THE CLOSURE OF THE SPACE OF HOMEOMORPHISMS ON A MANIFOLD

BY

WILLIAM E. HAVER⁽¹⁾

ABSTRACT. The space, $\bar{H}(M)$, of all mappings of the compact manifold M onto itself which can be approximated arbitrarily closely by homeomorphisms is studied. It is shown that $\bar{H}(M)$ is homogeneous and weakly locally contractible. If M is a compact 2-manifold without boundary, then $\bar{H}(M)$ is shown to be locally contractible.

1. Let M be a compact manifold and $H(M)$ denote the space of all homeomorphisms of M onto itself. We shall study the space, $\bar{H}(M)$, of all continuous functions of M onto itself which can be approximated arbitrarily closely by elements of $H(M)$. All function spaces on compact spaces will be assumed to have the supremum metric, ρ ; i.e., if X and Y are spaces with d the metric on Y and f and g are functions from X into Y , then $\rho(f, g) = \sup_{x \in X} \{d(f(x), g(x))\}$. Since M is compact, the topology thus generated agrees with the compact-open topology.

A mapping of an n -manifold, M^n , onto itself is said to be cellular if for each $y \in M^n$, $f^{-1}(y)$ can be expressed as the intersection of a nested sequence of n -cells. Armentrout ($n \leq 3$) [4] and Siebenmann ($n \geq 5$) [20] have recently shown that $\bar{H}(M^n)$, $n \neq 4$, is precisely the space of all cellular mappings of M^n onto itself. Hence most of the results of this paper could be stated in terms of spaces of cellular mappings. Cellular mappings have been studied extensively (cf., Lacher [15], [16]).

Let $H_\partial(M)$ denote the space of all homeomorphisms of M onto itself which equal the identity when restricted to the boundary of M and, following our previous notation, let $\bar{H}_\partial(M)$ denote the space of all continuous functions of M onto itself which can be approximated arbitrarily closely by elements of $H_\partial(M)$. We shall state some of the major results concerning $H(M)$ and $H_\partial(M)$ and then indicate which of the analogous theorems can be proven for $\bar{H}(M)$ and $\bar{H}_\partial(M)$:

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(i) It is well known that for any compact manifold M each of $H(M)$ and $H_\partial(M)$ is a separable metric space, a topological group under composition of functions and topologically complete.

(ii) Let B^n be the standard n -ball. The Alexander isotopy [1], first used in 1923, is very useful in dealing with $H_\partial(B^n)$ and combined with the fact that $H_\partial(B^n)$ is a topological group provides a trivial proof that $H_\partial(B^n)$ is locally contractible. Mason [19] showed that $H_\partial(B^2)$ is an absolute retract and Anderson [3] proved that $H_\partial(B^1)$ is homeomorphic to l_2 (separable Hilbert space).

(iii) Recently Černavskii [6] and Edwards-Kirby [7] proved that for any compact manifold each of $H(M)$ and $H_\partial(M)$ is locally contractible.

(iv) Of current interest is the problem of whether $H(M)$ is an l_2 -manifold (i.e., locally homeomorphic to l_2). Geoghegan [9] has shown that $H(M) \times l_2 \approx H(M)$ and $H_\partial(M) \times l_2 \approx H_\partial(M)$.

We shall discuss in this paper the state of the corresponding statements for $\bar{H}(M)$ and $\bar{H}_\partial(M)$:

(i) $\bar{H}(M)$ and $\bar{H}_\partial(M)$ are obviously separable metric spaces. In addition they are not merely topologically complete, but are complete under the supremum metric. Neither space is a topological group, under composition of functions, since the inverse of a cellular map need not be even well defined. However, we do prove (§2) that $\bar{H}(M)$ and $\bar{H}_\partial(M)$ are homogeneous.

(ii) Making use of an Alexander-type homotopy and the fact that $\bar{H}_\partial(B^n)$ is homogeneous, we give a simple proof (§3) that this space is locally contractible. Elsewhere the author [13] has shown that $\bar{H}_\partial(B^2)$ is an AR and Geoghegan [9] has proven that $\bar{H}_\partial(B^1)$ is homeomorphic to l_2 .

(iii) It is unknown whether $\bar{H}(M)$ is locally contractible for an arbitrary compact manifold. In §4, it is shown that $\bar{H}(M)$ and $\bar{H}_\partial(M)$ are weakly locally contractible. Then by modifying slightly the techniques of Edwards-Kirby we show in §5 that if M^2 is a compact 2-manifold then $\bar{H}_\partial(M^2)$ is locally contractible.

(iv) Geoghegan and Henderson [10] proved that $\bar{H}(M) \times l_2 \approx \bar{H}(M)$. In §4, using a theorem of Anderson [2], we give an easy proof of the fact that if $\bar{H}(M)$ is an l_2 -manifold for a particular compact manifold M , then $H(M)$ is an l_2 -manifold.

If $f \in \bar{H}(M)$ and $\epsilon > 0$, let $N_\epsilon(f) = \{g \in \bar{H}(M) \mid \rho(g, f) < \epsilon\}$. When we wish to speak of a neighborhood in $H(M)$ we write $N_\epsilon(f) \cap H(M)$ to denote $\{g \in H(M) \mid \rho(g, f) < \epsilon\}$. If X and Y are spaces with $X \subset Y$, the complement of X in Y will be denoted by $\tilde{X}$ when there is no possibility of confusion. The boundary of X is written ∂X , the closure of X is written $\bar{X}$, and 1_X denotes the identity map on X . We use the symbol $\text{Int } X$ to denote the interior of the set X and I to denote the closed unit interval.

The work of this paper is an extension of a portion of the author's doctoral

dissertation written under the direction of Louis F. McAuley at State University of New York at Binghamton; the proof of Lemma 5.1 was contained in that dissertation.

2. In this section we shall use a result of Edwards-Kirby [7] and an isotopy control device motivated by a technique of Mason [18] to prove that if M is a compact manifold, then $\bar{H}(M)$ and $\bar{H}_g(M)$ are homogeneous. (A space, X , is homogeneous if given $x, y \in X$, there exists a homeomorphism $\phi: X \rightarrow X$ such that $\phi(x) = y$.)

Lemma 2.1. Suppose $g_0 \in \bar{H}(M)$, $\{\epsilon_i\}_{i=1}^\infty$ is a sequence of positive numbers such that $\epsilon_{i+1} < \epsilon_i/2$, $\epsilon_i < 1/2^i$ for each i , and $\{\phi_i: \bar{H}(M) \rightarrow \bar{H}(M)\}_{i=0}^\infty$ is a sequence of homeomorphisms satisfying:

- (a) $\rho(\phi_i \circ \dots \circ \phi_0(g_0), 1_M) < \epsilon_{i+2}$, for $i \geq 0$.
- (b) If $\rho(f, \phi_{i-1} \circ \dots \circ \phi_0(g_0)) \geq \epsilon_i$, then $\phi_i(f) = f$, for $i \geq 1$.
- (c) If $\rho(f, \phi_{i-1} \circ \dots \circ \phi_0(g_0)) \leq \epsilon_{i+1}$, then $\rho(\phi_i(f), \phi_i \circ \dots \circ \phi_0(g_0)) = \rho(f, \phi_{i-1} \circ \dots \circ \phi_0(g_0))$, for $i \geq 1$.
- (d) If $\rho(f, \phi_{i-1} \circ \dots \circ \phi_0(g_0)) \geq \epsilon_{i+1}$, then $\rho(\phi_i(f), \phi_i \circ \dots \circ \phi_0(g_0)) \geq \epsilon_{i+1}$, for $i \geq 0$.
- (e) If $f \in \bar{H}(M)$, $\rho(f, g_0) = \rho(\phi_0(f), \phi_0(g_0))$.

Then $\phi = \lim_{i \rightarrow \infty} \phi_i \circ \dots \circ \phi_0$ is a homeomorphism of $\bar{H}(M)$ onto itself taking g_0 to 1_M .

Proof. Property (a) implies that $\phi(g_0) = 1_M$. To see that ϕ is onto let $f \in \bar{H}(M)$ and suppose $f \neq 1_M$. Choose a large enough integer, i , so that $\rho(f, 1_M) \geq \epsilon_i$. Since $\phi_i \circ \dots \circ \phi_0$ is a homeomorphism, there is an element $\tilde{f}$ of $\bar{H}(M)$ so that $\phi_i \circ \dots \circ \phi_0(\tilde{f}) = f$.

However, $\rho(f, \phi_i \circ \dots \circ \phi_0(g_0)) \geq \rho(f, 1_M) - \rho(\phi_i \circ \dots \circ \phi_0(g_0), 1_M) \geq \epsilon_i - \epsilon_{i+2} > \epsilon_{i+1}$ and hence, by property (b), $\phi(\tilde{f}) = \phi_i \circ \dots \circ \phi_0(\tilde{f}) = f$.

Similarly, since $\phi|(\phi_i \circ \dots \circ \phi_0)^{-1}(\{f | \rho(f, 1_M) > \epsilon_i\}) = \phi_i \circ \dots \circ \phi_0|(\phi_i \circ \dots \circ \phi_0)^{-1}(\{f | \rho(f, 1_M) > \epsilon_i\})$, ϕ is 1-1 and continuous on $\phi^{-1}(\bar{H}(M) - \{1_M\})$. To show that ϕ is indeed 1-1, we need to show that if $f \neq g_0 \in \bar{H}(M)$, then $\phi(f) \neq 1_M$. Let i be the smallest integer so that $\epsilon_{i+1} \leq \rho(f, g_0)$. By properties (c) and (e), $\rho(\phi_i \circ \dots \circ \phi_0(f), \phi_i \circ \dots \circ \phi_0(g_0)) \geq \epsilon_{i+1}$. And hence by property (d), $\rho(\phi_i \circ \dots \circ \phi_0(f), \phi_i \circ \dots \circ \phi_0(g_0)) \geq \epsilon_{i+1}$. But this implies that

$$\begin{aligned} \rho(\phi(f), 1_M) &\geq \rho(\phi(f), \phi_i \circ \dots \circ \phi_0(g_0)) - \rho(\phi_i \circ \dots \circ \phi_0(g_0), 1_M) \\ &\geq \epsilon_{i+1} - \epsilon_{i+2} > \epsilon_{i+1}/2, \end{aligned}$$

which shows that $\phi(f) \neq 1_M$.

To show that ϕ is continuous at g_0 , note that if $\rho(f, g_0) < \epsilon_i$, then

$$\begin{aligned}
\rho(\phi(f), 1_M) &\leq \rho(\phi(f), \phi_{i-1} \circ \dots \circ \phi_0(f)) + \rho(\phi_{i-1} \circ \dots \circ \phi_0(f), \phi_{i-1} \circ \dots \circ \phi_0(g_0)) \\
&\quad + \rho(\phi_{i-1} \circ \dots \circ \phi_0(g_0), \phi(g_0)) \\
&< \left(\sum_{j=i}^{\infty} 2\epsilon_j \right) + \epsilon_i + \epsilon_{i+1} < 2 \left(\sum_{j=i}^{\infty} \frac{1}{2^j} \right) + \frac{1}{2^i} + \frac{1}{2^{i+1}} < \frac{1}{2^{i-3}}.
\end{aligned}$$

Finally, ϕ^{-1} is obviously continuous on $\bar{H}(M) - \{1_M\}$ and we have shown that if $\rho(f, g_0) \geq \epsilon_{i+1}$, then $\rho(\phi(f), 1_M) > \epsilon_{i+1}/2$. Hence ϕ^{-1} is continuous.

As a corollary to their main theorem Edwards-Kirby [7, p. 80] obtain the following result: Let $\{B_i | 1 \leq i \leq p\}$ be an open cover of M . Then there exists a neighborhood, Q , of 1_M in $H(M)$ and a map $\phi: Q \times [0, p] \rightarrow H(M)$ such that: For each $b \in Q$ and each $j, j = 1, \dots, p$, if $j-1 \leq t \leq j$, then $\phi(b, t)|\tilde{B}_j = \phi(b, j)|\tilde{B}_j$; $\phi(b, 0) = b$ for all $b \in H(M)$; $\phi(b, p) = 1_M$ for all $b \in H(M)$; and $\phi(1_M, t) = 1_M$ for each $t \in [0, p]$. We will make use of the following immediate corollary to the above statement.

Lemma 2.2 (Edwards-Kirby). *Given $\eta > 0$, there is a $\delta > 0$ such that if $b \in H(M)$ and $\rho(b, 1_M) < \delta$, then there is a map $H: [0, p] \rightarrow H(M)$ such that $b_0 = b, b_p = 1_M, \rho(b_t, 1_M) < \eta$ for all $t \in [0, p]$ and for each $j, j = 1, 2, \dots, p$, if $j-1 \leq t \leq j$, then $b_t|_{\tilde{B}_j} = b_j|_{\tilde{B}_j}$ (where $H(t)$ is denoted b_t).*

The map H can be defined so that in addition if $b|_{\partial M} = 1_{\partial M}$, then $b_t|_{\partial M} = 1_{\partial M}$ [7, p. 64].

Lemma 2.3. *Let M be a compact manifold. For every $\epsilon > 0$ there exists a cover $\{B_1, \dots, B_p\}$ of M and an $\epsilon' > 0$ such that if $f \in \bar{H}(M)$ and $(f, 1_M) > \epsilon$, then for each $j, j = 1, 2, \dots, p, \rho(f|_{\tilde{B}_j}, 1_{\tilde{B}_j}) \geq \epsilon'$.*

Proof. It suffices to prove the following statement: Let M be a compact manifold. Given $\epsilon > 0$ there is an $\epsilon' > 0$ such that if $b \in H(M)$ with $\rho(b, 1_M) > \epsilon$, then there exist $x, y \in M$ so that $d(x, y) \geq \epsilon', d(H(x), x) \geq \epsilon'$ and $d(H(y), y) \geq \epsilon'$. (To complete the proof of the lemma, first notice that it suffices to deal only with elements of $H(M)$, and then choose an open cover $\{B_1, \dots, B_p\}$ of M of mesh less than ϵ' .)

The above statement is obviously true if M is a compact 0-manifold. Assume inductively that it has been demonstrated for all compact manifolds of dimension $\leq n-1$. Let M be a compact n -manifold and suppose $\epsilon > 0$ is given. Pick $\eta > 0$ such that $\eta < \epsilon/4$ and such that if S is a subset of M of diameter $< 2\eta$, then S is contained in a ball of diameter less than $\epsilon/8$.

Using the inductive hypothesis and the fact that M can be covered by a finite number of coordinate patches we next choose $\epsilon' > 0$ such that

(a) $\epsilon' < \eta/4$;

(b) For each $x \in M$, there is a ball, B_x , containing x such that if $w \in \partial B_x \cap \text{Int } M$, then $\epsilon' < d(x, w)$ and if $w \in \partial B_x$, $d(x, w) < \eta/2$;

(c) If $g \in H(\partial M)$ with $\rho(g, 1_{\partial M}) > \eta/2$, then there exist elements $x, y \in \partial M$ so that $d(x, y) \geq \epsilon'$, $d(b(x), x) \geq \epsilon'$, and $d(b(y), y) \geq \epsilon'$.

Now, suppose $b \in H(M)$ with $\rho(b, 1_M) > \epsilon$. Pick $x \in M$ such that $d(b(x), x) > \epsilon$. Choose a ball, B_x , so that $x \in B_x$, if $w \in \partial B_x \cap \text{Int } M$ then $d(x, w) > \epsilon'$ and if $w \in \partial B_x$, then $d(x, w) < \eta/2$.

Now either (i) there is a $y \in \partial B_x \cap \partial M$ such that $d(b(y), x) \geq \eta$ or (ii) there is a $y \in \partial B_x \cap \text{Int } M$ such that $d(b(y), x) \geq \eta$ or (iii) $b(\partial B_x) \subset N_\eta(x)$.

In case (i), $d(b(y), y) \geq d(b(y), x) - d(x, y) \geq \eta - \eta/2 > \eta/2$. Hence by property (c) of the definition of ϵ' , the conclusion of the inductive statement is satisfied.

In case (ii), $d(x, y) > \epsilon'$ and $d(b(y), y) > \eta/2 > \epsilon'$. Hence x and y are the desired points.

In case (iii), let B be a ball of diameter less than $\epsilon/8$ bounded by $b(\partial B_x)$. Now, $b(x) \notin B$, for otherwise $d(x, b(x)) < d(x, B) + \text{diam } B < \eta + \epsilon/8 < \epsilon/4 + \epsilon/8 < \epsilon$. Therefore, $b(\tilde{B}_x) = B$ and we can choose $y \in \tilde{B}_x$ such that $d(y, x) \geq \epsilon/2$ (this is possible since $\text{diam } M > \epsilon$). Then $d(y, b(y)) \geq d(y, x) - d(x, b(y)) \geq \epsilon/2 - \eta - \epsilon/8 > \epsilon/2 - \epsilon/4 - \epsilon/8 = \epsilon/8 > \epsilon'$.

Lemma 2.4. *Let M be a compact manifold and let $\epsilon > 0$ be given. Then there is a $\delta = \delta(\epsilon) > 0$ such that if $b \in H(M)$ and $\rho(b, 1_M) < \delta$, then there is a homeomorphism $\psi: \bar{H}(M) \rightarrow \bar{H}(M)$ such that*

(a) *if $\rho(f, 1_M) \geq \epsilon$, then $\psi(f) = f$;*

(b) *if $\rho(f, 1_M) < \delta$, then $\psi(f) = fb^{-1}$.*

Proof. By Lemma 2.3 we can choose a cover $\{B_1, \dots, B_p\}$ and a number $\eta > 0$ such that if $\rho(f, 1_M) \geq \epsilon$, then $\rho(f|_{\tilde{B}_j}, 1_{\tilde{B}_j}) \geq 4\eta$, for $j = 1, \dots, p$.

Then by Lemma 2.2, there is a δ , $0 < \delta \leq \eta$ such that if $\rho(b, 1_M) < \delta$ then there exists a map $H: [0, p] \rightarrow H(M)$ such that $b_0 = b$, $b_p = 1_M$, $\rho(b_t, 1_M) < \eta$ for every $t \in [0, p]$ and for each j , $j = 1, \dots, p$, if $j-1 \leq t \leq j$, then $h_t|_{\tilde{B}_j} = b_j|_{\tilde{B}_j}$. Next for each j , we define a map $\lambda_j: \bar{H}(M) \rightarrow [0, 1]$ by

$$\begin{aligned} \lambda_j(f) &= 0, & \text{if } \rho(f|_{\tilde{B}_j}, 1_{\tilde{B}_j}) \leq 3\eta, \\ &= \frac{\rho(f|_{\tilde{B}_j}, 1_{\tilde{B}_j})}{\eta} - 3, & \text{if } 3\eta \leq \rho(f|_{\tilde{B}_j}, 1_{\tilde{B}_j}) \leq 4\eta, \\ &= 1, & \text{if } \rho(f|_{\tilde{B}_j}, 1_{\tilde{B}_j}) \geq 4\eta. \end{aligned}$$

For each j , $1 \leq j \leq p$, define $\psi_j: \bar{H}(M) \rightarrow \bar{H}(M)$ by

$$\psi_j(f) = fb_{j-1}^{-1}b_{j-\lambda_j(f)}.$$

Define $\psi: \bar{H}(M) \rightarrow \bar{H}(M)$ by $\psi = \psi_p \circ \dots \circ \psi_1$.

To prove that ψ is a homeomorphism it suffices to show that for each j , ψ_j is a homeomorphism. The fact that if $j-1 \leq t \leq j$, then $b_t|_{\tilde{B}_j} = b_j|_{\tilde{B}_j}$ implies that for each positive number, s , $\psi_j|_{\{f \in \bar{H}(M) | \rho(f|_{\tilde{B}_j}, 1_M) = s\}}$ is a homeomorphism of $\{f \in \bar{H}(M) | \rho(f|_{\tilde{B}_j}, 1_M) = s\}$ onto itself. Therefore ψ_j is 1-1 and onto. This fact and the continuity of λ_j proves that ψ_j and its inverse are continuous.

To see that condition (a) is met, note that if $\rho(f, 1_M) \geq \epsilon$, then for each j , $\rho(f|_{\tilde{B}_j}, 1_M) \geq 4\eta$ and hence $\psi_j(f) = f$.

Now suppose $\rho(f, 1_M) < \delta$. Then $\rho(f, 1_M) < \eta$ and hence $\psi_1(f) = fb^{-1}b_1$. Assume inductively that $\psi_{j-1} \circ \dots \circ \psi_1(f) = fb^{-1}b_{j-1}$; then since

$$\rho(fb^{-1}b_{j-1}|_{\tilde{B}_j}, 1_M|_{\tilde{B}_j}) \leq \rho(fb^{-1}b_{j-1}, 1_M) \leq \rho(f, 1_M) + \rho(b^{-1}, 1_M) + \rho(b_{j-1}, 1_M) \leq 3\eta,$$

$$\psi_j(fb^{-1}b_{j-1}) = fb^{-1}b_{j-1}b_{j-1}^{-1}b_j = fb^{-1}b_j.$$

Therefore

$$\psi(f) = \psi_p(\psi_{p-1}(\dots(\psi_1(f))\dots)) = fb^{-1}b_p = fb^{-1}.$$

It is interesting to note that the statement and proof of Lemma 2.4 remain valid if $\bar{H}(M)$ is replaced throughout by $H(M)$.

Lemma 2.5. *Given $a > 0$ there exists $b' = b'(a) > 0$ such that if $b < b'$, $g \in \bar{H}(M)$ such that $\rho(g, 1_M) < b$, and $c > 0$ are given then there exists a homeomorphism $\psi: \bar{H}(M) \rightarrow \bar{H}(M)$ such that*

- (i) $\rho(\psi(g), 1_M) < c$;
- (ii) if $\rho(f, g) \geq a$, then $\psi(f) = f$;
- (iii) if $\rho(f, g) \leq b$, then $\rho(\psi(f), \psi(g)) = \rho(f, g)$;
- (iv) if $\rho(f, g) \geq b$, then $\rho(\psi(f), \psi(g)) \geq b$.

Proof. In Lemma 2.4, let $\epsilon = a/2$. Then let $b' = \min(\delta(\epsilon)/2, a/2)$. Suppose $b < b'$ and $g \in \bar{H}(M)$ are given with $\rho(g, 1_M) < b$. Choose $b \in H(M)$ such that $\rho(b, g) < \min(b, c)$ and $\rho(b, 1_M) < b$. Then by Lemma 2.4, there exists a homeomorphism $\psi: \bar{H}(M) \rightarrow \bar{H}(M)$ such that if $\rho(f, 1_M) \geq a/2$, then $\psi(f) = f$ and if $\rho(f, 1_M) < 2b$, then $\psi(f) = fb^{-1}$. It is trivial to check that ψ satisfies conditions (i)–(iv).

Theorem 2.6. *If M is a compact manifold, $\bar{H}(M)$ is homogeneous.*

Proof. Let g_0 be an arbitrary element of $\bar{H}(M)$. It is sufficient to show that there exist sequences $\{\epsilon_i\}_{i=1}^\infty$ and $\{\phi_i: \bar{H}(M) \rightarrow \bar{H}(M)\}_{i=0}^\infty$ satisfying the hypothesis

of Lemma 2.1 and hence that there exists a homeomorphism $\phi: \bar{H}(M) \rightarrow \bar{H}(M)$ taking g_0 to 1_M .

Choose a sequence of positive numbers $\{\epsilon_i\}_{i=1}^\infty$ such that for each i , $\epsilon_{i+1} < \epsilon_i/2$, $\epsilon_i < 1/2^i$ and $\epsilon_{i+1} \leq b'(\epsilon_i)$, where $b'(\epsilon_i)$ is the number promised in Lemma 2.5 for $a = \epsilon_i$. Let $b \in H(M)$ be chosen so that $\rho(b, g_0) < \epsilon_2$. Define the homeomorphism $\phi_0: \bar{H}(M) \rightarrow \bar{H}(M)$ by $\phi_0(f) = fb^{-1}$. Note that $\rho(\phi_0(g_0), 1_M) = \rho(g_0b^{-1}, 1_M) = \rho(g_0, b) < \epsilon_2$ and that if $f \in \bar{H}(M)$, $\rho(\phi_0(f), \phi_0(g_0)) = \rho(f, g_0)$.

Now assume inductively that homeomorphisms $\phi_0, \phi_1, \dots, \phi_{j-1}$ have been defined satisfying conditions (a)–(d) of Lemma 2.1.

In Lemma 2.5, let $a = \epsilon_j$, $b = \epsilon_{j+1}$, $c = \epsilon_{j+2}$ and $g = \phi_{j-1} \circ \dots \circ \phi_0(g_0)$. By the inductive hypothesis, $\rho(g, 1_M) = \rho(\phi_{j-1} \circ \dots \circ \phi_0(g_0), 1_M) < \epsilon_{j+1}$. Therefore, since $\epsilon_{j+1} < b'(\epsilon_j)$, by Lemma 2.5, there exists a homeomorphism $\phi_j: \bar{H}(M) \rightarrow \bar{H}(M)$ such that:

- (i) $\rho(\phi_j \circ \dots \circ \phi_0(g_0), 1_M) < \epsilon_{j+2}$;
- (ii) if $\rho(f, \phi_{j-1} \circ \dots \circ \phi_0(g_0)) \geq \epsilon_j$, then $\phi_j(f) = f$;
- (iii) if $\rho(f, \phi_{j-1} \circ \dots \circ \phi_0(g_0)) \leq \epsilon_{j+1}$, then;

$$\rho(\phi_j(f), \phi_j \circ \dots \circ \phi_0(g_0)) = \rho(f, \phi_{j-1} \circ \dots \circ \phi_0(g_0));$$

- (iv) if $\rho(f, \phi_{j-1} \circ \dots \circ \phi_0(g_0)) \geq \epsilon_{j+1}$, then $\rho(\phi_j(f), \phi_j \circ \dots \circ \phi_0(g_0)) \geq \epsilon_{j+1}$.

But these are precisely conditions (a)–(d) of Lemma 2.1 that the homeomorphism ϕ_j was to satisfy (condition (e) refers only to ϕ_0). The proof of Theorem 2.6 is completed.

In order to simplify notation we used the symbol $\bar{H}(M)$ throughout this section. The identical proofs also show that $\bar{H}_\theta(M)$ is homogeneous (recall the comment following the statement of Lemma 2.2).

Theorem 2.7. *Let M be a compact manifold. Then $\bar{H}_\theta(M)$ is homogeneous.*

3. In this section we consider $\bar{H}_\theta(B^n)$, where B^n is the Euclidean n -ball.

Theorem 3.1. *$\bar{H}_\theta(B^n)$ is locally contractible.*

Proof. Since $\bar{H}_\theta(B^n)$ is homogeneous, it suffices to show that $\bar{H}_\theta(B^n)$ is locally contractible at 1_{B^n} . We show, using an Alexander-type homotopy, that $N_\epsilon(1_{B^n})$ is contractible within itself to 1_{B^n} .

For any $f \in \bar{H}_\theta(B^n)$, define $\hat{f}: R^n \rightarrow R^n$ by

$$\begin{aligned} \hat{f}(x) &= f(x), & x \in B^n, \\ &= x, & x \notin B^n. \end{aligned}$$

Next define $A: \bar{H}_\theta(B^n) \times I \rightarrow \bar{H}_\theta(B^n)$ by

$$A(f, t)(x) = \frac{1-t}{1+t} \hat{f}\left(\frac{1+t}{1-t}x\right), \quad 0 \leq t < 1, \\ = x, \quad t = 1.$$

We note that A is continuous, $A(f, 0) = f$, $A(f, 1) = 1_{B^n}$ and $A(f, t) \in \bar{H}_\partial(B^n)$ for all $f \in \bar{H}_\partial(B^n)$ and $t \in I$. Furthermore, since $\rho(f, 1_{B^n}) < \epsilon$ implies that $\rho(A(f, t), 1_{B^n}) < \epsilon$, A contracts $N_\epsilon(1_{B^n})$ within itself to 1_{B^n} for every $\epsilon > 0$.

Actually it is possible to prove that $\bar{H}_\partial(B^n)$ is locally contractible without knowing that $\bar{H}_\partial(B^n)$ is homogeneous. See [12].

For the special case, $n = 4$, it is not known whether $\bar{H}_\partial(B^4)$ is equal to $Ce_\partial(B^4) = \{f: B^4 \rightarrow B^4 \mid f|_{\partial B^4} = 1_{\partial B^4} \text{ and } f \text{ is cellular}\}$. However, the map A does show that $Ce_\partial(B^4)$ is locally contractible at 1_{B^4} .

It was mentioned in the introduction that $\bar{H}_\partial(B^2)$ is an AR [13]. The following theorem is also contained in [13]:

Theorem 3.2. *Let α be an open cover of $\bar{H}_\partial(B^2)$. Then there exists a locally finite polyhedron, P , and maps $b: \bar{H}_\partial(B^2) \rightarrow P$, $g: P \rightarrow \bar{H}_\partial(B^2)$ and $\theta: \bar{H}_\partial(B^2) \times I \rightarrow \bar{H}_\partial(B^2)$ such that*

- (a) *for each $f \in H_\partial(B^2)$ there is an element, U_f , of α such that $\theta(f, t) \in U_f$, for each $t \in I$;*
- (b) *$\theta(f, 1) = f$, for each $f \in \bar{H}_\partial(B^2)$;*
- (c) *$\theta(f, 0) = gb(f)$, for each $f \in \bar{H}_\partial(B^2)$;*
- (d) *$\theta(f, t) \in H_\partial(B^2)$ for each $f \in \bar{H}_\partial(B^2)$ and $t \in [0, 1)$.*

This theorem will be used in §5. Theorem 3.2 implies that the inclusion map $i: H_\partial(B^2) \rightarrow \bar{H}_\partial(B^2)$ is a homotopy equivalence. Siebermann [20] has asked whether $i: H(M) \rightarrow \bar{H}(M)$ is a homotopy equivalence, for an arbitrary compact manifold M .

4. In this section we obtain some general topological results concerning the closure of a uniformly locally contractible space (compare with [8]). These results are then used in order to give partial solutions to the following unsolved problems:

- (i) Let M be a compact manifold. Given $\delta > 0$ does there exist a continuous function $\phi_\delta: \bar{H}(M) \rightarrow H(M)$ with the property that for each $g \in \bar{H}(M)$, $\rho(g, \phi_\delta(g)) < \delta$?
- (ii) Let M be a compact manifold. Is $\bar{H}(M)$ locally contractible?

Proposition 4.1. *Let Y be a metric space and X be a uniformly locally contractible subset of Y . Let $\delta > 0$ be given and let $f: P \rightarrow \bar{X}$ be a map of an arbitrary locally finite polyhedron, P , into $\bar{X}$. Then there exists a map $\phi: P \times I \rightarrow \bar{X}$ so that for each $p \in P$:*

- (a) $\phi(p, 0) = f(p)$;
- (b) if $t \neq 0$, $\phi(p, t) \in X$;
- (c) if $t \in I$, $d(f(p), \phi(p, t)) < \delta$.

Proof. Let $\delta_1, \delta_2, \dots$ be a decreasing sequence of positive numbers such that $\delta_1 \leq \delta/3$, $\delta_n \leq 1/3n$, and if $A_n \subset X$ with $\text{diam}(A_n) < 3\delta_{n+1}$, then $i: A_n \rightarrow X$ is null-homotopic in a subset of X of diameter less than δ_n .

Suppose $P \times (0, 1]$ has a fixed locally finite triangulation. If τ is a simplex of $P \times (0, 1]$, define the two positive integers, m_τ and n_τ as follows:

$$m_\tau = \max \{ \dim \sigma \mid \tau < \sigma \},$$

$$n_\tau = \min \{ n \mid n \text{ is an integer and if } \tau < \sigma, \text{ then } \sigma \subset P \times [1/n, 1] \}.$$

Note that if $\tau' < \tau$, $m_{\tau'} \geq m_\tau$ and $n_{\tau'} \geq n_\tau$.

Consider $P \times (0, 1]$ to have a locally finite triangulation such that if τ is a simplex of $P \times (0, 1]$, then $\text{diam}(f(\pi_1(\tau))) < \delta_{m_\tau + n_\tau}$ (where π_1 is projection on the first coordinate).

We will define a map $\psi: P \times (0, 1] \rightarrow X$ by induction on the skeleta of $P \times (0, 1]$. Define $\psi_0: (P \times (0, 1])^0 \rightarrow X$ as follows. If σ is a 0-simplex of $P \times (0, 1]$, let $\psi_0(\sigma)$ be an element of X such that $\rho(f(\pi_1(\sigma)), \psi_0(\sigma)) < \delta_{m_\sigma + n_\sigma}$.

Assume inductively that there exist maps $\psi_1, \dots, \psi_{k-1}$ with the following properties for $j = 1, \dots, k-1$:

- (i) ψ_j maps $(P \times (0, 1])^j$ into X ;
- (ii) ψ_j extends ψ_{j-1} ;
- (iii) if τ is a j -simplex of $P \times (0, 1]$, then $\text{diam} \psi_j(\tau) < \delta_{m_\tau + n_\tau - j}$.

We define $\psi_k: (P \times (0, 1])^k \rightarrow X$ as follows: If τ is a j -simplex of $(P \times (0, 1])^k$, $j < k$, let $\psi_k|_\tau = \psi_{k-1}|_\tau$. If τ is a k -simplex of $(P \times (0, 1])^k$, we note that $\text{diam}(\psi_{k-1}(\text{bdry } \tau)) < 3\delta_{m_\tau + n_\tau - (k-1)}$. If $k \neq 1$, this is true by the inductive hypothesis since if $\tau' < \tau$, $\text{diam}(\psi_{k-1}(\tau')) < \delta_{m_{\tau'} + n_{\tau'} - (k-1)} < \delta_{m_\tau + n_\tau - (k-1)}$. (Remember, if $\tau' < \tau$, then $m_{\tau'} \geq m_\tau$ and $n_{\tau'} \geq n_\tau$.) In the special case $k = 1$, suppose $\tau = \langle \tau', \tau'' \rangle$. Then

$$\begin{aligned} \text{diam}(\psi_{k-1}(\text{bdry } \tau)) &= \rho(\psi_0(\tau'), \psi_0(\tau'')) \\ &\leq \rho(\psi_0(\tau'), f(\pi_1(\tau'))) + \rho(f(\pi_1(\tau')), f(\pi_1(\tau''))) + \rho(f(\pi_1(\tau'')), \psi_0(\tau'')) \\ &< \delta_{m_{\tau'} + n_{\tau'}} + \text{diam}(f(\pi_1(\tau))) + \delta_{m_{\tau''} + n_{\tau''}} \\ &< \delta_{m_\tau + n_\tau} + \delta_{m_\tau + n_\tau} + \delta_{m_\tau + n_\tau} = 3\delta_{m_\tau + n_\tau}. \end{aligned}$$

In either case, $\psi_k|_{\text{bdry } \tau} = \psi_{k-1}|_{\text{bdry } \tau}$ can be extended to a map, $\psi_k|_\tau$, in such a way that $\psi_k(\tau)$ is a subset of X of diameter less than $\delta_{m_\tau + n_\tau - (k-1) - 1} = \delta_{m_\tau + n_\tau - k}$. We have shown that ψ_k satisfies the inductive hypothesis.

Then define $\psi: P \times (0, 1] \rightarrow X$ by $\psi(p, t) = \lim_{j \rightarrow \infty} \psi_j(p, t)$.

Finally define $\phi: P \times [0, 1] \rightarrow X$ by

$$\begin{aligned}\phi(p, t) &= \psi(p, t), & \text{if } t \neq 0, \\ &= f(p), & \text{if } t = 0.\end{aligned}$$

For $t \neq 0$, local continuity is assured since $P \times (0, 1]$ is locally finite. If $(p, t) \in P \times (0, 1]$ and $t < 1/n$, let σ be a simplex containing (p, t) and suppose $\dim \sigma = s$. The diameter of $\phi(\sigma)$ is less than $\delta_{m_\sigma + n_\sigma - s}$ by property (c) of the inductive statement. But, $s \leq m_\sigma$ and $n_\sigma \geq n$; hence $\text{diam}(\phi(\sigma)) < \delta_{n_\sigma} < \delta_n$. Also, if σ' is a vertex of σ , $\rho(f(\pi_1(\sigma')), \phi(\sigma')) < \delta_{n_\sigma + m_\sigma} < \delta_n$. Therefore

$$\begin{aligned}\rho(\phi(p, t), \phi(p, 0)) &\leq \rho(\phi(p, t), \phi(\sigma')) + \rho(\phi(\sigma'), f(\pi_1(\sigma'))) + \rho(f(\pi_1(\sigma')), f(p)) \\ &< 3\delta_n < 3(1/3n) = 1/n.\end{aligned}$$

We have thereby shown that ϕ is continuous.

Finally, if (p, t) is any element of $P \times (0, 1]$, $\rho(\phi(p, t), f(p)) = \rho(\phi(p, t), \phi(p, 0)) < 3\delta_1 < \delta$.

Proposition 4.2. *Let γ be a metric space and X be a uniformly locally contractible subset of γ . Let $\delta > 0$ be given and let $f: A \rightarrow \bar{X}$ be a map of an arbitrary ANR, A , into $\bar{X}$. Then given $\delta > 0$, there exists a map $\psi: A \times I \rightarrow \bar{X}$ so that for each $a \in A$:*

- (a) $\psi(a, 0) = f(a)$;
- (b) $\psi(a, 1) \in X$;
- (c) $d(f(a), \psi(a, t)) < \delta$ for all $t \in I$.

Proof. Choose a cover, $\mathcal{B}$, of A with the property that if $B \in \mathcal{B}$, then $\text{diam } f(B)$ is less than $\delta/2$.

Since A is an ANR, by a theorem of Hanner [11], there are a locally finite polyhedron P , maps $g: A \rightarrow P$ and $w: P \rightarrow A$ and a homotopy $H: A \times I \rightarrow A$ such that $H(a, 0) = a$, $H(a, 1) = wg(a)$ and for each $a \in A$, $H(a, I) \subset B$, for some $B \in \mathcal{B}$.

By Proposition 4.1, there exists a homotopy $\phi: P \times I \rightarrow \bar{X}$ such that $\phi(p, 0) = fw(p)$, $\phi(p, 1) \in X$ and $d(fw(p), \phi(p, t)) < \delta/2$. Define $\psi: A \times I \rightarrow X$ by

$$\begin{aligned}\psi(a, t) &= f(H(a, 2t)), & 0 \leq t \leq \frac{1}{2}, \\ &= \phi(g(a), 2t - 1), & \frac{1}{2} \leq t \leq 1.\end{aligned}$$

Note that ψ is continuous, since $f(H(a, 1)) = fw(g(a)) = \phi(g(a), 0)$. Also, $\psi(a, 0) = f(H(a, 0)) = f(a)$ and $\psi(a, 1) = \phi(g(a), 1) \in X$. By the definition of H , if $0 \leq t \leq \frac{1}{2}$, then $d(\psi(a, 0), \psi(a, t)) < \delta/2$ and by the definition of ϕ , if $\frac{1}{2} \leq t \leq 1$, then $d(\psi(a, \frac{1}{2}), \psi(a, t)) < \delta/2$.

Proposition 4.3. *Let Y be a metric space and X be a uniformly locally contractible subset of Y . Given $\epsilon > 0$, there is a $\delta > 0$ such that if $y \in \bar{X}$ and $f: A \rightarrow N_\delta(y) \cap \bar{X}$ is a map of an arbitrary ANR, A , into $N_\delta(y) \cap \bar{X}$, then there is a map $G: A \times I \rightarrow N_\epsilon(y) \cap \bar{X}$ such that for all $a \in A$, $G(a, 0) = f(a)$ and $G(a, 1) = y$.*

Proof. Let $\epsilon > 0$ be given and choose $\delta > 0$ small enough so that there exists a homotopy $H: (N_{2\delta}(y) \cap X) \times I \rightarrow N_\epsilon(y) \cap \bar{X}$ such that $H(x, 0) = x$ and $H(x, 1) = y$, for all $x \in N_{2\delta}(y) \cap X$. Now suppose A is an arbitrary ANR and $f: A \rightarrow N_\delta(y) \cap \bar{X}$ is given. We will make use of the map ψ defined in Proposition 4.2 to define $G: A \times I \rightarrow N_\epsilon(y)$.

Let

$$\begin{aligned} G(a, t) &= \psi(a, 2t), & 0 \leq t \leq \frac{1}{2}, \\ &= H(\psi(a, 1), 2t - 1), & \frac{1}{2} \leq t \leq 1. \end{aligned}$$

Then G is continuous, maps $A \times I$ into $N_\epsilon(y) \cap \bar{X}$, $G(a, 0) = \psi(a, 0) = f(a)$ for all $a \in A$ and $G(a, 1) = H(\psi(a, 1), 1) = y$ for all $a \in A$.

As mentioned in the introduction, if M is a compact manifold, Černavskii [6] and Edwards-Kirby [7] have shown that $H(M)$ and $H_\partial(M)$ are locally contractible spaces. Using the fact that if f, g, b are arbitrary elements of $H(M)$ then $\rho(f, g) = \rho(fb^{-1}, gb^{-1})$, it is trivial to show that $H(M)$ and $H_\partial(M)$ are uniformly locally contractible. Therefore, Propositions 4.1, 4.2, and 4.3 hold where X is replaced by $H(M)$ (or $H_\partial(M)$) and $\bar{X}$ by $\bar{H}(M)$ (or $\bar{H}_\partial(M)$). A space Z is said to be weakly locally contractible at $z \in Z$ if given any open set U containing z , there exists an open set V with $z \in V \subset U$ such that if P is any locally finite polyhedron and $f: P \rightarrow V$ any mapping, then there is a map $G: P \times I \rightarrow U$ such that $G(p, 0) = f(p)$ and $G(p, 1) = z$ for all $p \in P$. To indicate our partial solutions to the problems discussed at the beginning of this section we will restate some of the results of the preceding propositions in the following theorem:

Theorem 4.4. *Let M be a compact manifold.*

(i) *Given $\delta > 0$ and a map $F: A \rightarrow \bar{H}(M)$ of an ANR, A , into $\bar{H}(M)$, then there exists a continuous function $\phi_\delta: A \rightarrow H(M)$ with the property that $\rho(F(a), \phi_\delta(a)) < \delta$, for all $a \in A$. (This statement also holds if $\bar{H}(M)$ is replaced by $\bar{H}_\partial(M)$ and $H(M)$ by $H_\partial(M)$.)*

(ii) *$\bar{H}(M)$ and $\bar{H}_\partial(M)$ are weakly locally contractible.*

A closed set K of a space X is called Z -set if for any nonempty homotopically trivial open set U in X , $U - K$ is nonempty and homotopically trivial. R. D. Anderson [2] has shown that if $\{Z_i\}_{i \geq 0}$ is a countable collection of Z -sets in l_2 , then $l_2 - \bigcup_{i \geq 0} Z_i$ is homeomorphic to l_2 .

Proposition 4.5. *If M is a compact manifold and $\bar{H}(M)$ is locally homeomorphic to l_2 , then $H(M)$ is locally homeomorphic to l_2 .*

Proof. Suppose N is a neighborhood of 1_M in $\bar{H}(M)$ that is homeomorphic to l_2 . We will show that $N \cap H(M)$ is homeomorphic to l_2 , thereby demonstrating the proposition.

For each positive integer i , let $Z_i = \{f \in N \mid \text{there exists } x \in M \text{ with } \text{diam } f^{-1}(x) \geq 1/i\}$. Now, $N \cap H(M) = N - \bigcup_{i>0} Z_i$. So to show that $N \cap H(M)$ is homeomorphic to l_2 it suffices, by Anderson's theorem, to prove that for each i , Z_i is a Z -set. By standard arguments (cf., [14, p. 57]), Z_i is a closed (rel N) subset of N . Suppose U is a nonempty homotopically trivial open subset of N and let $f: S^{n-1} \rightarrow U - Z_i$ be given. Then choose a map $g: B^n \rightarrow U$ so that $g|_{S^{n-1}} = f$. By Proposition 4.1 (applied to the case where $X = H(M)$, $P = B^n$ and $\delta = \rho(g(B^n), \tilde{U})$) there exists a map $\phi: B^n \times I \rightarrow U$ such that, for each $w \in B^n$, $\phi(w, t) \in H(M)$ if $t \neq 1$ and $\phi(w, 1) = g(w)$. Now label the points of B^n radially so that $B^n = \{tx \mid x \in S^{n-1}, 0 \leq t \leq 1\}$. Define $F: B^n \rightarrow U - Z_i$ by $F(tx) = \phi(tx, t)$. Note that $F|_{S^{n-1}} = f$ and for each $t < 1$, $F(tx) = \phi(tx, t) \in H(M) \cap U \subset U - Z_i$.

5. In this section we show that if M^2 is a compact 2-manifold, then $\bar{H}_\partial(M^2)$ is locally contractible. Lemma 5.1 will be proven using a slight modification of the lifting process of Edwards-Kirby (and is valid in all dimensions). We then make use of the canonical approximation result for $\bar{H}_\partial(B^2)$ (Theorem 3.2) to show that $\bar{H}_\partial(M^2)$ is locally contractible.

If U is a subset of a manifold M , a proper imbedding of U into M is an imbedding $b: U \rightarrow M$ such that $b^{-1}(\partial M) = U \cap \partial M$. If C and U are compact subsets of M with $C \subset U$, let $I(U, C; M)$ denote the set of proper imbeddings of U into M which are the identity when restricted to C . Let $\bar{I}(U, C; M)$ denote the set of all mappings of U into M which can be approximated arbitrarily closely by elements of $I(U, C; M)$.

The statement of Lemma 5.1 corresponds to that of Lemma 4.1 of [7] except that in the situation under consideration it is not possible to obtain a homotopy by using the Alexander isotopy as in the Edwards-Kirby paper. (The inversion device of Siebenmann (see [20, Main Idea]) was developed to handle a similar situation and is valid in all dimensions. Unfortunately, it also does not lead to the desired homotopy.) We are, however, able to obtain a homotopy for the 2-manifold case (Proposition 5.3).

We have omitted the details of the proof of Lemma 5.1 in those places where the argument parallels that of [7].

Lemma 5.1. *Let positive numbers a, b, δ be given with $1 < a < b \leq 2$. Then there exists a positive number $\epsilon_{(a,b,\delta)} \leq \delta$ so that if*

$$\begin{aligned}\bar{T}_{(a,b,\delta)} = \{f \in \bar{T}(B^k \times 4B^n, \partial B^k \times 4B^n; B^k \times R^n) | \\ \rho(f|B^k \times ((a+b)/2)B^n, 1_{B^k \times ((a+b)/2)B^n}) < \epsilon_{(a,b,\delta)}\}\end{aligned}$$

then there exists a continuous function

$$\phi_{(a,b,\delta)}: \bar{T}_{(a,b,\delta)} \rightarrow \bar{T}(D^k - 4B^n, \partial B^k \times 4B^n; B^k \times R^n)$$

such that for all $f \in \bar{T}_{(a,b,\delta)}$:

- (1) $\phi_{(a,b,\delta)}(f)|B^k \times (4B^n - bB^n) = 1_{B^k \times (4B^n - bB^n)}$;
- (2) $\phi_{(a,b,\delta)}(f)|B^k \times aB^n = f|B^k \times aB^n$;
- (3) $\rho(\phi_{(a,b,\delta)}(f), 1_{B^k \times 4B^n}) < \delta/2$.

Proof. As in the proof of Lemma 4.1 of [7] it suffices to show that there exists an $\epsilon'_{(a,b,\delta)} > 0$ and a map $\phi: \bar{T}'_{(a,b,\delta)} \rightarrow \bar{T}(B^k \times 4B^n, \partial B^k \times 4B^n; B^k \times R^n)$ satisfying conditions (1)–(3), where

$$\begin{aligned}\bar{T}'_{(a,b,\delta)} = \{f \in \bar{T}(B^k \times 4B^n, (\partial B^k \times 4B^n) \cup ([\delta/16, 1]B^k \times 3B^n); B^k \times R^n) | \\ \rho(f|B^k \times ((a+b)/2)B^n, 1_{B^k \times ((a+b)/2)B^n}) < \epsilon'_{(a,b,\delta)}\}\end{aligned}$$

plays the role of $\bar{T}_{(a,b,\delta)}$.

We shall produce such a map ϕ by assigning to each pair (b, i) , where $b \in \bar{T}'_{(a,b,\delta)} \cap I(B^k \times 4B^n, (\partial B^k \times 4B^n) \cup ([\delta/16, 1]B^k \times 3B^n); B^k \times R^n) = \bar{T}'_{(a,b,\delta)} \cap I$ and i is a positive integer, an imbedding $b^i: B^k \times 4B^n \rightarrow B^k \times R^n$ in such a way that:

- (i) $b^i|B^k \times aB^n = b|B^k \times aB^n$;
- (ii) $b^i|(\partial B^k \times 4B^n) \cup B^k \times (4B^n - bB^n) = 1_{(\partial B^k \times 4B^n) \cup B^k \times (4B^n - bB^n)}$;
- (iii) given $\eta > 0$ there exists $\delta > 0$ and an integer N such that if $i, j > N$ and $d(b(x), g(x)) < \delta$ for all $x \in B^k \times ((a+b)/2)B^n$, then $d(b^i(x), g^j(x)) < \eta$ for all $x \in B^k \times 4B^n$;

- (iv) if $b|B^k \times ((a+b)/2)B^n = 1_{B^k \times ((a+b)/2)B^n}$, then $\rho(b^i, 1_{B^k \times 4B^n}) < \delta/4$.

The construction of a collection of such imbeddings would complete the proof of Lemma 5.1 as the following argument indicates: Let $f \in \bar{T}'_{(a,b,\delta)}$ and choose a sequence of elements of $\bar{T}'_{(a,b,\delta)} \cap I, \{b_i\}$, which converges to f . Then define $\phi(f)$ to be $\lim_{i \rightarrow \infty} b_i^i$. By property (i), $\phi(f)|B^k \times aB^n = f|B^k \times aB^n$ since for any $x \in B^k \times aB^n$, $b_i^i(x) = b_i(x)$ and $\{b_i(x)\}$ converges to $f(x)$. Similarly, property (ii) assures that $\phi(f)|(\partial B^k \times 4B^n) \cup B^k \times (4B^n - bB^n)$ is the identity map. Property (iii) guarantees that ϕ is well defined (independent of the choice of the sequence $\{b_i\}$), $\phi(f)$ is an element of $\bar{T}(B^k \times 4B^n, \partial B^k \times 4B^n; B^k \times R^n)$, and that ϕ is continuous. Property (iv) guarantees that if $f|B^k \times ((a+b)/2)B^n =$

$1_{B^k \times ((a+b)/2)B^n}$, then $\rho(\phi(f), 1_{B^k \times 4B^n}) \leq \delta/4$. Hence, we can choose $\epsilon'_{(a,b,\delta)}$ small enough (making use of the continuity of ϕ) and thereby redefine $\bar{I}'_{(a,b,\delta)}$ so that $\rho(\phi(f), 1_{B^k \times 4B^n}) < \delta/2$ for all $f \in \bar{I}'_{(a,b,\delta)}$.

Given a pair (b, i) , we shall make use of a modification of the lifting diagram of [7] to obtain the map b^i .

$$\begin{array}{ccc}
 B^k \times R^n & \xrightarrow{h^i} & B^k \times R^n \\
 \uparrow \gamma^{-1} & & \uparrow \gamma^{-1} \\
 B^k \times R^n & \xrightarrow{\bar{h}^i} & B^k \times R^n \\
 \downarrow e & & \downarrow e \\
 B^k \times T^n & \xrightarrow{\bar{h}^i} & B^k \times T^n \\
 \uparrow \text{id} & & \uparrow w_i \\
 B^k \times T^n & \xrightarrow{\bar{h}} & B^k \times T^n \\
 \uparrow i' & & \uparrow i' \\
 (B^k \times T^n) - (2D^k \times 2D^n) & \xrightarrow{\tilde{h}} & (B^k \times T^n) - (D^k \times D^n) \\
 \uparrow i & & \uparrow i \\
 B^k \times (T^n - 2D^n) & \xrightarrow{\hat{h}} & B^k \times (T^n - D^n) \\
 \downarrow \alpha & & \downarrow \alpha \\
 B^k \times 4B^n & \xrightarrow{h} & B^k \times R^n
 \end{array}$$

Let T^n denote the n -fold product of S^1 and identify an n -cell in T^n with $2B^n$. Let $\bar{e}: R^n \rightarrow T^n$ be a covering projection such that $\bar{e}|_{2B^n}$ is the identity and let $e: B^k \times R^n \rightarrow B^k \times T^n$ be equal to $\text{id} \times \bar{e}$. Let $D^n, 2D^n, 3D^n, 4D^n$ be concentric n -cells in $T^n - 2B^n$ such that $jD^n \subset \text{Int}(j+1)D^n$ for $j = 1, 2, 3$. Also let $D^k, 2D^k, 3D^k, 4D^k$ be concentric k -cells in $\text{Int } B^k$ such that $\delta/16B^k \subset D^k$ and $jD^k \subset \text{Int}(j+1)D^k$ for $j = 1, 2, 3$. In addition, let $4D^n$ and $4D^k$ be chosen small enough so that the diameter of each component of $e^{-1}(4D^k \times 4D^n)$ is less than $\delta/4$. Then let $\bar{\alpha}: T^n - D^n \rightarrow \text{Int}((a+b)/2)B^n$ be a fixed immersion with the property that $\bar{\alpha}$ restricted to $((3a+b)/4)B^n$ is the identity [17]. We shall choose $\epsilon'_{(a,b,\delta)}$ small enough so that $b(B^k \times aB^n) \subset B^k \times ((3a+b)/4)B^n$ for all $b \in \bar{I}'_{(a,b,\delta)} \cap I$. Let α denote the product immersion $\text{id} \times \bar{\alpha}: B^k \times (T^n - D^n) \rightarrow B^k \times \text{Int}((a+b)/2)B^n$. If $\epsilon'_{(a,b,\delta)}$ is chosen small enough, for each $b \in \bar{I}'_{(a,b,\delta)} \cap I$ we can canonically choose an embedding $\hat{h}: B^k \times (T^n - 2D^n) \rightarrow B^k \times (T^n - D^n)$ so that the lower square of the diagram commutes.

We note that $\hat{h}|_{(B^k - 2D^k) \times (T^n - 2D^n)}$ is the identity map, since b is the identity on $[\delta/16, 1]B^k \times 3B^n$ and $\delta/16B^k \subset D^k$. Therefore, to obtain $\hat{h}$, we extend $\hat{h}$ to be the identity on $(B^k - 2D^k) \times T^n$. If $\epsilon'_{(a,b,\delta)}$ is chosen small enough, then if $x \in (3D^k \times 3D^n) - (2D^k \times 2D^n)$, then $\hat{h}(x) \in (3 + 1/2)D^k \times (3 + 1/2)D^n$. Consider the restriction of $\hat{h}$ to $(B^k \times T^n) - (3D^k \times 3D^n)$. By the Schoenflies theorem [5] we can extend this restriction of $\hat{h}$ to a homeomorphism $\bar{h}: B^k \times T^n \rightarrow B^k \times T^n$. This extension may not be canonical, i.e., if $\{\hat{h}_i\}$ is a Cauchy sequence of imbeddings, it does not follow that $\{\bar{h}_i\}$ is a Cauchy sequence of imbeddings.

Until this point the construction of the diagram is independent of i and varies only with the imbedding b . Consider $4D^k \times 4D^n$ to be $\{tx | x \in \partial(4D^k \times 4D^n), 0 \leq t \leq 4\}$. We then define the homeomorphism $w_i: B^k \times T^n \rightarrow B^k \times T^n$ which takes $(3 + 1/2)D^k \times (3 + 1/2)D^n$ to $(1/i)D^k \times (1/i)D^n$ by

$$(a) \quad w_i|_{B^k \times T^n - (4D^k \times 4D^n)} = 1_{B^k \times T^n - (4D^k \times 4D^n)},$$

$$(b) \quad \begin{aligned} w_i(tx) &= [(t - (3 + \frac{1}{2}))(2) - (4 - 1/i) + 1/i]x, & 3 + \frac{1}{2} \leq t \leq 4, \\ &= \frac{t}{(3 + \frac{1}{2})i} x, & 0 \leq t \leq 3 + \frac{1}{2}. \end{aligned}$$

Then $\bar{b}^i: B^k \times T^n \rightarrow B^k \times T^n$ is defined by $\bar{b}^i(x) = w_i \bar{b}(x)$. Then $\bar{b}^i$ lifts to the homeomorphism $\tilde{b}^i: B^k \times R^n \rightarrow B^k \times R^n$. We note that $\tilde{b}^i$ has the property that for some constant, M , $d(\tilde{b}^i, \text{id}) < M$. Finally, let $\gamma: \text{Int}(bB^k \times bB^n) \rightarrow R^k \times R^n$ be a homeomorphism which is a radial expansion and is the identity on $((3a + b)/4)B^k \times ((3a + b)/4)B^n$. We extend $\tilde{b}^i$ by the identity to a homeomorphism $\tilde{h}^i: R^k \times R^n \rightarrow R^k \times R^n$ and define $h^i: B^k \times 4B^n \rightarrow B^k \times R^n$ by

$$\begin{aligned} h^i(x) &= \gamma^{-1} \tilde{h}^i \gamma(x), & x \in B^k \times bB^n, \\ &= x, & x \in B^k \times (4B^n - bB^n). \end{aligned}$$

Since $d(\tilde{b}^i, \text{id}) < M$, h^i is continuous and therefore is a homeomorphism.

To check property (i) note that $\alpha i^{-1} i'^{-1} w_i^{-1} e \gamma(x) = x$ for all $x \in B^k \times ((3a + b)/4)B^n$, and that ϵ' was chosen small enough so that if $x \in B^k \times aB^n$, then $b(x) \in B^k \times ((3a + b)/4)B^n$. That property (ii) is satisfied was guaranteed by the choice of the homeomorphism γ .

Each stage in the construction of h^i is canonical except the use of the Schoenflies theorem. Therefore, to show that property (iii) is satisfied we must only show that given $\eta' > 0$, there exists $\delta' > 0$ and an integer N such that if $d(\tilde{b}^i(x), \tilde{g}^i(x)) < \delta'$ for all $x \in (B^k \times T^n) - (2D^k \times 2D^n)$ and if $i, j > N$, then $\rho(\bar{b}^i, \bar{g}^i) < \eta'$. Let N be chosen so that $2/N < \eta'$ and $\delta' < \eta'/16$. If $x \in 3D^k \times 3D^n$, then $\tilde{b}^i(x)$ and $\tilde{g}^i(x)$ are elements of $(3 + 1/2)D^k \times (3 + 1/2)D^n$ and hence $d(\tilde{b}^i(x), \tilde{g}^i(x)) < 2/N < \eta'$. If $x \notin 3D^k \times 3D^n$, $\bar{b}^i(x) = \tilde{b}^i(x)$ and $\bar{g}^i(x) = \tilde{g}^i(x)$ and hence $d(\bar{b}^i(x), \bar{g}^i(x)) = d(w_i \bar{b}(x), w_i \bar{g}(x)) < 16\delta' < \eta'$.

Finally, if $h|_{B^k \times ((a + b)/2)B^n} = 1_{B^k \times ((a + b)/2)B^n}$ and i is any positive integer, $\bar{b}^i|_{(B^k \times T^n) - (4D^k \times 4D^n)} = 1_{(B^k \times T^n) - (4D^k \times 4D^n)}$. But, $4D^k$ and $4D^n$ were chosen small enough so that the diameter of each component of $e^{-1}(4D^k \times 4D^n)$ is less than $\delta/4$. Therefore, $\rho(h^i, 1_{B^k \times 4B^n}) < \delta/4$.

Lemma 5.2. *Let $n + k = 2$. Let positive numbers a, b, δ be given with $1 < a < b \leq 2$. Then there exists a positive number $\epsilon_{(a,b,\delta)} \leq \delta$ so that if*

$$\bar{I}_{(a,b,\delta)} = \{f \in \bar{I}(B^k \times 4B^n, \partial B^k \times 4B^n; B^k \times R^n) \mid \rho(f|B^k \times ((a+b)/2)B^n, 1_{B^k \times ((a+b)/2)B^n}) < \epsilon_{(a,b,\delta)}\}$$

and $\eta > 0$ is given then there exists a map $G_{(a,b,\delta,\eta)}: \bar{I}_{(a,b,\delta)} \rightarrow H_\partial(B^k \times 4B^n)$ such that for all $f \in \bar{I}_{(a,b,\delta)}$:

- (1) $G_{(a,b,\delta,\eta)}(f)|B^k \times (4B^n - bB^n) = 1_{B^k \times (4B^n - bB^n)}$;
- (2) $\rho(G_{(a,b,\delta,\eta)}(f)|B^k \times aB^n, f|B^k \times aB^n) < \eta$;
- (3) $\rho(G_{(a,b,\delta,\eta)}(f), 1_{B^k \times 4B^n}) < \delta$.

Proof. Let $\epsilon_{(a,b,\delta)}$ be the positive number obtained in Lemma 5.1. By Theorem 3.2 there is a mapping $\theta^1: \bar{H}_\partial(B^k \times bB^n) \rightarrow H_\partial(B^k \times 4B^n)$ such that $\theta^1(f)|B^k \times (4B^n - bB^n) = 1_{B^k \times (4B^n - bB^n)}$ and $\rho(\theta^1(f), f) < \min(\delta/2, \eta)$, for all $f \in \bar{H}_\partial(B^k \times bB^n)$. (This follows from applying Theorem 3.2 to the 2-ball $B^k \times bB^n$ and an arbitrary open cover of diameter less than $\min(\delta/2, \eta)$, obtaining a map of $\bar{H}_\partial(B^k \times bB^n)$ into $H_\partial(B^k \times bB^n)$ and then extending the elements of $H_\partial(B^k \times bB^n)$ by the identity on $B^k \times (4B^n - bB^n)$.) Define $G_{(a,b,\delta,\eta)}: \bar{I}_{(a,b,\delta)} \rightarrow H_\partial(B^k \times 4B^n)$ by

$$G_{(a,b,\delta,\eta)}(f) = \theta^1(\phi_{(a,b,\delta)}(f)|B^k \times bB^n),$$

where $\phi_{(a,b,\delta)}$ is the mapping obtained in Lemma 5.1.

It is easy to check that this map satisfies properties (1)–(3).

Proposition 5.3. *If $n + k = 2$, then there exists a neighborhood Q of $1_{B^k \times 4B^n}$ in $\bar{I}(B^k \times 4B^n, \partial B^k \times 4B^n; B^k \times R^n)$ and a homotopy*

$$\psi: Q \times [0, 1] \rightarrow \bar{I}(B^k \times 4B^n, \partial B^k \times 4B^n; B^k \times R^n)$$

such that $\psi(f, 0) = f$, $\psi(f, 1) \in \bar{I}(B^k \times 4B^n, (\partial B^k \times 4B^n) \cup (B^k \times B^n); B^k \times R^n)$, $\psi(f, t)|\partial(B^k \times 4B^n) = f|\partial(B^k \times 4B^n)$, for all $f \in Q$ and $t \in [0, 1]$.

Proof. For each positive integer, i , let $v_i = 1/2^i$. Choose the neighborhood Q small enough so that if $f \in Q$, then $\rho(f, 1_{B^k \times 4B^n}) < \epsilon_{(1+v_2, 1+v_1, v_4)}$.

Assume inductively that mappings $f_1, \dots, f_j$ have been defined so that for each i , $1 \leq i \leq j$, $f_i \in H_\partial(B^k \times 4B^n)$, f_i depends canonically on f , and

$$(1) \quad f_i|B^k \times (4B^n - (1 + v_i)B^n) = 1_{B^k \times (4B^n - (1 + v_i)B^n)},$$

$$(2) \quad \rho(f_i, 1_{B^k \times 4B^n}) < v_{i+3},$$

$$(3) \quad \rho(f_i|B^k \times (1 + v_{i+1})B^n, f|B^k \times (1 + v_{i+1})B^n) < \epsilon_{(1+v_{i+2}, 1+v_{i+1}, v_{i+4})},$$

$$(4) \quad ff_1^{-1} \dots f_i^{-1} \in \bar{I}(B^k \times 4B^n, \partial B^k \times 4B^n; B^k \times R^n),$$

$$(5) \quad \rho\left(ff_1^{-1} \dots f_i^{-1} | B^k \times \left(\frac{1+v_{i+2}+1+v_{i+1}}{2}\right) B^n, \right. \\ \left. 1_{(B^k \times ((1+v_{i+2}+1+v_{i+1})/2) B^n)}\right) < \epsilon_{(1+v_{i+2}, 1+v_{i+1}, v_{i+4})}.$$

Now applying Lemma 5.2, for each $f \in Q$, let f_{j+1} be defined by

$$f_{j+1} = G_{(1+v_{j+1}, 1+v_{j+1}, v_{j+4})} \circ \epsilon_{(1+v_{j+3}, 1+v_{j+2}, v_{j+5})} (ff_1^{-1} \dots f_j^{-1}).$$

[The map G is well defined by Condition (5) of the inductive hypothesis—in the case $j+1=1$, Q was chosen small enough to meet this requirement.]

Conditions (1)–(4) of the inductive hypothesis are guaranteed by the application of Lemma 5.2. Condition (5) follows from (2) and (3): If $x \in B^k \times ((1+v_{j+3}+1+v_{j+2})/2) B^n$, then $f_{j+1}(x) \in B^k \times (1+v_{j+2}) B^n$, since

$$\rho(f_{j+1}, 1_{B^k \times 4B^n}) < v_{j+4} \quad \text{and} \quad \frac{v_{j+3}}{2} + \frac{v_{j+2}}{2} + v_{j+4} = \frac{1}{2^{j+4}} + \frac{1}{2^{j+3}} + \frac{1}{2^{j+4}} = 1/2^{j+2} = v_{j+2}.$$

Therefore,

$$d(ff_1^{-1} \dots f_{j+1}^{-1}(x), x) = d(ff_1^{-1} \dots f_j^{-1}(f_{j+1}^{-1}(x)), f_{j+1}(f_{j+1}^{-1}(x))) \\ < \epsilon_{(1+v_{j+3}, 1+v_{j+2}, v_{j+5})}.$$

Let $g = \lim_{j \rightarrow \infty} ff_1^{-1} \dots f_j^{-1}$. Now, g is continuous since if $x \in B^k \times B^n$, then $g(x) = x$ and if $x \in B^k \times [1+v_{j+1}, 1+v_j] B^n$, then $d(g(x), x) < 1/2^j$. (Assume $x \in B^k \times [1+v_{j+1}, 1+v_j] B^n$. Note that $f_{j-1}^{-1} f_j^{-1}(x) \in B^k \times ((1+v_j+1+v_{j-1})/2) B^n$. By property (1), $g(x) = ff_1^{-1} \dots f_j^{-1}(x)$. Hence

$$d(g(x), x) = d(ff_1^{-1} \dots f_j^{-1}(x), x) \\ \leq d(ff_1^{-1} \dots f_{j-2}^{-1}(f_{j-1}^{-1}(f_j^{-1}(x))), f_{j-1}^{-1}(f_j^{-1}(x))) + d(f_{j-1}^{-1}(f_j^{-1}(x)), x) \\ \leq \epsilon_{(1+v_j, 1+v_{j-1}, v_{j+2})} + (v_{j+3} + v_{j+2}) \\ \leq v_{j+2} + v_{j+3} + v_{j+2} = 1/2^{j+2} + 1/2^{j+3} + 1/2^{j+2} < 1/2^j.$$

For each j , $j = 0, 1, \dots$, we can define an Alexander homotopy $\psi_j: Q \times [0, 1] \rightarrow \bar{I}(B^k \times 4B^n, \partial B^k \times 4B^n; B^k \times R^n)$ by $\psi_j(f, t) = ff_1^{-1} \dots f_{j-1}^{-1} A(f_j^{-1}, 1-t)$, where A is the Alexander isotopy defined in §3 with $B^k \times (1+v_j) B^n$ playing the role of B^n . Note that for each $t \in [0, 1]$, $\rho(\psi_j(f, t), g) < 1/2^{j-1}$. For if $x \in B^k \times 4B^n - (1+v_j) B^n$, $\psi_j(f, t)(x) = g(x)$ and if $x \in B^k \times (1+v_j) B^n$, $d(\psi_j(f, t)(x), g(x)) \leq$

$d(\psi_j(f, t)(x), x) + d(g(x), x) < 1/2^j + 1/2^j = 1/2^{j-1}$, by the previous argument. Also, $\psi_j(f, 0) = f f_1^{-1} \cdots f_{j-1}^{-1}$ and $\psi_j(f, 1) = f f_1^{-1} \cdots f_j^{-1}$. We obtain the desired map $\psi: Q \times [0, 1] \rightarrow \bar{T}(B^k \times 4B^n, \partial B^k \times 4B^n; B^k \times R^n)$ by composing the homotopies $\psi_0, \psi_1, \dots$ in the following manner:

$$\psi(f, t) = \psi_j \left(f, 2^{j+1} \left(t - \frac{2^j - 1}{2^j} \right) \right), \quad \frac{2^j - 1}{2^j} \leq t \leq \frac{2^{j+1} - 1}{2^{j+1}},$$

and letting $\psi(f, 1) = g$.

Theorem 5.4. *If M^2 is a compact 2-manifold, then $\bar{H}_\partial(M^2)$ is locally contractible, and hence if $\partial M^2 = \phi$, $\bar{H}(M^2)$ is locally contractible.*

Proof. Since $\bar{H}_\partial(M^2)$ is homogeneous, it suffices to check local contractibility at 1_{M^2} . But this follows from Proposition 5.3 exactly as in [7].

The only place the fact that $n + k = 2$ is used is in the application of Theorem 3.2. Therefore, an affirmative answer to the following question would show that $\bar{H}_\partial(M^m)$ is locally contractible for any compact m -manifold, M^m : Given $\delta > 0$ does there exist a continuous mapping $\theta: \bar{H}_\partial(B^m) \rightarrow H_\partial(B^m)$ such that $\rho(f, \theta(f)) < \delta$, for all $f \in \bar{H}_\partial(B^m)$?

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TENNESSEE, KNOXVILLE, TENNESSEE 37916