algebra, and let B be a W^* algebra containing Z as a sub-W^*-algebra of its center. Then B is naturally a normed Z-module. Let B_*, Z_* be the respective preduals, and let $B_\#$ be the space of all bounded W^* continuous module homomorphisms of B into Z. We show that then B can be identified with the space $(B_\#)^\#$ of all Z-functionals on $B_\#$, in analogy with the identification $B = (B_*)^*$. This is the theorem of Halpern [5, Theorem 3] in case Z is the center of B. Our proof follows similar lines, the main difference being that as in the body of this paper we avoid reference to Hilbert space representations of the algebras. As in [5], the key point is that W^* continuous functionals on B can be factored through Z, as follows:

5.1. **Factorization Lemma.** *Any $\theta \in B_*$ can be factored in the form $\theta = \omega \circ \phi$ where $\omega \in Z_*$, $\phi \in B_\#$ and $\|\theta\| = \|\omega\| \, \|\phi\|$.*

Proof. By the polar decomposition of functionals [11, Theorem 1.14.4], we can write $\theta = L_u\theta^+$ where $u \in B$ is a partial isometry and $\theta^+ \in B_*$ is positive. If $\theta^+ = \omega \circ \phi^+$ is a factorization of the required type for θ^+, then $\theta = \omega \circ L_u\phi^+$ is a factorization of the required type for θ. Hence we may assume $\theta = \theta^+$ is positive.

So let $\theta \in B_*$ be positive, and let $\omega = \theta \mid Z$ be its restriction to Z. I claim that then $\|z\theta\| = \|z\omega\|$ for all $z \in Z$. For $|z|\theta$ and $|z|\omega$ are positive, and hence $\|z\theta\| = \| |z|\theta\| = |z|\theta(1) = |z|\omega(1) = \|z\omega\|$. Thus the map $z \cdot \omega \mapsto z\theta$ is a Z-linear isometry of $Z\omega$ onto $Z\theta$ and extends to a Z-linear isometry $\phi_0\colon \overline{Z\omega} \to \overline{Z\theta}$ of the uniform closures. Now the annihilator $(Z\omega)^\perp \subset Z$ is a W^* closed ideal of Z, and hence of the form $(Z\omega)^\perp = (1-p)Z$, where $p \in Z$ is a projection, the *support* of ω. It follows that $\overline{Z\omega} = (Z\omega)^{\perp\perp} = pZ_*$. Hence we obtain a Z-linear map $\phi_*\colon Z_* \to B_*$ of norm one by putting $\phi_*(\omega') = \phi_0(p\omega')$ for $\omega' \in Z_*$. Let $\phi = (\phi_*)^*\colon B \to Z$ be the adjoint of ϕ_*. Then $\|\phi\| = \|\phi_*\| = 1$, ϕ is a w^* continuous Z-functional, and by definition of the adjoint we have $\omega \circ \phi = \phi_*\omega = \theta$. Q.E.D.

It is convenient to rephrase this result in terms of a certain tensor product of Z-modules . Let X and Y be normed Z-modules. We define a seminorm on the algebraic tensor product $X \otimes_Z Y$ by

$$\|\xi\| = \inf\{\textstyle\sum_i \|x_i\|\,\|y_i\|\}$$

the infimum being taken over all representations $\xi = \sum_i x_i \otimes y_i$ of ξ as a finite sum of elementary tensors. We denote this seminormed space by $X \otimes Y$.

5.2. **Corollary.** *Composition of functions defines a Z-linear isometry $Z_* \otimes B_\#$ $\simeq B_*$.*

Proof. It is immediate that composition gives a norm-decreasing Z-linear map $Z_* \otimes B_\# \to B_*$. The factorization lemma shows that this is a quotient map. To see that it is an isometry we must verify that its kernel is trivial. So let $\xi = \sum \omega_i \otimes \phi_i$ be an element of the tensor product.

Let $\omega_i = u_i|\omega_i|$ be the polar decomposition of ω_i in Z_*. Then each $|\omega_i|$ is dominated by the functional $\omega = \sum |\omega_i|$, and so by the Radon-Nikodym theorem we can write $|\omega_i| = z_i\omega$ for some $z_i \in Z^+$. Then $\omega_i = u_iz_i\omega$ and so $\xi = \sum \omega_i \otimes \phi_i = \sum u_iz_i\omega \otimes \phi_i = \omega \otimes (\sum u_iz_i\phi_i)$. Thus it suffices to show that no nontrivial tensor $\omega \otimes \phi$ is in the kernel of our map. Suppose then that $\omega \circ \phi = 0$. Then $z\omega \circ \phi = 0$ for all $z \in Z$, so $\phi(B) \subset (Z\omega)^\perp = (1-p)Z$, where p is the support of ω. Hence $p\phi = 0$, while $\omega = p\omega$ and so $\omega \otimes \phi = p\omega \otimes \phi = \omega \otimes p\phi = 0$, as asserted.

5.3. **Theorem.** *The natural map $i\colon B \to (B_\#)^\#\colon b \to b^{\#\#}$ defined by $b^{\#\#}(\phi) = \phi(b)$ is an isometry onto.*

Proof. It is immediate that i is Z-linear and norm-decreasing. On the other hand by the factorization lemma we have

$$\begin{aligned}\|b\| &= \sup\{|\theta(b)|: \theta \in B_*, \|\theta\| \le 1\} \\ &= \sup\{|\omega \circ \phi(b)|: \omega \in Z_*, \phi \in B_\#, \|\omega\| \le 1 \text{ and } \|\phi\| \le 1\} \\ &= \sup\{\|\phi(b)\|: \phi \in B_\#, \|\phi\| \le 1\} \\ &= \|b^{\#\#}\|.\end{aligned}$$

Thus i is an isometry. To see that it is onto, let $\lambda: B_\# \to Z$ be a Z-functional. We then obtain a linear functional $\hat{\lambda}$ on the algebraic tensor product $Z_* \otimes_Z B_\#$ by putting $\hat{\lambda}(\sum \omega_i \otimes \phi_i) = \sum \omega_i(\lambda(\phi_i))$. We see immediately that $\hat{\lambda}$ is bounded with respect to the seminorm defined above. By Corollary 5.2, therefore, there is a $b \in (B_*)^* = B$ with $(\omega \circ \phi)(b) = \omega(\lambda(\phi))$ for all $\omega \in Z_*$, $\phi \in B_\#$. In other words $\omega \circ \lambda = \omega \circ b^{\#\#}$ for all $\omega \in Z_*$, and so $\lambda = b^{\#\#}$. Q.E.D.

Bibliography

1. J. Dixmier, *Sur certains espaces considérés par M.H. Stone*, Summa Brasil. Math. **2** (1951), 151–182. MR **14**, 69.

2.———, *Formes linéaires sur un anneau d'opérateurs*, Bull. Soc. Math. France **81** (1953), 9–39. MR **15**, 539.

3. N. Dunford and B.J. Pettis, *Linear operations on summable functions*, Trans. Amer. Math. Soc. **47** (1940), 323–392. MR **1**, 338.

4. H. Halpern, *Module homomorphisms of a von Neumann algebra into its center*, Trans. Amer. Math. Soc. **140** (1969), 183–193. MR **39** #3321.

5.———, *Irreducible module homomorphisms of a von Neumann algebra into its center*, Trans. Amer. Math. Soc. **140** (1969), 195–221. MR **39** #3322.

6. I. Kaplansky, *Projections in Banach algebras*, Ann. of Math. (2) **53** (1951), 235–249. MR **13**, 48.

7.———, *Algebras of type* I, Ann. of Math. (2) **56** (1952), 460–472. MR **14**, 291.

8.———, *Modules over operator algebras*, Amer. J. Math. **75** (1953), 839–858. MR **15**, 327.

9. S. Sakai, *A characterization of W* algebras*, Pacific J. Math. **6** (1956), 763–773. MR **18**, 811.

10.———, *On the reduction theory of von Neumann*, Bull. Amer. Math. Soc. **70** (1964), 393–398. MR **28** #3334.

11.———, *C* algebras and W* algebras*, Springer, Berlin, 1971.

12. R. Schatten, *A theory of cross-spaces*, Ann. Math. Studies, no. 26, Princeton Univ. Press, Princeton, N.J., 1950. MR **12**, 186.

13. S. Sherman, *The second adjoint of a C*-algebra*, Proc. Internat. Congress Math. (Cambridge, Mass., 1950), vol. I, Amer. Math. Soc., Providence, R.I., 1952, p. 470.

14. M. Takesaki, *On the cross-norm of the direct product of C* algebras*, Tôhoku Math. J. (2) **16** (1964), 111–122. MR **29** #2668.

15.———, *On the Hahn-Banach type theorem and the Jordan decomposition of module linear mappings over some operator algebras*, Kōdai Math. Sem. Rep. **12** (1960), 1–10. MR **22** #5914.

16. J. von Neumann, *On rings of operators. Reduction theory*, Ann. of Math. (2) **50** (1949), 401–485. MR **10**, 548.

17. H. Widom, *Embedding in algebras of Type* I, Duke Math. J. **23** (1956), 309–324. MR **17**, 1228.

Department of Mathematics, University of California, Berkeley, California 94720

Current address: Mathematics Department, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139