$SU(n)$ ACTIONS ON DIFFERENTIABLE MANIFOLDS WITH VANISHING FIRST AND SECOND INTEGRAL PONTRJAGIN CLASSES⁽¹⁾

BY

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ABSTRACT. In this paper we determine the connected component of the identity of the isotropy subgroups of a given action of $SU(n)$ on a connected manifold whose first and second integral Pontrjagin classes are zero and whose dimension is less than $n^2 - 8n/3 - 1$.

1. General introduction. Let $SU(n)$ be a differentiable transformation group acting on a connected manifold M with $\dim M \leq n^2 - 8n/3 - 2$, and assume that the first two integral Pontrjagin classes of M are zero. The purpose of this paper is to determine the orbit structure of M under the action of $SU(n)$.

To be more precise, in Theorem 2.3 we determine those closed connected simple subgroups H of $SU(n)$ such that the first integral Pontrjagin class of $SU(n)/H$, $P_1(SU(n)/H; \mathbb{Z})$, is zero; in Theorem 2.5 we determine those closed connected simple subgroups H of $SU(n)$ such that $P_1(SU(n)/H; \mathbb{Z}) = 0 = P_2(SU(n)/H; \mathbb{Z})$; in Theorem 2.9 we determine the connected component of the identity of a regular isotropy subgroup of the given above actions of $SU(n)$ on M . In Theorem 3.1 we then show how the connected component of the identity of all isotropy subgroups of $SU(n)$ is completely determined by the connected component of the identity of the principal isotropy subgroup, and we give a list of such possible subgroups.

Wu-Chung Hsiang and Wu-Yi Hsiang considered this problem in [4]. One of the key steps in their approach was their knowledge of large subgroups of $SU(n)$. This forced them to put a severe restriction on M . Our approach is more general in nature, and allows us a much less restrictive dimension requirement.

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In this section we recall several well-known facts. Let G be a compact con-

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nected Lie group acting differentiably on a manifold M . For each $x \in M$, let $G_x = \{g \in G : g \cdot x = x\}$ and let $G(x) = \{g \cdot x : g \in G\}$. Then if $f: G/G_x \approx G(x) \subseteq M$ is the natural map,

$$f^*r(M) = r(G/G_x) \oplus \nu(G/G_x) = \alpha_{\xi_1}(\iota_{G/G_x}) \oplus \alpha_{\xi_1}(\phi_x)$$

where ξ_1 is the principal bundle $(G_x \rightarrow G \rightarrow G/G_x)$, ι_{G/G_x} is the isotropy representation of G_x in G , and ϕ_x is the slice representation of G_x at x . Suppose now that $x \in M$ is a regular element. Then ϕ_x is a trivial representation [5], and so $f^*r(M) = \alpha_{\xi_1}(\iota_{G/G_x}) \oplus \theta$ where θ is a trivial vector bundle. The following result is trivial to prove but is frequently useful.

Proposition 1.1. *Let $\eta = (H \rightarrow E \rightarrow B)$ and $\eta' = (G \rightarrow E' \rightarrow B')$ be principal bundles, and let*

$$\begin{array}{ccc} G & \xrightarrow{\Lambda} & G' \\ \downarrow & & \downarrow \\ E & \xrightarrow{\quad} & E' \\ \downarrow & & \downarrow \\ B & \xrightarrow{x} & B' \\ \eta & & \eta' \end{array}$$

be a homomorphism of principal bundles. Then

$$\begin{array}{ccc} R(G) & \xleftarrow{\Lambda^!} & R(G') \\ \downarrow a_\eta & & \downarrow a_{\eta'} \\ K(B) & \xleftarrow{x^!} & K(B') \end{array}$$

is a commutative diagram.

So if $F: G/G_x^0 \rightarrow G/G_x$ is the natural map and ξ_2 is the principal bundle $(G_x^0 \rightarrow G \rightarrow G/G_x^0)$, we have

$$F^!f^*r(M) = \alpha_{\xi_2}(\iota_{G/G_x^0}) \oplus \theta = r(G/G_x^0) \oplus \theta,$$

and so $F^*(f^*(P_j(M; Z))) = P_j(G/G_x^0; Z)$ (where $P_j(N; Z)$ is the j th integral Pontrjagin class of N).

So to study G actions on a manifold M satisfying $P_1(M; Z) = 0 = P_2(M; Z)$, we must first find all those closed connected subgroups H of G such that $P_1(G/H; Z) = 0 = P_2(G/H; Z)$.

2. Subgroups of $SU(n)$ with $P_1(SU(n)/H; Z) = 0 = P_2(SU(n)/H; Z)$. Let G be a compact Lie group, H a closed connected subgroup, and let T be a maximal torus in H . Then the natural map $\pi: G/T \rightarrow G/H$ is a fibre map with fibre H/T , and hence the kernel of the map $\pi^*: H^*(G/H; Z) \rightarrow H^*(G/T; Z)$ consists only of torsion elements. Let ξ be the principal bundle $(H \rightarrow G \rightarrow G/H)$ and ξ' the principal bundle $(T \rightarrow G \rightarrow G/T)$. Then if ϕ is a real representation of H , $\pi^!(\alpha_\xi(\phi)) = \alpha_{\xi'}(\phi|T)$ and hence $\pi^*(P_j(\alpha_\xi(\phi); Z)) = P_j(\alpha_{\xi'}(\phi|T); Z)$. Let $\eta(G) = (G \rightarrow E_G \rightarrow B_G)$ be a classifying bundle for G . Then $\eta(T) = (T \rightarrow E_G \rightarrow B_T)$ is a classifying bundle for T . Letting $i: G/T \rightarrow B_T$ be the classifying map of ξ' , we have that $\pi^!(\alpha_\xi(\phi)) = i^!(\alpha_{\eta(T)}(\phi|T))$, and hence from [2] that

$$\begin{aligned} \pi^*(P_j(\alpha_\xi(\phi); Z)) &= i^*(P_j(\alpha_{\eta(T)}(\phi|T); Z)) \\ &= (-1)^j i^* \sigma_{2j}(\tau_{\eta(T)}(W_1), \dots, \tau_{\eta(T)}(W_N), -\tau_{\eta(T)}(W_1), \dots, -\tau_{\eta(T)}(W_N)) \end{aligned}$$

where σ_{2j} is the $2j$ th elementary symmetric function, $\tau_{\eta(T)}$ is the transgression in the Serre spectral sequence $E(\eta(T))$ associated with $\eta(T)$, and $\{W_1, \dots, W_N\}$ are the positive weight vectors of ϕ .

Recall now that $r(G/H) = \alpha_\xi(\iota_{G/H}) = \alpha_\xi(\text{Ad}_G|H - \text{Ad}_H)$. By Proposition 1.1, $\alpha_\xi(\text{Ad}_G|H)$ is a trivial bundle, and hence $r(G/H) \oplus \alpha_\xi(\text{Ad}_H) = \theta$ where θ is a trivial vector bundle.

Corollary 2.1 [2]. *Let T be a torus in G . Then G/T is stably parallelizable.*

From $\pi^!(r(G/H)) = i^!\alpha_{\eta(T)}(\iota_{G/H}|T)$ and $\pi^!(\alpha_\xi(\text{Ad}_H)) = i^!\alpha_{\eta(T)}(\text{Ad}_H|T)$, we have the following corollary.

Corollary 2.2. $\sum_{\alpha+\beta=j} \pi^*(P_\alpha(G/H; Z)) \cdot \pi^*(P_\beta(\alpha_\xi(\text{Ad}_H); Z)) = 0$.

As E_G is contractible, $E_G \times_T G \rightarrow G/T$ is a homotopy equivalence, and hence we may take $\gamma = (G \rightarrow G/T \xrightarrow{i} B_T)$ to be a principal G bundle. Thus we see that $i^*: H^*(B_T; Z) \rightarrow H^*(G/T; Z)$ is just the edge homomorphism

$$H^*(B_T; Z) = E_2^{*,0}(\gamma) \rightarrow E_\infty^{*,0}(\gamma) \subset H^*(G/T; Z),$$

and hence $\ker i^* = \langle \text{im } d(\gamma)^+ \rangle$, the ideal in $H^*(B_T; Z)$ generated by the elements of positive degree in the image of the differential of $E(\gamma)$.

Suppose that U is a closed connected normal subgroup of H . Then by using Proposition 1.1 together with a standard argument, it follows that $P_j(G/H; Z) = 0$ implies that $P_j(G/U; Z) = 0$.

So to determine those closed connected subgroups H of G with $P_1(G/H; Z) = 0 = P_2(G/H; Z)$, we need only determine those closed simple subgroups U of G with $P_1(G/U; Z) = 0 = P_2(G/U; Z)$ and then, by using the classification theorem

of compact Lie groups, determine whatever restrictions there are on "piecing" the various simple (we shall see later that if $P_1(\mathrm{SU}(n)/H; \mathbb{Z}) = 0$, then H is either semisimple or a torus) groups together to form an acceptable H .

We now let $G = \mathrm{SU}(n)$ and let $\lambda: H \rightarrow \mathrm{SU}(n)$ be the embedding of H as a closed subgroup, i.e. λ is an injective homomorphism of the compact connected Lie group H into $\mathrm{SU}(n)$. By examining the spectral sequence $E(\gamma)$ of γ , we see that in low dimensions at least, $\ker i^* = \langle (\lim \tau_\gamma)^+ \rangle$ where τ_γ is the transgression of γ . Thus we have (at least in low dimensions)

$$\ker i^* = \langle S^+(\tau_{\eta(T)}(\mu_1), \dots, \tau_{\eta(T)}(\mu_n)) \rangle$$

where $\{\mu_1, \dots, \mu_n\}$ is the set of weight vectors of λ , and $S^+(x_1, \dots, x_n)$ is the ring of symmetric polynomials with zero constant term.

We now apply the above to the representation $\phi = \mathrm{Ad}_H$. Note that as $\lambda: H \rightarrow \mathrm{SU}(n)$, $\mu_n = -(\mu_1 + \dots + \mu_{n-1})$, and hence

$$2\sigma_2(\tau_{\eta(T)}(\mu_1), \dots, \tau_{\eta(T)}(\mu_n)) = - \sum_{i=1}^n (\tau_{\eta(T)}(\mu_i))^2.$$

Recall that, by Corollary 2.2,

$$\begin{aligned} \pi^*(P_1(\mathrm{SU}(n)/H; \mathbb{Z})) &= -\pi^*P_1(\alpha_\xi(\mathrm{Ad}_H); \mathbb{Z}) \\ &= i^*(\sigma_2(\tau_{\eta(T)}(w_1), \dots, \tau_{\eta(T)}(w_N), -\tau_{\eta(T)}(w_1), \dots, -\tau_{\eta(T)}(w_N))) \\ &= -i^* \sum_{j=1}^N (\tau_{\eta(T)}(w_j))^2. \end{aligned}$$

So $\pi^*(P_1(\mathrm{SU}(n)/H; \mathbb{Z})) = 0$ if and only if for some integer K ,

$$K \sum_{j=1}^n (\tau_{\eta(T)}(\mu_j))^2 = 2 \sum_{j=1}^N (\tau_{\eta(T)}(w_j))^2$$

So if U is a closed, simple, normal subgroup of H , it follows that $\pi^*(P_1(\mathrm{SU}(n)/U; \mathbb{Z})) = 0$ if and only if there exists an integer K such that

$$K \sum_{j=1}^n |\mu_j|^2 = 2 \sum_{j=1}^N |w_j|^2$$

where $||$ is the Cartan-Killing norm.

We are now ready to determine those closed simple subgroups G of $\mathrm{SU}(n)$ for which $P_1(\mathrm{SU}(n)/G; \mathbb{Z}) = 0$. First note that if $\lambda: G \rightarrow \mathrm{SU}(n)$, then $\mathrm{SU}(n)/\lambda[G]$ is diffeomorphic to $\mathrm{SU}(n)/\lambda^*[G]$. Hence in the following theorem if $\lambda: G \rightarrow \mathrm{SU}(n)$ is such that $P_1(\mathrm{SU}(n)/\lambda[G]; \mathbb{Z}) = 0$, then we shall either include (G, λ) or (G, λ^*) in the list, but not both.

Theorem 2.3. *Suppose $\lambda: G \rightarrow \text{SU}(n)$ is an almost faithful embedding, where G is a simply connected, compact, simple Lie group such that $P_1(\text{SU}(n)/\lambda[G]; \mathbb{Z}) = 0$. Then modulo trivial representations, (G, λ) appears on the following list:*

G	λ	
SU(4)	$b\Lambda^2\mu_4$	$b = 1, 2, 4$
	$a\mu_4 \oplus 3\Lambda^2\mu_4 \oplus d\mu_4^*$	$a + d = 2$
SU(5)	$\mu_5 \oplus b\Lambda^2\mu_5 \oplus c\Lambda^2\mu_5^*$	$b + c = 3$
SU(6)	$b\Lambda^2\mu_6 \oplus c\Lambda^2\mu_6^*$	$b + c = 1, 3$
	$b\Lambda^3\mu_6$	$b = 1, 2$
	$a\mu_6 \oplus b\Lambda^2\mu_6 \oplus \Lambda^3\mu_6 \oplus c\Lambda^2\mu_6^* \oplus d\mu_6^*$	$a + d + 4(b + c) = 6$
SU(7)	$a\mu_7 \oplus b\Lambda^2\mu_7 \oplus c\Lambda^2\mu_7^* \oplus d\mu_7^*$	$a + d = 4, b + c = 1$
SU(8)	$\mu_8 \oplus b\Lambda^3\mu_8 \oplus c\Lambda^3\mu_8^*$	$b + c = 1$
SU(k), $k \geq 2$	$\text{Ad}_{\text{SU}(k)}$	
SU(k), $k \geq 2$	$a\mu_k \oplus d\mu_k^*$	$a + d \mid 2k$
SU(k), $k \geq 2$	$a\mu_k \oplus b\Lambda^2\mu_k \oplus c\Lambda^2\mu_k^* \oplus d\mu_k^*$	$a + d = k - 2, b + c = 1$
SU(k), $k \geq 4$	$a\mu_k \oplus b\Lambda^2\mu_k \oplus c\Lambda^2\mu_k^* \oplus d\mu_k^*$	$a + d = 2 + k, b + c = 1$
		$a + d = 4, b + c = 2$
		$a + d = 2, b + c = 1$
SU(k), $k \geq 4$	$\Lambda^2\mu_k \oplus \Lambda^2\mu_k^*$	
$S_p(3)$	$a\Lambda^2\nu_6 \oplus b(\Lambda^3\nu_6 - \nu_6)$	$a + b = 1, 2$
	$4\nu_6 \oplus (\Lambda^3\nu_6 - \nu_6)$	
$S_p(k)$, $k \geq 3$	$a\nu_{2k}$	$a \mid 2(k + 1)$
$S_p(k)$, $k \geq 3$	$4\nu_{2k} \oplus \Lambda^2\nu_{2k}$	
$S_p(k)$, $k \geq 3$	$\text{Ad}_{S_p(k)}$	
$S_{\text{PIN}}(5)$	$a\rho_5 \oplus b\Delta_5$	$2a + b \mid 6$
$S_{\text{PIN}}(7)$	$a\rho_7 \oplus b\Delta_7$	$a + b \mid 5$
$S_{\text{PIN}}(9)$	$a\rho_9 \oplus b\Delta_9$	$a + 2b \mid 7$
$S_{\text{PIN}}(11)$	$a\rho_{11} \oplus b\Delta_{11}$	$a + 4b \mid 9$

$S_{\text{PIN}}(13)$	$a\rho_{13} \oplus b\Delta_{13}$	$a + 8b \mid 11$
$S_{\text{PIN}}(2k+1), k \geq 2$	$a\rho_{2k+1}$	$a \mid (2k-1)$
$S_{\text{PIN}}(2k+1), k \geq 2$	$\text{Ad}_{S_{\text{PIN}}(2k+1)}$	
$S_{\text{PIN}}(8)$	$a\rho_8 \oplus b\Delta_8^+ \oplus c\Delta_8^-$	$a + b + c \mid 6$
$S_{\text{PIN}}(10)$	$a\rho_{10} \oplus b\Delta_{10}^+ \oplus c\Delta_{10}^-$	$a + 2(b+c) \mid 8$
$S_{\text{PIN}}(12)$	$a\rho_{12} \oplus b\Delta_{12}^+ \oplus c\Delta_{12}^-$	$a + 4(b+c) \mid 10$
$S_{\text{PIN}}(14)$	$a\rho_{14} \oplus b\Delta_{14}^+ \oplus c\Delta_{14}^-$	$a + 8(b+c) \mid 12$
$S_{\text{PIN}}(2k), k \geq 4$	$a\rho_{2k}$	$a \mid 2(k-1)$
$S_{\text{PIN}}(2k), k \geq 4$	$\text{Ad}_{S_{\text{PIN}}(2k)}$	
G_2	$\phi_2 = \text{Ad}_{G_2}, \phi_1, 2\phi_1, 4\phi_1$	
F_4	$\phi_4 = \text{Ad}_{F_4}, \phi_1, 3\phi_1$	
E_6	$a\phi_1 \oplus b\phi_5$	$a + b = 1, 2, 4$
	$\text{Ad}_{E_6} = \phi_6$	
E_7	$\phi_1, 3\phi_1, \text{Ad}_{E_7} = \phi_6$	
E_8	Ad_{E_8}	

Remark. Note that if $\bar{G}$ is a compact simple Lie group and $\lambda: \bar{G} \rightarrow G_1$ is any nontrivial homomorphism of Lie groups, then $\text{Ker } \lambda$ is a finite central subgroup. So any embedding of $\bar{G}$ lifts to an almost faithful embedding of $\bar{G}$'s universal covering group $\tilde{\lambda}: \tilde{G} \rightarrow G_1$. If $G_1 = \text{SU}(n)$ and $P_1(\text{SU}(n)/\bar{G}) = 0$, then $(\tilde{G}, \tilde{\lambda})$ will be on the above list.

Proof of Theorem 2.3. Suppose $G = \text{SU}(k)$, $2 \leq k \leq n$. If $\lambda: \text{SU}(k) \rightarrow \text{SU}(n)$ and $\Sigma(\lambda) = \{(j, \mu_j): 1 \leq j \leq n\}$ is the set of weight vectors of λ , let $n(\lambda) = \sum_{j=1}^n |\mu_j|^2$. For $1 \leq p, q \leq k$, let e_q^p be the $k \times k$ matrix $e_q^p = [\delta_p^i \delta_q^j]$. Let $\lambda_j = i(e_j^j - k^{-1} \sum_{i=1}^k e_i^i)$ for $1 \leq j \leq k$. Then $\{\lambda_j: 1 \leq j \leq k-1\}$ forms a basis for $\mathfrak{h}$, the standard Cartan subalgebra of $\text{SU}(k)$. We may take our invariant, nondegenerate inner-product for the real Lie algebra of $\text{SU}(k)$ so that $(\lambda_i, \lambda_i) = (k-1)/k$ and $(\lambda_i, \lambda_j) = -1/k$ if $i \neq j$. Note that

$$n(\text{Ad}_{\text{SU}(k)}) = 2 \sum_{1 \leq i < j \leq k} |\lambda_i - \lambda_j|^2 = 2 \sum_{1 \leq i < j \leq k} 2 = 2k(k-1).$$

So as we have shown, $P_1(\text{SU}(n)/\lambda[\text{SU}(k)]) = 0$ implies $2k(k-1)$ is an integral multiple of $n(\lambda)$.

We let $\pi = \{\lambda_j - \lambda_{j+1}: 1 \leq j \leq k-1\}$ be our system of simple roots. If $\phi:$

$SU(k) \rightarrow SU(n)$ is irreducible, let Λ_ϕ be the highest weight of ϕ , and let

$$q_j = \frac{2(\Lambda_\phi, \lambda_j - \lambda_{j+1})}{(\lambda_j - \lambda_{j+1}, \lambda_j - \lambda_{j+1})} = (\Lambda_\phi, \lambda_j - \lambda_{j+1}).$$

Then recall q_j is a nonnegative integer.

For each $1 \leq i \leq k-1$, let ϕ_i be the unique irreducible representation satisfying

$$\frac{2(\Lambda_{\phi_i}, \lambda_j - \lambda_{j+1})}{(\lambda_j - \lambda_{j+1}, \lambda_j - \lambda_{j+1})} = \delta_j^i, \quad \text{i.e. } \phi_i \leftrightarrow \circ \circ \dots \circ \overset{1}{\circ} \circ \dots \circ$$

Recall that $\phi_1 = \mu_k$ and $\phi_j = \Lambda^j \mu_k$, $1 \leq j \leq k$. So

$$\sum(\phi_j) = \{\lambda_{i_1} + \dots + \lambda_{i_j} : 1 \leq i_1 < i_2 < \dots < i_j \leq k\}.$$

Hence

$$\begin{aligned} n(\phi_j) &= \sum_{1 \leq i_1 < \dots < i_j \leq k} |\lambda_{i_1} + \dots + \lambda_{i_j}|^2 = \sum_{1 \leq i_1 < \dots < i_j \leq k} j \binom{k-j}{k} \\ &= \frac{j}{k} (k-j) \frac{k(k-1) \dots (k-j+1)}{1 \cdot 2 \dots j} \\ &= \frac{(k-1)(k-2) \dots (k-j+1)}{2 \cdot 3 \dots j} j(k-j) \quad \text{if } j > 1 \\ &= \frac{(k-1)(k-2) \dots (k-j+2)}{2 \cdot 3 \dots (j-1)} (k-j+1)(k-j) \quad \text{if } j > 2. \end{aligned}$$

Note that $n(\phi_{j+1}) = n(\phi_j) \cdot (k-j-1)/j$. So if $1 \leq j \leq (k-1)/2$, then $n(\phi_{j+1}) \geq n(\phi_j)$. Also, $1 \leq j \leq (k-1)/2$ implies $\sum(\phi_j) = -\sum(\phi_{k-j})$ (i.e. $\phi_j = \phi_{k-j}^*$) which implies $n(\phi_j) = n(\phi_{k-j})$. We have

$$n(\phi_1) = k-1, \quad n(\phi_2) = (k-1)(k-2),$$

$$n(\phi_3) = \frac{1}{2}(k-1)(k-2)(k-3) > 2k(k-1) \quad \text{if } k > 9.$$

So if $4 \leq j \leq [(k+1)/2]$ and $n(\phi_j) \leq 2k(k-1)$, it follows that $k \leq 8$, and hence $k=8$, $j=4$. By a direct check, this is impossible. So if $1 \leq j \leq [(k+1)/2]$ and $n(\phi_j) \leq 2k(k-1)$, we have $j=1$, $j=2$, or $j=3$ and $6 \leq k \leq 8$.

Lemma 2.4. Suppose $\phi: SU(k) \rightarrow SU(n)$ is an irreducible map with $n(\phi) \leq 2k(k-1)$. Then up to conjugation, ϕ appears on the following list:

$$\Lambda^3 \mu_k, \quad 6 \leq k \leq 8,$$

$$\mu_k, S^2 \mu_k, \Lambda^2 \mu_k, \text{Ad}_{\text{SU}(k)}, \quad k \geq 2.$$

Proof. Let

$$\phi \leftrightarrow \overset{q_1}{\circ} \overset{q_2}{\circ} \overset{q_3}{\circ} \dots \overset{q_{k-1}}{\circ}.$$

Suppose that there exist at least three distinct indices a, b, c so that q_a, q_b, q_c all are different from zero. If ϕ_1, ϕ_2 are two different irreducible representations, we say $\phi_1 \geq \phi_2$ if $q_i(\phi_1) \geq q_i(\phi_2)$ for $i = 1, 2, \dots, k-1$. It follows easily that $\phi_1 \geq \phi_2$ implies $(\Lambda_{\phi_1}, \Lambda_{\phi_1}) \geq (\Lambda_{\phi_2}, \Lambda_{\phi_2})$.

Using this fact, one easily shows $(\Lambda_\phi, \Lambda_\phi) > 1$.

Since q_a, q_b, q_c are all different from zero, it follows easily by applying the Weyl group that $\Sigma(\phi)$ contains at least $(k-1)(k-2)(k-3)$ distinct vectors Λ all with $(\Lambda, \Lambda) = (\Lambda_\phi, \Lambda_\phi)$. So $n(\phi) \geq (k-1)(k-2)(k-3)$. So if $k > 6$, $n(\phi) > 2k(k-1)$.

Note $4 \leq k$ as there exist $q_a, q_b, q_c \neq 0$. If $4 \leq k \leq 6$, one shows by direct calculation that $(\Lambda_\phi, \Lambda_\phi) \geq 5$ and hence

$$n(\phi) \geq 5(k-1)(k-2)(k-3) > 2k(k-1) \quad \text{if } 4 \leq k \leq 6.$$

Suppose next that there exist exactly 2 indices a, b with $q_a, q_b \neq 0$. Suppose $k > 9$. Further suppose that $3 \leq a \leq k-3$. Then $\phi \geq \Lambda^a \mu_k = \phi_a$. So $(\Lambda_\phi, \Lambda_\phi) \geq (\Lambda_{\phi_a}, \Lambda_{\phi_a})$. Also, by applying the Weyl group, it is clear that we introduce at least as many new vectors Λ into $\Sigma(\phi)$ with $(\Lambda, \Lambda) = (\Lambda_\phi, \Lambda_\phi)$, as we do vectors Λ^1 into $\Sigma(\Lambda^a \mu_k)$ with $(\Lambda^1, \Lambda^1) = (\Lambda_{\phi_a}, \Lambda_{\phi_a})$, and so $n(\phi) \geq n(\Lambda^a \mu_k) > 2k(k-1)$ (as $3 < a < k-3$).

It follows that we may assume (up to conjugation)

- (1) $a = 1, b = 2,$
- (2) $1 \leq a \leq 2, k-2 \leq b \leq k-1,$
- (3) $1 \leq a \leq 2, b = 3, 6 \leq k \leq 8,$
- (4) $1 \leq a \leq 3, k-3 \leq b \leq k-1, 7 \leq k \leq 8.$

By direct computation, we now easily arrive at the result.

With this lemma, the results with $G = \text{SU}(k)$ follow easily. One obtains the results for $G = B_k, C_k, D_k, G_2, F_4, E_6, E_7, E_8$ similarly.

Theorem 2.5. Suppose $\lambda: G \rightarrow \text{SU}(n)$ is a nontrivial representation, where G is a simply connected, compact, simple Lie group. Suppose $P_1(\text{SU}(n)/\lambda[G]; Z) = 0 = P_2(\text{SU}(n)/\lambda[G]; Z)$. Then (G, λ) (modulo an arbitrary trivial representation) appears on the following list:

G	λ	
$SU(2)$	$4\mu_2$	
$SU(3)$	$\mu_3 \oplus bS^2\mu_3 \oplus cS^2\mu_3^*$	$b + c = 1$
$SU(4)$	$\Lambda^2\mu_4$	
	$2\Lambda^2\mu_4$	
	$\Lambda^2\mu_4 \oplus S^2\mu_4^*$	
$SU(6)$	$\Lambda^2\mu_6$	
$SU(k), k \geq 2$	$\text{Ad}_{SU(k)}$	
$SU(k), k \geq 2$	$a\mu_k \oplus b\mu_k^*$	$a + b = 1$
		$a + b = 2$
		$a + b = 3, 3 k$
		$a + b = 6, 3 k$
$S_p(3)$	$\Lambda^2\nu_6$	
$S_p(k), k \geq 3$	ν_{2k}	
	$3\nu_{2k}$	$3 k + 1$
	$\text{Ad}_{S_p(k)}$	
$S_{PIN}(5)$	$a\rho_5 \oplus b\Delta_5$	$2a + b 6, \quad b \neq 4$
$S_{PIN}(2k+1), k \geq 2$	ρ_{2k+1}	
	$3\rho_{2k+1}$	$3 2k - 1$
	$\text{Ad}_{S_{PIN}(2k+1)}$	
$S_{PIN}(8)$	$a\rho_8 \oplus b\Delta_8^+ \oplus c\Delta_8^-$	$a + b + c 6, \quad b + c > 0$
$S_{PIN}(10)$	Δ_{10}^+	
$S_{PIN}(2k), k \geq 4$	ρ_{2k}	
	$2\rho_{2k}$	
	$3\rho_{2k}$	$3 k - 1$
	$6\rho_{2k}$	$3 k - 1$
	$\text{Ad}_{S_{PIN}(2k)}$	

$$\begin{array}{ll}
G_2 & \phi_2 = \text{Ad}_{G_2}, \phi_1 \\
F_4 & \phi_4 = \text{Ad}_{F_4}, \phi_1 \\
E_6 & \text{Ad}_{E_6} \\
E_7 & \text{Ad}_{E_7} \\
E_8 & \text{Ad}_{E_8}
\end{array}$$

Proof. Let $\{(j, \mu_j) : 1 \leq j \leq n\} = \Sigma(\lambda)$, and let $\sigma_j(\lambda) = \sigma_j(r_{\eta(T)}(\mu_1), \dots, r_{\eta(T)}(\mu_n))$ for $2 \leq j \leq 4$. Recall that

$$\begin{aligned}
(\ker i^*)^8 &= (Zv_1 \oplus \dots \oplus Zv_l)^2 \sigma_2(\lambda) \oplus (Zv_1 \oplus \dots \oplus Zv_l) \sigma_3(\lambda) \\
&\oplus Z\sigma_4(\lambda) \quad \text{where } l = \text{rank } G.
\end{aligned}$$

Now

$$P_2(\text{SU}(n)/\text{im } \lambda; Z) + P_1(\text{SU}(n)/\text{im } \lambda; Z) \cdot P_1(\alpha_{\xi}(\text{Ad}_U); Z) + P_2(\alpha_{\xi}(\text{Ad}_U); Z) \equiv 0$$

(modulo 2 torsion). So as $P_1(\text{SU}(n)/\text{im } \lambda; Z) = 0 = P_2(\text{SU}(n)/\text{im } \lambda; Z)$ by hypothesis, we see that $\pi^*(P_1(\alpha_{\xi}(\text{Ad}_U); Z)) = 0 = \pi^*(P_2(\alpha_{\xi}(\text{Ad}_U); Z))$. Hence

$$\left(\sum_{j=1}^l a_j v_j^2 + \sum_{1 \leq i < j \leq l} b_{ij} v_i v_j \right) \sigma_2(\lambda) + \left(\sum_{j=1}^l d_j v_j \right) \sigma_3(\lambda) + m \cdot \sigma_4(\lambda) = \sigma_4(\text{Ad}_G)$$

where $a_j, b_{ij}, d_j, m \in \mathbb{Z}$ for $1 \leq i \leq l, 1 \leq j \leq l$. Suppose $G = \text{SU}(k)$. By using the two facts

$$\sigma_4(y_1, -y_1, \dots, y_p, -y_p) = \sigma_2(y_1^2, \dots, y_p^2)$$

and

$$\sigma_j(X_1, \dots, X_p, Y_1, \dots, Y_q) = \sum_{\alpha + \beta = j} \sigma_{\alpha}(X_1, \dots, X_p) \cdot \sigma_{\beta}(Y_1, \dots, Y_q),$$

it follows that

$$\begin{aligned}
\sigma_4(\text{Ad}_{\text{SU}(k)}) &= (2k^2 - k - 6) \left(\sigma_1(v_1^4, \dots, v_{k-1}^4) + 3\sigma_2(v_1^2, \dots, v_{k-1}^2) \right. \\
&\quad \left. + 2 \sum_{j=1}^{k-1} v_j^2 \sigma_2(v_1, \dots, \hat{v}_j, \dots, v_{k-1}) \right) \\
&\quad + (8k^2 - 6k - 24) \sum_{j=1}^{k-1} v_j \sigma_3(v_1, \dots, \hat{v}_j, \dots, v_{k-1}) \\
&\quad + (12k^2 - 12k - 36) \sigma_4(v_1, \dots, v_{k-1})
\end{aligned}$$

where $v_j = r_{\eta(T)}(\lambda_j)$ as usual.

As an example of the method, consider $\lambda = a\mu_k \oplus b\mu_k^*$, $a + b \mid 2k$. One shows easily that

$$\begin{aligned}\sigma_2(\mu_k) &= -(\sigma_1(v_1^2, \dots, v_{k-1}^2) + \sigma_2(v_1, \dots, v_{k-1})), \\ \sigma_3(\mu_k) &= -\left(\sum_{j=1}^{k-1} v_j^2 \sigma_1(v_1, \dots, \hat{v}_j, \dots, v_{k-1}) + 2\sigma_3(v_1, \dots, v_{k-1})\right), \\ \sigma_4(\mu_k) &= -\left(\sum_{j=1}^{k-1} v_j^2 \sigma_2(v_1, \dots, \hat{v}_j, \dots, v_{k-1}) + 3\sigma_4(v_1, \dots, v_{k-1})\right), \\ \sigma_4(\lambda) &= (a+b)\sigma_4(\mu_k) + \frac{1}{2}(a+b)(a+b-1)(\sigma_2(\mu_k))^2.\end{aligned}$$

So

$$\begin{aligned}\sigma_4(\text{Ad}_{SU(k)}) &= -\left(\sum_{j=1}^{k-1} a_j v_j^2 + \sum_{1 \leq i < j \leq k-1} b_{ij} v_i v_j\right) \\ &\quad \cdot (a+b)(\sigma_1(v_1^2, \dots, v_{k-1}^2) + \sigma_2(v_1, \dots, v_{k-1})) \\ &\quad - (a-b) \sum_{j=1}^{k-1} d_j v_j \left(\sum_{j=1}^{k-1} v_j^2 \sigma_1(v_1, \dots, \hat{v}_j, \dots, v_{k-1}) + 2\sigma_3(v_1, \dots, v_{k-1})\right) \\ &\quad + m \left(\frac{1}{2}(a+b)(a+b-1)\sigma_1(v_1, \dots, v_{k-1})\right. \\ &\quad \quad \left.+ \frac{3}{2}(a+b)(a+b-1)\sigma_2(v_1^2, \dots, v_{k-1}^2)\right. \\ &\quad \quad \left.+ (a+b)(a+b-1) \sum_{j=1}^{k-1} v_j^3 \sigma_1(v_1, \dots, \hat{v}_j, \dots, v_{k-1})\right. \\ &\quad \quad \left.+ (2(a+b)(a+b-1) - (a+b)) \sum_{j=1}^{k-1} v_j^2 \sigma_2(v_1, \dots, \hat{v}_j, \dots, v_{k-1})\right. \\ &\quad \quad \left.+ (3(a+b)(a+b-1) - 3(a+b))\sigma_4(v_1, \dots, v_{k-1})\right).\end{aligned}$$

Clearly, $a_i = a_j$ and $d_i = d_j$ if $1 \leq i, j \leq k-1$. Also, $b_{ij} = b_{\alpha\beta}$ if $1 \leq i < j \leq k-1$, $1 \leq \alpha < \beta \leq k-1$. Note $-(a+b)a_1 + \frac{1}{2}(a+b)(a+b-1)m = 2k^2 - k - 6$.

Suppose $k = 2$. Then this is the only condition. So $-(a+b)a_1 + \frac{1}{2}(a+b)(a+b-1)m = 0$. This is clearly acceptable for $a + b \mid 4$.

Suppose $k > 2$.

$$-(a+b)(a_1 + a_2 + b_{12}) - (a-b)(d_1 + d_2) + 3(a+b)(a+b-1)m/2 = 6k^2 - 3k - 18.$$

So $-(a+b)b_{12} - (a-b)(d_1 + d_2) + \frac{1}{2}(a+b)(a+b-1)m = 2k^2 - k - 6$. Also, $-(a+b)(a_1 + b_{12}) - (a-b)d_1 + (a+b)(a+b-1)m = 4k^2 - 2k - 12$. So $-(a-b)d_2 = 0$.

Hence $-(a+b)b_{12} + \frac{1}{2}(a+b)(a+b-1)m = 2k^2 - k - 6$.

Suppose $k = 3$. Then this is the only condition. $-(a+b)a_1 + \frac{1}{2}(a+b)(a+b-1)m = 9$. So $a+b \mid 6$ is acceptable.

Suppose $k \geq 4$. $-(a+b)(a_1 + b_{12} + b_{13} + b_{23}) + (2(a+b)(a+b-1) - (a+b))m = 8k^2 - 6k - 24$ or $(a+b)m = 2k$. So

$$-(a+b)a_1 + (a+b-1)k = 2k^2 - k - 6,$$

$$-(a+b)a_1 = 2k^2 - (a+b)k - 6 = 2(k^2 - 3) - (a+b)k.$$

Hence $(a+b) \mid 2k^2 - 6$. Recall $(a+b) \mid 2k$. Hence $(a+b) \mid 2k^2$. So $(a+b) \mid 6$. So $a\mu_k \oplus b\mu_k^*$, $k \geq 4$ is acceptable when $(a+b=1)$, $(a+b=2)$, $(a+b=3 \text{ and } 3 \mid k)$, $(a+b=6 \text{ and } 3 \mid k)$. This concludes the proof.

Remark. Suppose $\lambda: G_1 \times G_2 \rightarrow \text{SU}(n)$ so that $P_1(\text{SU}(n)/\text{im } \lambda; Z) = 0$. Recall $\lambda = \sum_{j=1}^t \phi_j \otimes \psi_j$ where $\phi_1, \dots, \phi_t, \psi_1, \dots, \psi_t$ are either trivial or irreducible.

Let T_1 be a maximal torus for G_1 , and T_2 for G_2 . Let $v_1, \dots, v_{\text{rk } G_1}$ be a base for $H^2(B_{T_1}; Z)$, and $\bar{v}_1, \dots, \bar{v}_{\text{rk } G_2}$ for $H^2(B_{T_2}; Z)$. For some nonzero integer M , we have

$$\begin{aligned} \sigma_2(\text{Ad}_{G_1}) + \sigma_2(\text{Ad}_{G_2}) &= \sigma_2(\text{Ad}_{G_1 \times G_2}) = M \cdot \sigma_2(\lambda) = M \cdot \sum_{j=1}^t \sigma_2(\phi_j \otimes \psi_j) \\ &= M \cdot \sum_{j=1}^t (\dim \psi_j) \sigma_2(\phi_j) + (\dim \phi_j) \sigma_2(\psi_j) \\ &= M \cdot \sum_{j=1}^t (\dim \psi_j) \sigma_2(\phi_j) + M \sum_{j=1}^t (\dim \phi_j) \sigma_2(\psi_j). \end{aligned}$$

So as $\sigma_2(\text{Ad}_{G_1})$ and $\sigma_2(\phi_j)$ ($1 \leq j \leq t$) are polynomials in $v_1, \dots, v_{\text{rk } G_1}$, and $\sigma_2(\text{Ad}_{G_2})$ and $\sigma_2(\psi_j)$ ($1 \leq j \leq t$) are polynomials in $\bar{v}_1, \dots, \bar{v}_{\text{rk } G_2}$, we have

$$\begin{aligned} \sigma_2(\text{Ad}_{G_1}) &= M \sum_{j=1}^t (\dim \psi_j) \sigma_2(\phi_j), \text{ and} \\ (2.6) \quad \sigma_2(\text{Ad}_{G_2}) &= M \sum_{j=1}^t (\dim \phi_j) \sigma_2(\psi_j). \end{aligned}$$

So as $n(\text{Ad}_G) = 0$ if and only if G is a torus, we have the following.

Lemma 2.7. *Let H be a compact connected subgroup of $\text{SU}(n)$ so that $P_1(\text{SU}(n)/H) = 0$. Then either H is a torus or H is semisimple.*

So we may assume that G_1 and G_2 are simple. So if $P_1(\text{SU}(n)/\text{im } \lambda; Z) = 0 = P_2(\text{SU}(n)/\text{im } \lambda; Z)$, then as before

$$(2.8) \quad \sigma_4(\text{Ad}_{G_1}) + \sigma_2(\text{Ad}_{G_1})\sigma_2(\text{Ad}_{G_2}) + \sigma_4(\text{Ad}_{G_2}) = \sigma_4(\text{Ad}_{G_1 \times G_2}) \in \ker i^*.$$

Theorem 2.9. *Suppose that $\text{SU}(n)$ acts as a differentiable transformation group on a connected manifold M , where $P_1(M; \mathbb{Z}) = 0 = P_2(M; \mathbb{Z})$. Suppose also that $\dim M \leq n^2 - 8n/3 - 2$. Let G be the connected component of the identity of "the" principal isotropy subgroup of the action. Then (possibly modulo a low-dimensional trivial representation), G together with its embedding appears on the following list.*

G	λ	
$\text{SU}(k),$	$k \geq 4$	μ_k
	$k \geq 6$	$a\mu_k \oplus b\mu_k^* \quad a+b=2$
	$k \geq 9$	$a\mu_k \oplus b\mu_k^* \quad a+b=3, 3 k$
	$k \geq 18$	$a\mu_k \oplus b\mu_k^* \quad a+b=6, 3 k$
$\text{SO}(2k+1),$	$k \geq 2$	ρ_{2k+1}
	$k \geq 11$	$3\rho_{2k+1} \quad 3 2k-1$
$\text{S}_p(k),$	$k \geq 3$	γ_{2k}
	$k \geq 8$	$3\gamma_{2k} \quad 3 k+1$
$\text{SO}(2k),$	$k \geq 4$	ρ_{2k}
	$k \geq 6$	$2\rho_{2k}$
	$k \geq 10$	$3\rho_{2k} \quad 3 k-1$
	$k \geq 19$	$6\rho_{2k} \quad 3 k-1$

Proof. Note that $n^2 - 8n/3 - 2 \geq \dim M \geq \dim(\text{SU}(n)/\text{im } \lambda) = \dim \text{SU}(n) - \dim G = n^2 - 1 - \dim G$ and so $\dim G > 8n/3$; that is, the dimension of G as a Lie group is greater than $8/3$ times the dimension of λ as a representation.

By Lemma 2.7, G is either a torus or G is semisimple. Suppose G is a torus, say $G = T^k$. Then $k = \dim G > 8n/3$. This is impossible as $\text{rk } \text{SU}(n) = n-1$. So G is semisimple.

We first suppose $G = G_1 \times G_2$ where G_1 and G_2 are compact simple Lie groups. Recall $\lambda = \sum_{j=1}^t \phi_j \otimes \psi_j$. By 2.6 if for some j both ϕ_j and ψ_j are non-trivial, then there exists a positive integer M such that $n(\text{Ad}_{G_1}) \geq M \cdot \dim \psi_j \cdot n(\phi_j)$ and $n(\text{Ad}_{G_2}) \geq M \cdot \dim \phi_j \cdot n(\psi_j)$. This turns out to be a severe restriction on ϕ_j and ψ_j . Using this fact together with 2.6 and 2.8 in the usual way, we arrive at the list for $G_1 \times G_2$. We then use this list together with the additional

requirement $\dim G > 8n/3$ to prove that G must be simple, and that, in fact, (G, λ) lies on the above list.

3. The orbit types of $SU(n)$ actions on manifolds with vanishing first and second integral Pontrjagin classes.

Theorem 3.1. *Let $SU(n)$ act as a differentiable transformation group on a connected manifold M . Suppose that $P_1(M; \mathbb{Z}) = 0 = P_2(M; \mathbb{Z})$, and that $\dim M \leq n^2 - 8n/3 - 2$. Let $y \in M$ be a regular element and let $x \in M$. Then after conjugating G_y^0 if necessary, the inclusion $G_y^0 \subseteq G_x^0 \subseteq SU(n)$ appears on the following list:*

(1) If

$$(G_y^0) = \left(SU(k) \xrightarrow{a\mu_k \oplus b\mu_k^* \oplus (n-(a+b))\theta} SU(n) \right)$$

where $a + b = 1, 2, 3, 6$ and $3|k$ if $a + b = 3, 6$, then

$$(G_y^0 \subseteq G_x^0 \subseteq SU(n)) = \left(SU(k) \xrightarrow{\mu_k \oplus (l-k)\theta} SU(l) \xrightarrow{a\mu_l \oplus b\mu_l^* \oplus (n-(a+b)l)\theta} SU(n) \right).$$

(2) If

$$(G_y^0) = \left(SO(2k+1) \xrightarrow{a\rho_{2k+1} \oplus (n-(2k+1))\theta} SU(n) \right),$$

where $a = 1, 3$ and $3|(2k-1)$ if $a = 3$, then

$$(G_y^0 \subseteq G_x^0 \subseteq SU(n)) = \left(SO(2k+1) \xrightarrow{\rho_{2k+1} \oplus (l-(2k+1))\theta} SO(2l+1) \xrightarrow{a\rho_{2l+1} \oplus (n-(a(2l+1)))\theta} SU(n) \right).$$

(3) If

$$(G_y^0) = \left(S_p(k) \xrightarrow{a\gamma_{2k} \oplus (n-2ak)\theta} SU(n) \right)$$

where $a = 1, 3$ and $3|(k-1)$ if $a = 3$, then

$$(G_y^0 \subseteq G_x^0 \subseteq SU(n)) = \left(S_p(k) \xrightarrow{\gamma_{2k} \oplus 2(l-k)\theta} S_p(l) \xrightarrow{a\gamma_{2l} \oplus (n-2al)\theta} SU(n) \right).$$

(4) If

$$(G_y^0) = \left(SO(2k) \xrightarrow{a\rho_{2k} \oplus (n-2ak)\theta} SU(n) \right)$$

where $a = 1, 2, 3, 6$ and $3|(k-1)$ if $a = 3, 6$, then

$$(G_y^0 \subseteq G_x^0 \subseteq SU(n)) = \left(SO(2k) \xrightarrow{\rho_{2k} \oplus (n-2k)\theta} SO(2l) \xrightarrow{a\rho_{2l} \oplus (n-2al)\theta} SU(n) \right).$$

Proof. As an example of the method, we shall give the proof for

$$(G_y^0) = \left(\text{SU}(k) \xrightarrow{a\mu_k \oplus b\mu_k^* \oplus (n-2k)\theta} \text{SU}(n) \right)$$

where $a + b = 2$, $a \geq 1$, and $k \geq 6$.

As y is regular, after conjugating G_x by a suitable element of $\text{SU}(n)$, we may suppose $G_y^0 \subseteq G_x^0$. Hence we first must study the lattice of closed connected subgroups $G_y^0 \subseteq H \subseteq \text{SU}(n)$.

If V is a subspace of $\mathbb{C}^n$, let $\text{SU}(V)$ be the subgroup of $\text{SU}(n)$ which leaves $V^\perp$ fixed pointwise. Then in [5], the following theorem is proven.

Theorem. *Let V be a subspace of $\mathbb{C}^n$ with $\dim V \geq 3$. Let H be a subgroup of $\text{SU}(n)$ which contains $\text{SU}(V)$. Then there exists a subspace V_1 of $\mathbb{C}^n$ with $V \subseteq V_1$ so that $\text{SU}(V_1)$ is a normal subgroup of H .*

We shall determine a similar result for G_y^0 . Suppose $\phi: G_y^0 \rightarrow H$, $\psi: H \rightarrow \text{SU}(n)$, and $\lambda = \phi \circ \psi$. We may take $H = T^c \times U_1 \times \cdots \times U_d$ where T^c is a torus of rank c , U_j is a compact, simply connected, simple Lie group, and where ϕ and ψ have finite kernels. Then $\phi = \phi_T \oplus \phi_1 \oplus \cdots \oplus \phi_d$. Clearly ϕ_T is trivial as $\text{SU}(k)$ is simple and T^c is abelian. So we may take $V = U_1 \times \cdots \times U_d$, and $\phi = \phi_1 \oplus \cdots \oplus \phi_d$. Now $\psi = \sum_{j=1}^t \psi_{j_1} \otimes \cdots \otimes \psi_{j_d}$ where each ψ_{j_i} is either trivial or irreducible.

Let $(\psi \circ \phi)_j = (\psi_{j_1} \circ \phi_1) \otimes \cdots \otimes (\psi_{j_d} \circ \phi_d)$. Note that $(\psi \circ \phi)_j$ is a nontrivial representation for some j .

For some $1 \leq i \leq m$, $\psi_{j_i} \circ \phi_i: \text{SU}(k) \rightarrow \text{SU}(n)$ is nontrivial. Note that the nontrivial components of $\psi_{j_i} \circ \phi_i$ have dimension $\geq k$. Also if there exists $1 \leq \alpha \leq m$ with $i \neq \alpha$ and $\psi_{j_\alpha} \circ \phi_\alpha$ nontrivial, then the nontrivial components of $(\psi \circ \phi)_j$ have dimension $\geq k^2$ which is greater than $2k$.

So as $\psi \circ \phi = a\mu_k \oplus b\mu_k^* \oplus (n-2k)\theta$ and $a + b = 2$, it is clear that we may assume

(1) $(\psi \circ \phi)_j$ is trivial if $j > 2$.

(2) If $j = 1, 2$, then there exists at most one integer i_j with $1 \leq i_j \leq m$ so that $\psi_{j_{i_j}} \circ \phi_{i_j}$ is nontrivial.

So G_y^0 is contained in a normal subgroup V of H , where $V = U$, or $V = U_1 \times U_2$.

Suppose $V = \text{SU}(l)$. Then we may write $\phi: \text{SU}(k) \rightarrow \text{SU}(l)$, $\psi: \text{SU}(l) \rightarrow \text{SU}(n)$, and $\psi \circ \phi = \lambda = a\mu_k \oplus b\mu_k^* \oplus (n-2k)\theta$, where $a + b = 2$. So if we write $\psi = \psi_1 \oplus \cdots \oplus \psi_t \oplus L\theta: \text{SU}(l) \rightarrow \text{SU}(n)$, where ψ_j is irreducible for $1 \leq j \leq t$, then $(\psi_1 \circ \phi) \oplus \cdots \oplus (\psi_t \circ \phi) \oplus L\theta = \psi \circ \phi = a\mu_k \oplus b\mu_k^* \oplus (n-2k)\theta$.

Hence $\psi = \psi_1 \oplus L\theta$ or $\psi = \psi_1 \oplus \psi_2 \oplus L\theta$.

Now $l^2 - 1 \geq k^2 - 1 > 8n/3$. Hence ψ_1 and ψ_2 have drastic limitations on their dimensions. Using the fact that $\phi > \psi$ implies $\dim \phi > \dim \psi$, it follows

that the only possible irreducible components of ψ are μ_l and μ_l^* . Arguing similarly for ϕ , we see that the only possible irreducible components of ϕ are μ_k and μ_k^* .

Hence, it is clear that one of the following cases holds:

- (1) $\phi = a\mu_k \oplus b\mu_k^* \oplus (l-2k)\theta$, $\psi = \mu_l \oplus (n-l)\theta$.
- (2) $\phi = b\mu_k \oplus a\mu_k^* \oplus (l-2k)\theta$, $\psi = \mu_l^* \oplus (n-l)\theta$.
- (3) $\phi = \mu_k \oplus (l-k)\theta$, $\psi = a\mu_l \oplus b\mu_l^* \oplus (n-2l)\theta$.
- (4) $\phi = \mu_k^* \oplus (l-k)\theta$, $\psi = b\mu_l \oplus a\mu_l^* \oplus (n-2l)\theta$.

Arguing similarly for the case when V is some other simple compact Lie group (and using the fact that $\dim V > k^2 - 1 > 8n/3$), we get the following additional possibilities, where $\phi: \text{SU}(k) \rightarrow V$, $\psi: V \rightarrow \text{SU}(n)$, and $\psi \circ \phi = \lambda$.

V	ϕ	ψ
$S_p(l) \ (a = 1 = b)$	$\mu_k \oplus \mu_k^* \oplus 2(l-k)\theta$	$\gamma_{2l} \oplus (n-2l)\theta$
$\text{SO}(l) \ (a = 1 = b)$	$\mu_k \oplus \mu_k^* \oplus (l-2k)\theta$	$\rho_l \oplus (n-l)\theta$

Now suppose $V = U_1 \times U_2$. Then using arguments similar to the above, one shows that $V = \text{SU}(l_1) \times \text{SU}(l_2)$, and that we can write $\phi = \phi_1 \oplus \phi_2$ and $\psi = (\psi_1 \oplus \theta) \oplus (\theta \oplus \psi_2)$ where modulo trivial copies,

$$\begin{aligned}\phi_1 &= \mu_k, \mu_k^*, & \phi_2 &= \mu_k, \mu_k^*, \\ \psi_1 &= \mu_{l_1}, \mu_{l_1}^*, & \psi_2 &= \mu_{l_2}, \mu_{l_2}^*.\end{aligned}$$

We now specialize to the case $H = G_x^0$.

We shall show that $H = G_x^0$ implies $V = \text{SU}(l)$, and that in fact

$$(G_y^0 \subset V \subset \text{SU}(n)) = \left(\text{SU}(k) \xrightarrow{\mu_k \oplus (l-k)\theta} \text{SU}(l) \xrightarrow{a\mu_l \oplus b\mu_l^* \oplus (n-2k)\theta} \text{SU}(n) \right).$$

Let $\xi_1 = (G_x \rightarrow G \rightarrow G/G_x)$. Then for $j = 1, 2$,

$$0 = P_j(r(G/G_x) \oplus \nu(G/G_x)) = P_j(\alpha_{\xi_1}(\text{Ad}_G|_{G_x} - \text{Ad}_{G_x}) \oplus \alpha_{\xi_1}(\phi_x)).$$

Now V is a connected normal subgroup of G_x^0 so that

$$\begin{array}{c} G_y^0 \xrightarrow{\phi} V \subseteq G_x \subseteq \text{SU}(n) = G \\ \quad \quad \quad \searrow \psi \quad \nearrow \\ \quad \quad \quad \lambda \end{array}$$

i.e. $\psi \circ \phi = \lambda$.

Let $\pi: G/V \rightarrow G/G_x$ be the natural map.

Let $\xi_2 = (V \xrightarrow{\psi} G \rightarrow G/V)$. Then if Γ is a real representation of G_x ,

$\pi^!(\alpha_{\xi_1}(\Gamma)) = \alpha_{\xi_2}(\Gamma|V)$. So

$$\begin{aligned}\pi^!(\alpha_{\xi_1}(\text{Ad}_G|G_x - \text{Ad}_{G_x})) \oplus \alpha_{\xi_1}(\phi_x) &= \alpha_{\xi_2}(\text{Ad}_G|V - \text{Ad}_{G_x}|V) \oplus \alpha_{\xi_2}(\phi_x|V) \\ &= \alpha_{\xi_2}(\text{Ad}_G|V - \text{Ad}_V - \theta) \oplus \alpha_{\xi_2}(\phi_x|V) \\ &= \tau(G/V) \oplus \alpha_{\xi_2}(\phi_x|V) \in \tilde{KO}(G/V),\end{aligned}$$

where θ is some trivial representation of V . So modulo torsion,

$$P_1(G/V) + P_1(\alpha_{\xi_2}(\phi_x|V)) = 0,$$

and

$$P_2(G/V) + P_1(G/V) \cdot P_1(\alpha_{\xi_2}(\phi_x|V)) + P_2(\alpha_{\xi_2}(\phi_x|V)) = 0.$$

Recall that as G_y is the principal isotropy subgroup of ϕ_x , $\phi_x|_{G_y} = \text{Ad}_{G_x}|_{G_y} - \text{Ad}_{G_y} \oplus$ trivial copies. So

$$\begin{aligned}\phi_x|_{G_y}^0 &= \text{Ad}_{G_x}|_{G_y}^0 - \text{Ad}_{G_y}|_{G_y}^0 \oplus \text{trivial copies} \\ &= ((\text{Ad}_{G_x}|_{G_x}^0)|V)|_{G_y}^0 - \text{Ad}_{G_y}^0 \oplus \text{trivial copies} \\ &= \text{Ad}_V|_{G_y}^0 - \text{Ad}_{G_y}^0 \oplus \text{trivial copies} \\ &= \text{Ad}_V \circ \phi - \text{Ad}_{G_y}^0 \oplus \text{trivial copies}.\end{aligned}$$

Note also that $\dim(\iota_{\text{SU}(n)/G_x} \oplus \phi_x) = \dim M \leq n^2 - 8n/3 - 2$, and hence $\dim \phi_x \leq n^2 - 8n/3 - 2$.

We shall now examine the several possibilities for V , ϕ , and ψ .

Suppose $V = \text{SU}(l)$, and $\phi = a\mu_k \oplus b\mu_k^* \oplus (l - 2k)\theta$, $\psi = \mu_l \oplus (n - l)\theta$. Then

$$\begin{aligned}\phi_x|_{\text{SU}(k)} &= \phi_x \circ \phi = (\mu_l \otimes \mu_l^* - \theta) \circ \phi - \mu_k \otimes \mu_k^* \oplus \text{trivial copies} \\ &= ab(\mu_k \otimes \mu_k \oplus \mu_k^* \otimes \mu_k^*) \oplus (a^2 + b^2 - 1)\mu_k \otimes \mu_k^* \\ &\quad \oplus (a + b)(l - (a + b)k)(\mu_k \oplus \mu_k^*) \oplus \text{trivial copies}.\end{aligned}$$

Recall $k^2 - 1 > 8n/3$, and $l \geq 2k$. So $\dim \phi_x < n^2 < 9k^4/64 < 9l^4/1024$.

This is a severe limitation on ϕ_x . By listing all the irreducible representations Γ of $\text{SU}(l)$ with $\dim \Gamma < 9l^4/1024$ and then using our determination of $\phi_x \circ \phi$, we see easily that $\phi_x|_{\text{SU}(l)}$ can contain copies only of μ_l , μ_l^* , $\mu_l \otimes \mu_l$, $\mu_l \otimes \mu_l^*$, $\mu_l^* \otimes \mu_l^*$. Hence

$$\phi_x|_{\text{SU}(l)} = c(\mu_l \otimes \mu_l \otimes \mu_l^* \otimes \mu_l^*) \otimes d\mu_l \otimes \mu_l^* \otimes e(\mu_l \otimes \mu_l^*) \otimes F\theta,$$

and so $\phi_x|_{\text{SU}(k)} = \phi_x \circ \phi = (4abc + (a^2 + b^2)d)\mu_k \otimes \mu_k^* \oplus$ a sum of representations not containing $\mu_k \otimes \mu_k^*$. So $4abc + (a^2 + b^2)d = a^2 + b^2 - 1$. This is impossible.

In the other cases for V , ϕ, ψ , we argue similarly to the above. In some cases there will exist a representation $\phi_x|_V$ so that $\phi_x \circ \phi = \lambda$. In these cases, by using $P_1(G/V) + P_1(a_{\xi}(\phi_x|_V)) = 0$, we arrive at the conclusion that $V = \text{SU}(l)$, $\phi = \mu_k \otimes (l - k)\theta$, and $\psi = a\mu_l \otimes b\mu_l^*$, i.e.

$$(G_y^0 \subseteq V \subseteq \text{SU}(n)) = \left(\text{SU}(k) \xrightarrow{\mu_k \otimes (l-k)\theta} \text{SU}(l) \xrightarrow{a\mu_l \otimes b\mu_l^* \otimes (n-2k)\theta} \text{SU}(n) \right)$$

as was to be shown.

Finally we shall show $G_x^0 = V$.

So suppose $G_x^0 \neq V$. We shall derive a contradiction. We give here a modification of an argument found in [4].

We may suppose x was chosen so that $\dim G_x$ is minimal among these G_t with G_t^0 not simple. We see that after conjugating if necessary, we have $y \in S_x$, the slice at x . Let ϕ_x be the slice representation of G_x at x . Then $\phi_x|_{G_x^0} = \phi_1 \oplus \beta\theta$, where ϕ_1 is a nontrivial representation without trivial copies.

We may write $y = \bar{y} + \eta$ where $\bar{y} \in W_{\phi_1}$, the representation space of ϕ_1 , and where $\eta \in (W_{\phi_1})^\perp$ i.e. $\eta \in W_{\beta\theta}$.

Clearly $G_y = G_{\bar{y}}$ and so $\bar{y}$ is a regular element.

Let z be an element of the unit ball of $W_{\phi_1} \subseteq S_x \subseteq M$. To show: $(G_x^0(\phi_1)_z)^0 \subseteq V$. By the minimality condition

$$(G_z^0) = \left(\text{SU}(l') \xrightarrow{a\mu_{l'} \otimes b\mu_{l'}^* \otimes (n-2l')\theta} \text{SU}(n) \right).$$

So as $G_x^0 \subseteq G_z^0 \subseteq N_{\text{SU}(n)}(V)$, clearly $G_z^0 \subseteq V$. Hence $(G_x^0(\phi_1)_z)^0 = G_z^0 \subseteq V$. So we may apply the following proposition found in [4].

Proposition. *Let H be a closed, connected, proper normal subgroup of a compact, connected, Lie group K .*

Let $\psi: K \rightarrow O(m)$ be a representation, such that for each $x \in W_\psi - \{0\}$, $K_x^0 \subseteq H$. Then

- (1) $H \subseteq \ker \psi$.
- (2) $\text{rank } K/H = 1$.

Hence $V \subseteq \ker \phi_1 = \ker \phi_x|_{G_x^0}$ and $\text{rank } G_x^0/V = 1$. So $\phi_x|_{G_y^0}$ is trivial. But note that $\phi_x \circ \phi = (l - k)(\mu_k \oplus \mu_k^*) \oplus \text{trivial copies, as}$

$$\begin{aligned}\phi_x|_{G_y^0} &= (\phi_x|_{G_y})|_{G_y^0} = (\text{Ad}_{G_x}|_{G_y} - \text{Ad}_{G_y})|_{G_y^0} \\ &= (\text{Ad}_{G_x}|_V)|_{G_y^0} - \text{Ad}_{G_y}|_{G_y^0} \\ &= \text{Ad}_V|_{G_y^0} - \text{Ad}_{G_y^0} \oplus \theta.\end{aligned}$$

Hence $l = k$, and so $V = G_y^0$. So $G_x^0 \sim G_y^0 \times L$, where either $L = \text{SU}(2)$ or $L = T^1$.

Note that $\phi_x|_{G_x^0} = \theta \otimes \gamma$. Let $\{a_1\}$ be the standard base for "the" Cartan subalgebra of L . Suppose

$$(G_x^0 \rightarrow \text{SU}(n)) = \left(\text{SU}(k) \times L \xrightarrow{((a\mu_k \oplus b\mu_k^* \oplus t\theta) \otimes \Gamma) \otimes (\theta \otimes \delta)} \text{SU}(n) \right).$$

Then $\dim \Gamma = 1$. So $\Gamma = \theta$ or $L = T^1$. Suppose $\Gamma = \theta$. Let $\xi_3 = (G_x^0 \rightarrow \text{SU}(n) \rightarrow \text{SU}(n)/G_x^0)$. Then

$$P_1(\alpha_{\xi_3}(\iota_{\text{SU}(n)/G_x^0}) \oplus \alpha_{\xi_3}(\phi_x|_{G_x^0})) = 0.$$

So

$$\begin{aligned}M(2\sigma_2(\mu_k) + \sigma_2(\delta)) &= \sigma_2(\iota_{\text{SU}(n)/G_x^0}) + \sigma_2(\phi_x|_{G_x^0}) \\ &= \sigma_2(((a\mu_k \oplus b\mu_k^* \oplus t\theta) \otimes \theta) \oplus (\theta \otimes \delta)) \\ &\quad \otimes (((a\mu_k^* \oplus b\mu_k \oplus t\theta) \otimes \theta) \oplus (\theta \otimes \delta^*)) - \mu_k^* \otimes \mu_k^* - \text{Ad}_L) + \sigma_2(\gamma) \\ &= (6k + 4t + 4 \dim \delta)\sigma_2(\mu_k) + (4k + 2t)\sigma_2(\delta) + 2 \dim \delta \cdot \sigma_2(\delta) - \sigma_2(\text{Ad}_L) + \sigma_2(\gamma).\end{aligned}$$

Hence $2M = 6k + 4t + 4 \dim \delta$. So

$$(3k + 2t + 2 \dim \delta)\sigma_2(\delta) = (4k + 2t)\sigma_2(\delta) + 2 \dim \delta \cdot \sigma_2(\delta) - \sigma_2(\text{Ad}_L) + \sigma_2(\gamma)$$

or

$$-k\sigma_2(\delta) = \sigma_2(\gamma) - \sigma_2(\text{Ad}_L).$$

So as $\sigma_2(\delta)$, $\sigma_2(\gamma)$, $\sigma_2(\text{Ad}_L)$ are all polynomials in $r(a_1)$ with negative coefficients, we see $L = \text{SU}(2)$. So $\sigma_2(\text{Ad}_{\text{SU}(2)}) = \sigma_2(\gamma) + k\sigma_2(\delta)$. An easy check of $RO(\text{SU}(2))$ shows this is impossible.

So $\Gamma \neq \theta$. Hence $L = T^1$. By considering $L \hookrightarrow \text{SU}(n)$, we see $(2k + t)\Gamma \oplus \delta$:

$L \rightarrow \text{SU}(n)$. We easily have

$$\begin{aligned}
 M(2\sigma_2(\mu_k) + \sigma_2((2k+t)\Gamma \oplus \delta)) &= M\sigma_2(((a\mu_k \oplus b\mu_k^* \oplus t\theta) \otimes \Gamma) \oplus (\theta \otimes \delta)) \\
 &= \sigma_2((((a\mu_k \oplus b\mu_k^* \oplus t\theta) \otimes \Gamma) \oplus (\theta \otimes \delta)) \otimes (((a\mu_k^* \oplus b\mu_k \oplus t\theta) \otimes \Gamma^*) \oplus (\theta \otimes \delta^*)) - \mu_k \otimes \mu_k^*) + \sigma_2(\gamma) \\
 &= \sigma_2((a^2 + b^2 - 1)\mu_k \otimes \mu_k^* \oplus ab(\mu_k \otimes \mu_k \oplus \mu_k^* \otimes \mu_k^*) \oplus (a+b)t(\mu_k \otimes \mu_k^*) \\
 &\quad \oplus (a\mu_k \oplus b\mu_k^* \oplus t\theta) \otimes (\Gamma \otimes \delta^*) \oplus (a\mu_k^* \oplus b\mu_k \oplus t\theta) \otimes (\Gamma^* \otimes \delta) \oplus \theta \otimes (\delta \otimes \delta^*)) + \sigma_2(\gamma) \\
 &= (2k((a+b)^2 - 1) + 2(a+b)t + 2(a+b) \dim \delta)\sigma_2(\mu_k) \\
 &\quad + \sigma_2((2k+t)(\Gamma \otimes \delta^* \oplus \Gamma^* \otimes \delta) \oplus \theta \otimes (\delta \otimes \delta^*)) + \sigma_2(\gamma) \\
 &= (6k + 4t + 4 \dim \delta)\sigma_2(\mu_k) + \sigma_2(((2k+t)\Gamma \oplus \delta) \otimes ((2k+t)\Gamma^* \oplus \delta^*)) \\
 &= (6k + 4t + 4 \dim \delta)\sigma_2(\mu_k) + (4k + 2t + 2 \dim \delta)\sigma_2((2k+t)\Gamma \oplus \delta) + \sigma_2(\gamma).
 \end{aligned}$$

So $2M = 6k + 4t + 4 \dim \delta$, or $M = 3k + 2t + 2 \dim \delta$. Hence

$$(3k + 2t + 2 \dim \delta)\sigma_2((2k+t)\Gamma \oplus \delta) = (4k + 2t + 2 \dim \delta)\sigma_2((2k+t)\Gamma \oplus \delta) + \sigma_2(\gamma).$$

So $0 = k\sigma_2((2k+t)\Gamma \oplus \delta) + \sigma_2(\gamma)$. But as $\phi_x|_{G_x^0} = \theta \otimes \gamma$, we see that $\gamma \otimes \mathbb{C} : L \rightarrow \text{SU}(n)$, and hence $k\sigma_2((2k+t)\Gamma \oplus \delta)$ and $\sigma_2(\gamma)$ are polynomials in $\tau(a_1)$ with negative coefficients. This is impossible.

The other cases follow similarly.

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