IRREGULARITIES OF DISTRIBUTION. VIII⁽¹⁾

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ABSTRACT. If $x_1, x_2, \dots$ is a sequence in the unit interval $0 \leq x < 1$ and if S is a subinterval, write $C(n, S)$ for the number of elements among $x_1, \dots, x_n$ which lie in S , minus n times the length of S . For a well distributed sequence, $C(n, S)$ as a function of n will be small. It is shown that the lengths of the intervals S for which $C(n, S)(n = 1, 2, \dots)$ is bounded form at most a countable set.

1. Introduction. The present paper is independent of the preceding papers of this series. However, the reader would be advised to first read the sixth paper [3] of the series, which deals with a similar but rather simpler problem.

We shall be concerned with the distribution of an arbitrary given sequence $x_1, x_2, \dots$ of points in the unit cube of k -dimensional Euclidean space. This unit cube U^k consists of the points $x = (x_1, \dots, x_k)$ with $0 \leq x_i < 1 (i = 1, \dots, k)$.

Let S be a measurable subset of U^k of measure $\mu(S)$, and write $Z(n, S)$ for the number of points among $x_1, \dots, x_n$ which lie in S . The quantity

$$(1) \quad D(n, S) = |Z(n, S) - n\mu(S)|$$

tells us how far $Z(n, S)$ deviates from the "expected" number $n\mu(S)$. Put

$$(2) \quad E(S) = \sup_n D(n, S),$$

and call $E(S)$ the *error* of S . We shall show in the present paper that very few boxes B with sides parallel to the coordinate axes have a finite error $E(B)$.

By a *subinterval* of U^1 we shall mean a single point or an open, half-open or closed interval of positive length which is contained in U^1 . By a *box* contained in U^k we shall understand a set $B = I_1 \times \dots \times I_k$, where $I_1, \dots, I_k$ are subintervals of U^1 . Thus B consists of points $x = (x_1, \dots, x_k)$ with $x_j \in I_j (j = 1, \dots, k)$.⁽²⁾ Write $\mathfrak{B}_t$ for the class of sets which are unions of at most t boxes in U^k .

For $\kappa > 0$, let $M_t(\kappa)$ be the set of numbers μ of the type $\mu = \mu(A)$ where $A \in \mathfrak{B}_t$ and $E(A) < \kappa$. Let $M_t(\infty)$ be the set of numbers μ of the type $\mu = \mu(A)$ with $A \in \mathfrak{B}_t$ and $E(A) < \infty$; thus $M_t(\infty)$ is the union of the sets $M_t(\kappa)$ with $0 < \kappa < \infty$. Recall that a number γ is a *limit point* of a set M of reals if there is a sequence of distinct elements of M which converge to γ . The *derivative* $M^{(1)}$ of

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⁽²⁾ In the theory of uniform distribution, one usually studies the "discrepancy" function $D(n) = \sup D(n, B)$, with the supremum taken over all boxes B in U^k , rather than the error $E(B)$, as in the present paper.

M consists of all the limit points of M . The higher derivatives are defined inductively by $M^{(d)} = (M^{(d-1)})^{(1)}$ ($d = 2, 3, \dots$).

Theorem 1. *Suppose $d > 8\kappa > 0$. Then $(M_t(\kappa))^{(d)}$ is empty.*

Since a set M having $M^{(d)}$ empty for some d is nowhere dense and is at most countable, we obtain the

Corollary. *Each set $M_t(\kappa)$ is nowhere dense and is at most countable. The set $M_t(\infty)$ is at most countable.*

In [3] we proved a result like Theorem 1 in the one-dimensional case for intervals I whose left endpoint was 0. The generalization to arbitrary intervals causes considerable additional difficulties in the proof; the generalization to arbitrary dimension and the generalization to unions of t boxes are easy.

As far as I know, the only interesting sequences for which the boxes B with finite $E(B)$ have been determined are the sequences

$$(3) \quad x_n = \{an + \beta\} \quad (n = 1, 2, \dots)$$

in the one-dimensional case, where the notation $\{\xi\}$ denotes the fractional part of ξ . Let us define an *interval modulo 1* as either a subinterval I of U^1 or the union of two subintervals I_1, I_2 of U^1 such that $0 \in I_1$ and I_2 contains every number less than 1 which is sufficiently close to 1. (In particular, every interval modulo 1 lies in $\mathfrak{B}_2$. A suitably defined *box modulo 1* would lie in $\mathfrak{B}_{2\kappa}$; in fact we defined the class $\mathfrak{B}_t$ in order to allow boxes modulo 1.) Now for a sequence (3), $E(J)$ is finite where J is an interval modulo 1 if (Ostrowski [2]) and only if (Kesten [1]; this is the hard part) $\mu(J) = \{\alpha l\}$ for some integer l . In particular, the set $M_1(\infty)$ is infinite if α is irrational.

No particular importance attaches to the number 8κ in Theorem 1. But in [3] it was shown that the van der Corput sequence has $(M_1(d))^{(d)}$ nonempty for $d = 1, 2, \dots$.

Theorem 1 probably remains true if the class $\mathfrak{B}_t$ is replaced by the class $\mathfrak{P}_t$ of polyhedrons with at most t faces or by the class $\mathfrak{E}$ of ellipsoids contained in U^k . But Theorem 1 is not true for the class of convex subsets of U^k when $k > 1$:

Theorem 2. *Suppose $k > 1$. There is a sequence $x_1, x_2, \dots$ in U^k such that for every μ in $0 \leq \mu \leq 1$, there is a convex set S in U^k with $\mu(S) = \mu$ and with $E(S) \leq \frac{1}{2}$.*

The fairly easy proof of Theorem 2 is given in the last section and is independent of the rest of the paper. The proof of Theorem 1 is unfortunately rather long. It would be desirable to have a simpler proof.

2. Preliminaries. Every interval is of one of the following four types: (i) $\alpha \leq x \leq \beta$, (ii) $\alpha \leq x < \beta$, (iii) $\alpha < x \leq \beta$, (iv) $\alpha < x < \beta$. Now for a box $B = I_1 \times \dots \times I_k$, each interval I_j is of one of four possible types, and hence we have 4^k types of boxes. There are 4^{kt} types of t -tuples of boxes $B_1, \dots, B_t$, and

hence there is a finite number of types of elements of $\mathfrak{B}_t$. It clearly will suffice to prove Theorem 1 for each subclass of $\mathfrak{B}_t$ whose elements are of a given type. For the sake of simplicity we shall only deal with the type where each interval I used in the definition of boxes is of the type $\alpha \leq x < \beta$. Denote such an interval by $I[\alpha, \beta)$.

For $1 \leq i \leq k$, let $B_i(\alpha)$ be the set of points x in U^k with $\alpha \leq x_i < 1$, and for $k+1 \leq i \leq 2k$, let $B_i(\alpha)$ be the box of points x in U^k with $0 \leq x_{i-k} < \alpha$. For $\alpha_1, \dots, \alpha_{2k}$ with $0 \leq \alpha_j \leq 1$ ($j = 1, \dots, 2k$), put

$$B(\alpha_1, \dots, \alpha_{2k}) = B(\alpha_1) \cap \dots \cap B(\alpha_{2k}).$$

Then $B(\alpha_1, \dots, \alpha_{2k})$ is a box, and every box of the type described above may be written as $I[\alpha_1, \alpha_{k+1}) \times \dots \times I[\alpha_k, \alpha_{2k}) = B(\alpha_1, \dots, \alpha_{2k})$. For $\alpha_1, \dots, \alpha_{2kt}$ with $0 \leq \alpha_j \leq 1$ ($j = 1, \dots, 2kt$), put

$$A(\alpha_1, \dots, \alpha_{2kt}) = B(\alpha_1, \dots, \alpha_{2k}) \cup B(\alpha_{2k+1}, \dots, \alpha_{4k}) \cup \dots \\ \cup B(\alpha_{2k(t-1)+1}, \dots, \alpha_{2kt}).$$

Then $A(\alpha_1, \dots, \alpha_{2kt})$ is in $\mathfrak{B}_t$, and every element of $\mathfrak{B}_t$ of the type described above is a set $A(\alpha_1, \dots, \alpha_{2kt})$.

It will be convenient to write $2kt = m$, $\alpha = (\alpha_1, \dots, \alpha_{2kt})$ and $A(\alpha) = A(\alpha_1, \dots, \alpha_{2kt})$. Also put $\mu(\alpha) = \mu(A(\alpha))$. The vectors α will be restricted to the closed cube C in R^m defined by $0 \leq \alpha_i \leq 1$ ($i = 1, \dots, m$).

We shall call a finite or infinite sequence of real numbers $\alpha^{(1)}, \alpha^{(2)}, \dots$ *monotonic of the type $<$* if $\alpha^{(1)} < \alpha^{(2)} < \dots$, *monotonic of the type $=$* if $\alpha^{(1)} = \alpha^{(2)} = \dots$, and *monotonic of the type $>$* if $\alpha^{(1)} > \alpha^{(2)} > \dots$. Every infinite sequence of real numbers contains an infinite monotonic subsequence. A finite or infinite sequence of vectors $\alpha^{(1)} = (\alpha_1^{(1)}, \dots, \alpha_m^{(1)})$, $\alpha^{(2)} = (\alpha_1^{(2)}, \dots, \alpha_m^{(2)})$, $\dots$ will be called *monotonic of the type $(u_1, \dots, u_m)$* where each u_h is either $<$ or $=$ or $>$, if for $1 \leq h \leq m$, the sequence $\alpha_h^{(1)}, \alpha_h^{(2)}, \dots$ is monotonic of the type u_h .

Given subsets A, A' of U^k , the *symmetric difference*

$$A \hat{\ } A'$$

is the set of elements x which lie in A but not in A' , or in A' but not in A .

Lemma 1. Suppose $\alpha^{(1)}, \alpha^{(2)}, \dots$ is a monotonic sequence of vectors in C . Then no point x of U^k lies in more than $m = 2kt$ of the sets $A(\alpha^{(1)}) \hat{\ } A(\alpha^{(2)}), A(\alpha^{(2)}) \hat{\ } A(\alpha^{(3)}), \dots$.

Proof. $A(\alpha) = A(\alpha_1, \dots, \alpha_{2kt})$ is formed as a union and intersection of the $2kt$ sets $B_i(\alpha_{i+2kj})$ ($1 \leq i \leq 2k, 0 \leq j \leq t-1$). Therefore if for every i, j with $1 \leq i \leq 2k, 0 \leq j \leq t-1$, the point x behaves the same way with respect to $B_i(\alpha_{i+2kj})$ and $B_i(\alpha'_{i+2kj})$, i.e. lies in both or in neither of them, then x lies in either both $A(\alpha)$ and $A(\alpha')$ or in neither of them. Hence if $x \in A(\alpha) \hat{\ } A(\alpha')$, then there are i, j with $1 \leq i \leq 2k, 0 \leq j \leq t-1$, such that $x \in B_i(\alpha_{i+2kj}) \hat{\ } B_i(\alpha'_{i+2kj})$.

Thus to prove Lemma 1, it will suffice to show that for fixed i, j , a point x lies in at most one of the sets

$$(4) \quad B_i(\alpha_{i+2kj}^{(1)}) \wedge B_i(\alpha_{i+2kj}^{(2)}), \quad B_i(\alpha_{i+2kj}^{(2)}) \wedge B_i(\alpha_{i+2kj}^{(3)}), \quad \dots$$

But $B_i(\alpha)$ decreases with α if $1 \leq i \leq k$, and it increases with α if $k+1 \leq i \leq 2k$. The sequence $\alpha_{i+2kj}^{(1)}, \alpha_{i+2kj}^{(2)}, \dots$ is monotonic. Hence the sequence of sets $B_i(\alpha_{i+2kj}^{(1)}), B_i(\alpha_{i+2kj}^{(2)}), \dots$ is either increasing or decreasing (in the weak sense). Therefore x can lie in at most one of the sets (4).

The property enunciated in Lemma 1, together with the continuity of $\mu(\alpha)$, are the only properties of the sets $A(\alpha)$ which we shall need. It would be easy to construct other parameter families of sets with these properties, and hence other families of sets for which Theorem 1 holds.

3. Directed systems. Let $(i_1, \dots, i_d)$ and $(i'_1, \dots, i'_d)$ be d -tuples of positive integers. We shall write

$$(i_1, \dots, i_d) <_j (i'_1, \dots, i'_d)$$

if $i_1 = i'_1, \dots, i_{j-1} = i'_{j-1}$ and $i_j < i'_j$.

We are going to define *directed systems* of real numbers. A directed system of order 0 consists of a single real number α in $0 \leq \alpha \leq 1$. A directed system of order 1 is a finite monotonic sequence of reals $\alpha(1), \alpha(2), \dots, \alpha(l)$ with $l > 1$ terms in $0 \leq \alpha \leq 1$. Thus a directed system of order 1 is of some type (u) where u may be $<$, $=$ or $>$. For arbitrary $d \geq 1$, a directed system of order d is of some type $(u_1, \dots, u_d)$, where each u_i may be $<$, $=$ or $>$, and consists of integers $l_1, \dots, l_d$ greater than 1 and of real numbers $\alpha(i_1, \dots, i_d)$ ($1 \leq i_1 \leq l_1, \dots, 1 \leq i_d \leq l_d$) in the interval $0 \leq \alpha \leq 1$, such that if $1 \leq i_1, i'_1 \leq l_1, \dots, 1 \leq i_d, i'_d \leq l_d$ and $(i_1, \dots, i_d) <_j (i'_1, \dots, i'_d)$, then

$$\alpha(i_1, \dots, i_d) u_j \alpha(i'_1, \dots, i'_d).$$

For example, in a directed system of the type $(<, \dots, <)$, the numbers $\alpha(i_1, \dots, i_d)$ are ordered lexicographically.

Lemma 2. Suppose there exists a directed system of the type $(u_1, \dots, u_{j-1}, =, u_{j+1}, \dots, u_d)$ where $j < d$. Then the symbols $u_{j+1}, \dots, u_d$ are all the $=$ sign, i.e. the type is $(u_1, \dots, u_{j-1}, =, \dots, =)$.

Proof. Let $\alpha(i_1, \dots, i_d)$ belong to a directed system of the type $(u_1, \dots, u_{j-1}, =, u_{j+1}, \dots, u_d)$. Then $\alpha(1, \dots, 1, 1, i_{j+1}, \dots, i_d) = \alpha(1, \dots, 1, 2, i'_{j+1}, \dots, i'_d)$ for any numbers $1 \leq i_{j+1}, i'_{j+1} \leq l_{j+1}, \dots, 1 \leq i_d, i'_d \leq l_d$. Hence the $l_{j+1} \dots l_d$ numbers $\alpha(1, \dots, 1, i_{j+1}, \dots, i_d)$ with $1 \leq i_{j+1} \leq l_{j+1}, \dots, 1 \leq i_d \leq l_d$ are all equal, and the symbols $u_{j+1}, \dots, u_d$ must be the $=$ sign.

Next, we define directed systems of vectors α . A directed system of order zero consists of a single vector α in the cube C . A directed system of order d where $d \geq 1$ is of some type (u_{ih}) ($1 \leq i \leq d, 1 \leq h \leq m$), where each u_{ih} is either $<$

or $=$ or $>$, and consists of integers $l_1, \dots, l_d$ greater than 1 and of vectors $\alpha(i_1, \dots, i_d)$ ($1 \leq i_1 \leq l_1, \dots, 1 \leq i_d \leq l_d$) such that for each h in $1 \leq h \leq m$, the coordinates $\alpha_h(i_1, \dots, i_d)$ ($1 \leq i_1 \leq l_1, \dots, 1 \leq i_d \leq l_d$) form a directed system of reals of the type $(u_{1h}, \dots, u_{dh})$. That is, we have

$$\alpha_h(i_1, \dots, i_d) u_{jh} \alpha_h(i'_1, \dots, i'_d)$$

if $1 \leq i_1, i'_1 \leq l_1, \dots, 1 \leq i_d, i'_d \leq l_d$ and $(i_1, \dots, i_d) <_j (i'_1, \dots, i'_d)$.

4. A proposition which implies Theorem 1. By a *range* we shall understand a finite set of consecutive positive integers. Thus a range N will consist of integers $a + 1, a + 2, \dots, b$ where $0 \leq a < b$. The number $|N| = b - a$ will be called the *length* of the range, so that a range of length l contains exactly l integers. Now let $f(n)$ be a function defined on the positive integers, and let N be a range. We put

$$(5) \quad f^+(N) = \max_{n \in N} f(n), \quad f^-(N) = \min_{n \in N} f(n),$$

and

$$(6) \quad f^*(N) = f^+(N) - f^-(N).$$

For α in C , write

$$(7) \quad f(n, \alpha) = n\mu(\alpha) - Z(n, A(\alpha)).$$

The meaning of the notations $f^+(N, \alpha)$, $f^-(N, \alpha)$ and $f^*(N, \alpha)$ is then obvious.

Given a subset S of the cube C , write $M(S)$ for the set of numbers $\mu = \mu(\alpha)$ with $\alpha \in S$. For any set M of real numbers, put $M^{(0)} = M$.

Proposition.⁽³⁾ Suppose $d \geq 0$ and S is a subset of C such that $(M(S))^{(d)}$ contains a number μ in $0 < \mu < 1$. Suppose $\varepsilon > 0$. Then there is

(i) a positive integer r ,

(ii) a directed system of vectors $\alpha(i_1, \dots, i_d)$ ($1 \leq i_j \leq l_j$) of order d with $\alpha(i_1, \dots, i_d) \in S$ and $0 < \mu(\alpha(i_1, \dots, i_d)) < 1$, and

(iii) there are neighborhoods⁽⁴⁾ $N(i_1, \dots, i_d)$ of the numbers $\mu(\alpha(i_1, \dots, i_d))$, with the following property:

If N is a range with $|N| \geq r$, and if

$$\beta(i_1, \dots, i_d) \quad (1 \leq i_1 \leq l_1, \dots, 1 \leq i_d \leq l_d)$$

is a directed system with $\mu(\beta(i_1, \dots, i_d)) \in N(i_1, \dots, i_d)$, but not necessarily of the same type as the directed system $\alpha(i_1, \dots, i_d)$, then

$$(8) \quad (l_1 \dots l_d)^{-1} \sum_{i_1=1}^{l_1} \dots \sum_{i_d=1}^{l_d} f^*(N, \beta(i_1, \dots, i_d)) \geq \frac{1}{4}(d+1) + \frac{1}{12} - \varepsilon.$$

⁽³⁾ This proposition corresponds to the proposition in [3]. Also, Lemmas 3, 7 of the present paper correspond, respectively, to Lemmas 5, 4 of [3].

⁽⁴⁾ By a *neighborhood* of a real number μ we understand an open interval containing μ .

It might be in order to clarify the meaning of the proposition when $d = 0$. In this case the directed system consists of a single vector $\alpha \in S$. The hypothesis implies only that there is an $\alpha \in S$ with $0 < \mu(\alpha) < 1$. Hence this case may be restated as follows.

Case $d = 0$ of the proposition. Suppose $\alpha \in C$ with $0 < \mu(\alpha) < 1$, and suppose $\varepsilon > 0$. Then there exists an integer r and a neighborhood N of $\mu(\alpha)$ such that for every range N with $|N| \geq r$ and every β with $\mu(\beta) \in N$, we have

$$f^*(N, \beta) \geq \frac{1}{4} + \frac{1}{12} - \varepsilon.$$

The proof of the proposition will be postponed until later. At present we are going to show that the proposition implies Theorem 1. We have to show that $(M_i(\kappa))^{(d)}$ is empty if $d > 8\kappa > 0$. It will suffice to show that $(M_i(\kappa))^{(d)}$ contains no element μ with $0 < \mu < 1$ if $d > 8\kappa - 1$. Put differently, it will be enough to show that if $(M_i(\kappa))^{(d)}$ contains an element μ with $0 < \mu < 1$, then

$$(9) \quad \kappa \geq \frac{1}{8}(d + 1).$$

By what we said in §2, we may restrict ourselves to sets of $\mathfrak{B}$, of the type $A(\alpha)$ with $\alpha \in C$. Thus if $\overline{M}_i(\kappa)$ is the set of numbers $\mu = \mu(\alpha)$ with $E(A(\alpha)) < \kappa$, we have to show that (9) holds if there is a $\mu \in (\overline{M}_i(\kappa))^{(d)}$ with $0 < \mu < 1$. Let $S(\kappa)$ consist of the vectors α with $E(A(\alpha)) < \kappa$. Then $\overline{M}_i(\kappa) = M(S(\kappa))$. If $(\overline{M}_i(\kappa))^{(d)} = (M(S(\kappa)))^{(d)}$ contains an element μ with $0 < \mu < 1$, we apply the proposition with $\varepsilon = 1/12$, with a range N having $|N| \geq r$, and with $\beta(i_1, \dots, i_d) = \alpha(i_1, \dots, i_d)$ (or with $\beta = \alpha$ if $d = 0$), and we see that there is a $\beta \in S(\kappa)$ with

$$f^*(N, \beta) \geq \frac{1}{4}(d + 1).$$

Hence either $|f^+(N, \beta)| \geq \frac{1}{8}(d + 1)$ or $|f^-(N, \beta)| \geq \frac{1}{8}(d + 1)$, and there is an integer $n \in N$ with $|f(n, \beta)| \geq \frac{1}{8}(d + 1)$. Thus

$$D(n, A(\beta)) = |Z(n, A(\beta)) - n\mu(\beta)| = |f(n, \beta)| \geq \frac{1}{8}(d + 1),$$

and $E(A(\beta)) \geq \frac{1}{8}(d + 1)$. Since $\beta \in S(\kappa)$ and $E(A(\beta)) < \kappa$, we obtain (9).

5. An auxiliary lemma. If $f(n)$ is a function defined on the positive integers, and if N, N' are ranges, put

$$(10) \quad f^\nabla(N, N') = \max(0, f^-(N) - f^+(N'), f^-(N') - f^+(N)).$$

Lemma 3. Let $f(n), g(n)$ be defined on the positive integers, and let L, L' be subranges of a range N . Then

$$f^*(N) + g^*(N) \geq (f - g)^\nabla(L, L') + \frac{1}{2}(f^*(L) + g^*(L) + f^*(L') + g^*(L')).$$

Proof. Since both L, L' are contained in N , we have $f^*(N) \geq \max(f^*(L),$

$f^*(L')$ and $g^*(N) \geq \max(g^*(L), g^*(L'))$, so that the lemma is certainly true if $(f - g)^\nabla(L, L') = 0$. We therefore may assume without loss of generality that

$$(f - g)^\nabla(L, L') = (f - g)^-(L) - (f - g)^+(L') > 0.$$

Then we have for every $l \in L$ and every $l' \in L'$,

$$(11) \quad f(l) - g(l) - (f(l') - g(l')) \geq (f - g)^\nabla(L, L').$$

Let a_f, b_f, a_g, b_g be integers in L with

$$\begin{aligned} f(a_f) &= f^+(L), & f(b_f) &= f^-(L), \\ g(a_g) &= g^+(L), & g(b_g) &= g^-(L), \end{aligned}$$

so that

$$(12) \quad f(a_f) - f(b_f) = f^*(L),$$

$$(13) \quad g(a_g) - g(b_g) = g^*(L).$$

Similarly, choose a'_f, b'_f, a'_g, b'_g in L' with

$$(14) \quad f(a'_f) - f(b'_f) = f^*(L'),$$

$$(15) \quad g(a'_g) - g(b'_g) = g^*(L').$$

Applying (11) with $l = a_g, l' = a'_g$ we get

$$f(a_g) - g(a_g) - f(a'_g) + g(a'_g) \geq (f - g)^\nabla(L, L').$$

Applying (11) with $l = b_f, l' = b'_f$, we obtain

$$f(b_f) - g(b_f) - f(b'_f) + g(b'_f) \geq (f - g)^\nabla(L, L').$$

Adding these two inequalities and the four equations (12), (13), (14), (15), we obtain

$$\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 \geq 2(f - g)^\nabla(L, L') + f^*(L) + g^*(L) + f^*(L') + g^*(L'),$$

where

$$\begin{aligned} \varphi_1 &= f(a_f) - f(b'_f), & \varphi_2 &= g(a'_g) - g(b_g), \\ \varphi_3 &= f(a_g) - f(b'_g), & \varphi_4 &= g(a'_f) - g(b_f). \end{aligned}$$

Since $f^*(N) \geq \max(\varphi_1, \varphi_3)$ and $g^*(N) \geq \max(\varphi_2, \varphi_4)$, the lemma follows.

The lemma will not be used until §11.

6. The case $d = 0$ of the proposition. Write $\|\xi\|$ for the distance from a real number ξ to the nearest integer. Suppose $\mu = \mu(\alpha)$ lies in the open interval $0 < \mu < 1$. Then there is a positive integer q such that $\|\mu q\| \geq \frac{1}{3}$. This follows

from Kronecker's theorem if μ is irrational, and is easily proved if μ is rational, the worst case being when $\mu = \frac{1}{2}$ or $\frac{2}{3}$. Choose a neighborhood N of μ such that $|\mu' - \mu| < \varepsilon/q$ for every $\mu' \in N$. Then for every β with $\mu(\beta) \in N$,

$$\|\mu(\beta)q\| > \frac{1}{2} - \varepsilon.$$

Now put $r = q + 1$, and let $N = \{a + 1, a + 2, \dots, b\}$ be a range of length $|N| \geq r$. Then there are two integers n, n' in N , e.g. $n = a + 1$ and $n' = a + 1 + q$, such that

$$\|n\mu(\beta) - n'\mu(\beta)\| > \frac{1}{2} - \varepsilon.$$

Hence

$$|f(n', \beta) - f(n, \beta)| \geq \|n\mu(\beta) - n'\mu(\beta)\| > \frac{1}{2} - \varepsilon,$$

and in the notation of (6) we have

$$f^*(N, \beta) > \frac{1}{3} - \varepsilon = \frac{1}{4} + \frac{1}{12} - \varepsilon.$$

This finishes the proof of the case $d = 0$ of the proposition. The proposition in general will later be proved by induction on d .

7. Kronecker type lemmas.

Lemma 4. *There are positive valued functions $f_1(y_0), f_2(y_0, y_1), f_3(y_0, y_1, y_2), \dots$, defined for nonzero $y_0, y_1, y_2, \dots$, with the following property:*

Suppose l is a positive integer and

$$(16) \quad \begin{aligned} 0 < \varepsilon < 1, \quad 0 < |\delta_1| < f_1(\varepsilon), \quad 0 < |\delta_2| < f_2(\varepsilon, \delta_1), \quad \dots, \\ 0 < |\delta_l| < f_l(\varepsilon, \delta_1, \dots, \delta_{l-1}). \end{aligned}$$

Then there is a positive integer $p = p(l, \varepsilon, \delta_1, \dots, \delta_l)$ such that, for every range N with $|N| \geq p$ and arbitrary $\alpha_1, \dots, \alpha_l$, there is an $n \in N$ with

$$(17) \quad \{\delta_i n - \alpha_i\} < \varepsilon \quad (i = 1, \dots, l).$$

Proof. Put $f_1(y_0) = |y_0|$. Suppose $0 < \varepsilon < 1$ and $0 < |\delta_1| < f_1(\varepsilon) = \varepsilon$. Suppose at first that $\delta_1 > 0$, and put $p = [1/\delta_1] + 1$, where $[\xi]$ denotes the integer part of a real number ξ . The numbers $z_0 = 0, z_1 = \delta_1, \dots, z_{p-1} = [1/\delta_1]\delta_1$ lie in $0 \leq z \leq 1$, and given any α there is a z_i with $\{z_i - \alpha\} < \delta_1 < \varepsilon$. Thus there is an n with $0 \leq n \leq p - 1$ and $\{\delta_1 n - \alpha\} < \varepsilon$. Since this holds for every α , it is easily seen that in every range N with $|N| \geq p$, there is an n with $\{\delta_1 n - \alpha\} < \varepsilon$. The situation is similar if $\delta_1 < 0$.

Now suppose $l \geq 2$ and $f_1, \dots, f_{l-1}$ have been constructed and have the desired properties. Suppose

$$(18) \quad 0 < \varepsilon < 1, \quad 0 < |\delta_l| < f_l(\varepsilon), \quad \dots, \quad 0 < |\delta_{l-1}| < f_{l-1}(\varepsilon, \delta_1, \dots, \delta_{l-2}).$$

Put $p' = p(l-1, \varepsilon, \delta_1, \dots, \delta_{l-1})$ and

$$(19) \quad f_l(\varepsilon, \delta_1, \dots, \delta_{l-1}) = \varepsilon/2p'.$$

(Clearly, it does not matter how we define f_l if (18) is violated.)

Suppose

$$(20) \quad 0 < |\delta_l| < f_l(\varepsilon, \delta_1, \dots, \delta_{l-1}).$$

Then $0 < |\delta_l| < \frac{1}{2}\varepsilon = f_l(\frac{1}{2}\varepsilon)$, and by the case $l =$ of the lemma there is a $p'' = p(1, \frac{1}{2}\varepsilon, \delta_l)$ such that for every range N'' with $|N''| \geq p''$ and every α_l , there is an $n'' \in N''$ with $\{\delta_l n'' - \alpha_l\} < \frac{1}{2}\varepsilon$. Put

$$p = p(l, \varepsilon, \delta_1, \dots, \delta_l) = p' + p''.$$

Now let $\alpha_1, \dots, \alpha_l$ be arbitrary, and let N be a range with $|N| \geq p$. Assume at first that $\delta_l > 0$, and let N'' be the subrange of N with $|N''| = p''$ and with its smallest element the same as that of N . There is an $n'' \in N''$ with $\{\delta_l n'' - \alpha_l\} < \frac{1}{2}\varepsilon$. Let N' be the range $n'', n'' + 1, \dots, n'' + p' - 1$, so that $N' \subseteq N$. There is an $n \in N'$ with

$$\{\delta_l n - \alpha_l\} < \varepsilon \quad (i = 1, \dots, l-1).$$

Furthermore, $\{\delta_l n - \alpha_l\} = \{\delta_l(n - n'') + \delta_l n'' - \alpha_l\} = \delta_l(n - n'') + \{\delta_l n'' - \alpha_l\} < \delta_l p' + \frac{1}{2}\varepsilon < \varepsilon$ by (19) and (20). Hence (17) holds for $i = 1, \dots, l$.

The situation is analogous if $\delta_l < 0$. In this case we let the largest element of N'' coincide with that of N .

Lemma 5. Suppose l is a positive integer and suppose $\varepsilon, \delta_1, \dots, \delta_l$ satisfy (16). Let $p = p(l, \varepsilon, \delta_1, \dots, \delta_l)$ be the number p of Lemma 4. There are neighborhoods D_1 of $\delta_1, \dots, D_l$ of δ_l (which may depend on $l, \varepsilon, \delta_1, \dots, \delta_l$) with the following property:

Suppose N, N' are ranges with lengths $|N| \geq p, |N'| \geq p$. Suppose $\eta_1 \in D_1, \dots, \eta_l \in D_l$. Then there are integers $n \in N, n' \in N'$ such that

$$(21) \quad \|n\eta_i - n'\eta_i\| > \frac{1}{2} - \varepsilon \quad (i = 1, \dots, l).$$

Proof. Choose the neighborhood D_i of δ_i so small that

$$(22) \quad p|\eta_i - \delta_i| < \frac{1}{2}\varepsilon$$

for every $\eta_i \in D_i$ ($i = 1, \dots, l$). Now if N, N' are ranges with $|N| \geq p, |N'| \geq p$, pick n' arbitrary in N' and let n_0 be the smallest element in N .

Suppose $\eta_1 \in D_1, \dots, \eta_l \in D_l$ are given. Put

$$\alpha_i = (n' - n_0)\eta_i + n_0\delta_i + \frac{1}{2} - \frac{1}{2}\varepsilon \quad (i = 1, \dots, l).$$

By Lemma 4 there is an $n \in N$ with $n_0 \leq n \leq n_0 + p - 1$ and with $\{n\delta_i - \alpha_i\} < \varepsilon$ ($i = 1, \dots, l$). This is the same as

$$\frac{1}{2} - \frac{1}{2}\varepsilon \leq \{n\delta_i - (n' - n_0)\eta_i - n_0\delta_i\} < \frac{1}{2} + \frac{1}{2}\varepsilon.$$

Now

$$n\eta_i - n'\eta_i = n\delta_i - (n' - n_0)\eta_i - n_0\delta_i + (n - n_0)(\eta_i - \delta_i),$$

and since $|n - n_0| < p$ and since η_i in $\mathbf{D}_i$ satisfies (22) and hence $|n - n_0||\eta_i - \delta_i| < \frac{1}{2}\varepsilon$, we obtain $\frac{1}{2} - \varepsilon < \{n\eta_i - n'\eta_i\} < \frac{1}{2} + \varepsilon$, which is equivalent to (21).

We shall say that a function $g(n)$ is of the type η , where η is a real number, if

$$\{g(n+1) - g(n) - \eta\} = 0 \quad (n = 1, 2, \dots).$$

If N, N' are ranges of positive integers, we put

$$(23) \quad g^\square(N, N') = \min_{n \in N} \min_{n' \in N'} |g(n) - g(n')|.$$

This is not to be confused with the notation $f^\nabla(N, N')$ of (10).

Lemma 6. Suppose $l, \varepsilon, \delta_1, \dots, \delta_l$ and p and $\mathbf{D}_1, \dots, \mathbf{D}_l$ are as in Lemma 5. Let r be a positive integer, and assume that $\delta_1, \dots, \delta_l$ satisfy the inequalities

$$(24) \quad 0 < |\delta_i| < \varepsilon/(2r) \quad (i = 1, \dots, l),$$

and in fact that every $\eta_i \in \mathbf{D}_i$ satisfies

$$(25) \quad |\eta_i| < \varepsilon/(2r).$$

Then there is an integer $p^* = p^*(l, \varepsilon, \delta_1, \dots, \delta_l; r)$, such that if $\eta_1 \in \mathbf{D}_1, \dots, \eta_l \in \mathbf{D}_l$, and if $g_1(n), \dots, g_l(n)$ are functions of the types $\eta_1, \dots, \eta_l$ respectively, and if N, N' are ranges with $|N| \geq p^*, |N'| \geq p^*$, then there are subranges $L \subseteq N, L' \subseteq N'$ with

$$(26) \quad |L| = |L'| = r$$

and with

$$(27) \quad g_i^\square(L, L') > \frac{1}{2} - 2\varepsilon \quad (i = 1, \dots, l).$$

Proof. Put $p^* = \max(p, r)$. Suppose $\eta_1 \in \mathbf{D}_1, \dots, \eta_l \in \mathbf{D}_l$ and $|N| \geq p^*, |N'| \geq p^*$. By Lemma 5, there are integers $n \in N$ and $n' \in N'$ with (21). There is a range $L \subseteq N$ with $n \in L$ and $|L| = r$, and a range $L' \subseteq N'$ with $n' \in L'$ and $|L'| = r$. For $m \in L$ and $m' \in L'$ we have $|(m - n)\eta_i| \leq r|\eta_i| < \frac{1}{2}\varepsilon$ and $|(m' - n')\eta_i| < \frac{1}{2}\varepsilon$, so that, by (21),

$$\begin{aligned} |g_i(m) - g_i(m')| &\geq \|g_i(m) - g_i(m')\| = \|m\eta_i - m'\eta_i\| \\ &= \|n\eta_i - n'\eta_i + (m - n)\eta_i - (m' - n')\eta_i\| > \frac{1}{2} - \varepsilon - 2\varepsilon/2 > \frac{1}{2} - 2\varepsilon. \end{aligned}$$

Thus (27) holds.

8. Functions $f(n, \alpha, \beta)$. For α, β in C , put

$$(28) \quad \begin{aligned} f(n, \alpha, \beta) &= f(n, \alpha) - f(n, \beta) \\ &= n(\mu(\alpha) - \mu(\beta)) - (Z(n, A(\alpha)) - Z(n, A(\beta))). \end{aligned}$$

The function $f(n, \alpha, \beta)$ is of the type $\eta = \mu(\alpha) - \mu(\beta)$.

Lemma 7. Suppose $\alpha_1, \alpha_2, \dots$ are elements of C such that the numbers $\mu(\alpha_1), \mu(\alpha_2), \dots$ are all distinct and converge to a number μ with $0 < \mu < 1$. Suppose $0 < \varepsilon < 1, l \geq 1, r \geq 1$.

There is a finite subsequence $\alpha(1), \dots, \alpha(2l)$ of $\alpha_1, \alpha_2, \dots$ with $0 < \mu(\alpha(j)) < 1$ ($j = 1, \dots, 2l$), and there are neighborhoods N_j of $\mu(\alpha(j))$, and there is an integer q , with the following properties:

For any $\beta(1), \dots, \beta(2l)$ with $\mu(\beta(j)) \in N_j$ ($j = 1, \dots, 2l$), we have

$$(29) \quad |\mu(\beta(2i-1)) - \mu(\beta(2i))| < \varepsilon/(4r) \quad (i = 1, \dots, l).$$

Furthermore, if N, N' are ranges with $|N| \geq q, |N'| \geq q$, there are subranges $L \subseteq N, L' \subseteq N'$ with

$$(30) \quad |L| = |L'| = r$$

and with

$$(31) \quad f^\square(L, L', \beta(2i-1), \beta(2i)) > \frac{1}{2} - \varepsilon \quad (i = 1, \dots, l).$$

Proof. There is an integer j_1 with

$$0 < |\mu(\alpha_{j_1}) - \mu(\alpha_{j_1+1})| < \min(f_1(\frac{1}{2}\varepsilon), \varepsilon/(4r)),$$

where f_1 is the function of Lemma 4. Put $\delta_1 = \mu(\alpha_{j_1}) - \mu(\alpha_{j_1+1})$. There is a $j_2 > j_1 + 1$ such that

$$0 < |\mu(\alpha_{j_2}) - \mu(\alpha_{j_2+1})| < \min(f_2(\frac{1}{2}\varepsilon, \delta_1), \varepsilon/(4r)).$$

Put $\delta_2 = \mu(\alpha_{j_2}) - \mu(\alpha_{j_2+1})$. We continue in this manner, and choose integers $j_1 < j_1 + 1 < j_2 < j_2 + 1 < \dots < j_l < j_l + 1$, such that the numbers

$$\delta_i = \mu(\alpha_{j_i}) - \mu(\alpha_{j_i+1}) \quad (i = 1, \dots, l)$$

satisfy (16) and (24), with ε replaced by $\varepsilon/2$. Let $D_i = D_i(l, \varepsilon/2, \delta_1, \dots, \delta_l)$ ($i = 1, \dots, l$) and $p^* = p^*(l, \varepsilon/2, \delta_1, \dots, \delta_l; r)$ be as in Lemma 6. Put

$$\alpha(1) = \alpha_{j_1}, \quad \alpha(2) = \alpha_{j_1+1}, \quad \dots, \quad \alpha(2l-1) = \alpha_{j_l}, \quad \alpha(2l) = \alpha_{j_l+1},$$

so that $\mu(\alpha(2i-1)) - \mu(\alpha(2i)) = \delta_i$ ($i = 1, \dots, l$). Choose neighborhoods N_j of $\mu(\alpha(j))$ ($j = 1, \dots, 2l$) so small that $\mu_{2i-1} - \mu_{2i} \in D_i$ if $\mu_{2i-1} \in N_{2i-1}$ and μ_{2i}

$\in N_{2l}$. Then if $\beta(1), \dots, \beta(2l)$ are vectors with $\mu(\beta(j)) \in N_j (j = 1, \dots, 2l)$, (29) follows from the condition (25) (but with $\varepsilon/2$ in place of ε) on the neighborhoods $D_1, \dots, D_l$.

Now suppose that $|N| \geq q$, $|N'| \geq q$ where $q = p^*(l, \varepsilon/2, \delta_1, \dots, \delta_l; r)$. The function $g_i(n) = f(n, \beta(2i-1), \beta(2i))$ is of the type $\mu(\beta(2i-1)) - \mu(\beta(2i)) = \eta_i$, say, where $\eta_i \in D_i (i = 1, \dots, l)$. By Lemma 6, there are ranges $L \subseteq N$, $L' \subseteq N'$ with (26) and (27) (but with ε replaced by $\varepsilon/2$), i.e. with (30) and (31). This finishes the proof of Lemma 7.

Suppose again that $g(n)$ is a function of the type η . We shall say that g has a jump at n if either $n = 1$ and $g(1) \neq \eta$, or if $n > 1$ and $g(n) - g(n-1) - \eta \neq 0$. Recall that the function $f(n, \alpha, \beta)$ is defined in terms of a given sequence $x_1, x_2, \dots$.

Lemma 8. *The function $f(n, \alpha, \beta)$ of the type $\mu(\alpha) - \mu(\beta)$ has a jump at n if and only if*

$$x_n \in A(\alpha) \hat{\ } A(\beta).$$

Proof. $f(n, \alpha, \beta)$ has a jump at n precisely if

$$Z(n, A(\beta)) - Z(n, A(\alpha)) - (Z(n-1, A(\beta)) - Z(n-1, A(\alpha))) \neq 0.$$

(Here we set $Z(0, A) = 0$.) This holds if and only if $x_n \in A(\alpha) \hat{\ } A(\beta)$.

9. The construction of directed systems. Suppose the numbers $\alpha(i_1, \dots, i_d)$ with $1 \leq i_1 \leq l_1, \dots, 1 \leq i_d \leq l_d$ belong to a directed system of the type $(u_1, \dots, u_d)$. Let u be $<$ or $=$ or $>$. Put

$$\begin{aligned} (u_1^*, \dots, u_d^*, u_{d+1}^*) &= (u_1, \dots, u_d, u) && \text{if } u_d \text{ is not } =, \\ &= (u_1, \dots, u_k, u, \dots, u) && \text{if for some } k < d, \text{ the symbols} \\ &&& u_{k+1}, \dots, u_d \text{ are } =, \text{ and either } k = 0 \text{ or } u_k \text{ is not } =. \end{aligned}$$

We shall describe a process of constructing a directed system of order $d+1$ of the type $(u_1^*, \dots, u_d^*, u_{d+1}^*)$. Let K be the set of d -tuples $(i_1, \dots, i_d)$ with $1 \leq i_1 \leq l_1, \dots, 1 \leq i_d \leq l_d$. A subset H of K will be called a *segment* if $(i_1, \dots, i_d) \in H$ whenever $(i'_1, \dots, i'_d) \in H$ and $(i_1, \dots, i_d) <_j (i'_1, \dots, i'_d)$ for some j . Let l_{d+1} be an integer greater than 1. A *partial directed system* of order $d+1$ on H will mean a system of numbers $\alpha(i_1, \dots, i_d, i_{d+1})$ defined for $(i_1, \dots, i_d) \in H$ and $1 \leq i_{d+1} \leq l_{d+1}$, such that

(a) $\alpha(i_1, \dots, i_d, i_{d+1}) u_j^* \alpha(i'_1, \dots, i'_d, i'_{d+1})$ if $(i_1, \dots, i_d) \in H$, $(i'_1, \dots, i'_d) \in H$ and $(i_1, \dots, i_d, i_{d+1}) <_j (i'_1, \dots, i'_d, i'_{d+1})$ for some j in $1 \leq j \leq d+1$.

(b) $\alpha(i_1, \dots, i_d, i_{d+1}) u_j^* \alpha(i'_1, \dots, i'_d)$ if $(i_1, \dots, i_d) \in H$, $(i'_1, \dots, i'_d) \notin H$ and $(i_1, \dots, i_d) <_j (i'_1, \dots, i'_d)$ for some j in $1 \leq j \leq d$.

In particular, if H is empty, the empty set is a partial system defined on H . If $H = K$, a partial directed system on H is a directed system of order $d+1$.

Lemma 9. Suppose H is a segment, and H^* the segment which consists of H and a single further d -tuple $(t_1, \dots, t_d)$. Suppose

$$\alpha(i_1, \dots, i_d, i_{d+1}) \quad ((i_1, \dots, i_d) \in H, 1 \leq i_{d+1} \leq l_{d+1})$$

is a partial directed system defined on H .⁽⁵⁾ Suppose there is a sequence $\alpha_s(t_1, \dots, t_d) (s = 1, 2, \dots)$ which is monotonic of the type u and tends to $\alpha(t_1, \dots, t_d)$. Then if s_0 is sufficiently large, and if we put

$$(32) \quad \alpha(t_1, \dots, t_d, i) = \alpha_{s_0+i}(t_1, \dots, t_d) \quad (1 \leq i \leq l_{d+1}),$$

then the numbers

$$\alpha(i_1, \dots, i_d, i_{d+1}) \quad ((i_1, \dots, i_d) \in H^*, 1 \leq i_{d+1} \leq l_{d+1})$$

are a partial directed system defined on H^* .

Proof. The condition (a) is satisfied if $(i_1, \dots, i_d), (i'_1, \dots, i'_d) \in H$. In order that it also be satisfied for H^* , we have to satisfy the following two conditions.

$$(a_1^*) \quad \alpha(i_1, \dots, i_d, i_{d+1}) u_j^* \alpha(t_1, \dots, t_d, i'_{d+1}) \quad \text{if } (i_1, \dots, i_d) <_j (t_1, \dots, t_d)$$

for some j in $1 \leq j \leq d$.

$$(a_2^*) \quad \alpha(t_1, \dots, t_d, i_{d+1}) u_{d+1}^* \alpha(t_1, \dots, t_d, i'_{d+1}) \quad \text{if } i_{d+1} < i'_{d+1}.$$

Since $(t_1, \dots, t_d)$ is the only new element in H^* , (b) will be satisfied for H^* if

$$(b^*) \quad \alpha(t_1, \dots, t_d, i_{d+1}) u_j^* \alpha(i'_1, \dots, i'_d) \quad \text{whenever } (t_1, \dots, t_d) <_j (i'_1, \dots, i'_d).$$

Now since $\alpha_s(t_1, \dots, t_d) (s = 1, 2, \dots)$ is monotonic of the type $u = u_{d+1}^*$, the condition (a_2^*) will be satisfied if $\alpha(t_1, \dots, t_d, i)$ is given by (32), no matter how we choose s_0 .

Now suppose that $1 \leq j \leq d$ and u_j^* is not $=$. Suppose $(i_1, \dots, i_d) <_j (t_1, \dots, t_d)$. Since $(i_1, \dots, i_d)$ is in H and $(t_1, \dots, t_d)$ is not in H , we have $\alpha(i_1, \dots, i_d, i_{d+1}) u_j^* \alpha(t_1, \dots, t_d)$ by the hypothesis (b). Since $\alpha_s(t_1, \dots, t_d) (s = 1, 2, \dots)$ tends to $\alpha(t_1, \dots, t_d)$, (a_1^*) will be satisfied if s_0 in (32) is sufficiently large. Next, suppose that $1 \leq j \leq d$ and u_j^* is $=$. Then also u_j and u are $=$. Suppose $(i_1, \dots, i_d) <_j (t_1, \dots, t_d)$. Then $\alpha(i_1, \dots, i_d, i_{d+1}) = \alpha(t_1, \dots, t_d)$ by (b). Since $\alpha_s(t_1, \dots, t_d) = \alpha(t_1, \dots, t_d) (s = 1, 2, \dots)$, (a_1^*) will certainly be true. Thus (a_1^*) can always be satisfied.

Now suppose that $1 \leq j \leq d$ and u_j is not $=$. Then u_j^* equals u_j and is not $=$. Since $\alpha(i_1, \dots, i_d)$ is a directed system of the type $(u_1, \dots, u_d)$, we have $\alpha(t_1, \dots, t_d) u_j^* \alpha(i'_1, \dots, i'_d)$ if $(t_1, \dots, t_d) <_j (i'_1, \dots, i'_d)$. Since $\alpha_s(t_1, \dots, t_d) (s = 1, 2, \dots)$ tends to $\alpha(t_1, \dots, t_d)$, (b^*) will be true if s_0 in (32) is sufficiently large. Finally, suppose that $1 \leq j \leq d$ and u_j is $=$. Then u_j^* is u . Since $\alpha(i_1, \dots, i_d)$ is

⁽⁵⁾ This part of the hypothesis does not apply if H is empty. Obvious changes have to be made in the proof in this case.

a directed system of the type $(u_1, \dots, u_d)$, we have $\alpha(t_1, \dots, t_d) = \alpha(i'_1, \dots, i'_d)$ if $(t_1, \dots, t_d) < (i'_1, \dots, i'_d)$. Since $\alpha_s(t_1, \dots, t_d)$ is monotonic of the type u and tends to $\alpha(t_1, \dots, t_d)$, we have $\alpha(t_1, \dots, t_d, i_{d+1})u\alpha(t_1, \dots, t_d) = \alpha(i'_1, \dots, i'_d)$, i.e. $\alpha(t_1, \dots, t_d, i_{d+1})u_j^*\alpha(i'_1, \dots, i'_d)$. Thus (b*) will be satisfied.

This finishes the proof of Lemma 9. It is clear that by using an inductive argument and using Lemma 9 at each step, we can gradually build up a directed system of order $d + 1$ of the type $(u_1^*, \dots, u_{d+1}^*)$.

Now suppose that $\alpha(i_1, \dots, i_d)$ with $1 \leq i_1 \leq l_1, \dots, 1 \leq i_d \leq l_d$ is a directed system of vectors of the type $(u_{ih})(1 \leq i \leq d, 1 \leq h \leq m)$. Let $u_1, \dots, u_m$ be symbols $<, =$ or $>$. For each h in $1 \leq h \leq m$, put

$$\begin{aligned} (u_{1h}^*, \dots, u_{dh}^*, u_{d+1,h}^*) &= (u_{1h}, \dots, u_{dh}, u_h) && \text{if } u_{dh} \text{ is not } =, \\ &= (u_{1h}, \dots, u_{k_h,h}, u_h, \dots, u_h) && \text{if } u_{k_h+1,h}, \dots, u_{dh} \text{ are } =, \\ &&& \text{and if either } k_h = 0 \text{ or } u_{k_h,h} \text{ is not } =. \end{aligned}$$

We shall indicate a process to construct a directed system of vectors of order $d + 1$ of the type $(u_{ih}^*)(1 \leq i \leq d + 1, 1 \leq m \leq h)$. A *partial directed system of vectors* of order $d + 1$ on a segment H is defined in the obvious way.

Lemma 10. *Suppose H is a segment and H^* is the segment consisting of H and of a single further d -tuple $(t_1, \dots, t_d)$. Suppose*

$$\alpha(i_1, \dots, i_d, i_{d+1}) \quad ((i_1, \dots, i_d) \in H, 1 \leq i_{d+1} \leq l_{d+1})$$

is a partial directed system defined on H .⁽⁶⁾ Suppose there is a sequence of vectors $\alpha_s(t_1, \dots, t_d)(s = 1, 2, \dots)$ which converges to $\alpha(t_1, \dots, t_d)$, and which is monotonic of the type $(u_1, \dots, u_m)$. Then if s_0 is sufficiently large, and if we put

$$(33) \quad \alpha(t_1, \dots, t_d, i) = \alpha_{s_0+i}(t_1, \dots, t_d) \quad (1 \leq i \leq l_{d+1}),$$

then the vectors

$$\alpha(i_1, \dots, i_d, i_{d+1}) \quad ((i_1, \dots, i_d) \in H^*, 1 \leq i_{d+1} \leq l_{d+1})$$

are a partial directed system defined on H^ .*

Proof. We may use induction on the number m of components of our vectors, and use Lemma 9 at each step of the induction.

10. Inductive proof of the proposition. Let S be a subset of the cube C . A vector α will be called a *limit point* of S if there is a sequence $\alpha_1, \alpha_2, \dots$ of elements of S which converge to α and which have distinct values $\mu(\alpha_1), \mu(\alpha_2), \dots$. (This condition is more restrictive than the usual condition that $\alpha_1, \alpha_2, \dots$ be distinct.) Let $S^{(1)}$ be the set of limit points of S . Since $\mu(\alpha)$ is a continuous function of α , it is clear that $M(S^{(1)}) \subseteq M^{(1)}(S)$. Conversely, if $\mu \in M^{(1)}(S)$, there are distinct elements $\mu_1 = \mu(\alpha_1), \mu_2 = \mu(\alpha_2), \dots$ of $M(S)$ with $\alpha_1, \alpha_2, \dots$ in S and with

⁽⁶⁾ See the footnote to Lemma 9.

$\mu_i \rightarrow \mu$. There is a convergent subsequence of $\alpha_1, \alpha_2, \dots$; let us denote the limit of this subsequence by α . Then $\alpha \in S^{(1)}$ and $\mu = \mu(\alpha)$, so that $\mu \in M(S^{(1)})$. Hence

$$(34) \quad M^{(1)}(S) = M(S^{(1)}).$$

Every sequence $\alpha_1, \alpha_2, \dots$ has a subsequence which is monotonic of some type $(u_1, \dots, u_m)$. If the numbers $\mu(\alpha_1), \mu(\alpha_2), \dots$ are all distinct, this type cannot be $(=, \dots, =)$. Hence if $S^{(1)}(u_1, \dots, u_m)$ consists of the elements $\alpha \in S^{(1)}$ for which there is a monotonic sequence $\alpha_1, \alpha_2, \dots$ in S of the type $(u_1, \dots, u_m)$ which tends to α and has distinct $\mu(\alpha_1), \mu(\alpha_2), \dots$, then $S^{(1)}$ is the union of the $3^m - 1$ sets $S^{(1)}(u_1, \dots, u_m)$ with $(u_1, \dots, u_m)$ not $(=, \dots, =)$.

Now assume that $d \geq 0$ and that $M^{(d+1)}(S)$ contains an element μ with $0 < \mu < 1$. By (34) we have $M^{(d+1)}(S) = M^{(d)}(S^{(1)})$, so that μ lies in one of the $3^m - 1$ sets $M^{(d)}(S^{(1)}(u_1, \dots, u_m))$ with $(u_1, \dots, u_m)$ not $(=, \dots, =)$. Suppose $(u_1, \dots, u_m)$ is a particular m -tuple with

$$\mu \in M^{(d)}(S^{(1)}(u_1, \dots, u_m)).$$

We now assume the truth of the proposition for our particular value of d and apply it to $S^{(1)}(u_1, \dots, u_m)$. There is an integer $r = r^{(d)}$, a directed system $\alpha(i_1, \dots, i_d)$ with elements in $S^{(1)}(u_1, \dots, u_m)$, and there are neighborhoods $N(i_1, \dots, i_d)$ with the properties enunciated in the proposition. Suppose this directed system

$$\alpha(i_1, \dots, i_d) \quad (1 \leq i_1 \leq l_1, \dots, 1 \leq i_d \leq l_d)$$

is of the type $(u_{ih})(1 \leq i \leq d, 1 \leq h \leq m)$. Construct $(u_{ih}^*)(1 \leq i \leq d+1, 1 \leq h \leq m)$ as in §9. The goal of the present section is a proof of the following

Lemma 11. *Suppose $l_{d+1} = 2l > 0$, $r > 0$, $\varepsilon > 0$. There is a directed system*

$$\alpha(i_1, \dots, i_d, i_{d+1}) \quad (1 \leq i_1 \leq l_1, \dots, 1 \leq i_{d+1} \leq l_{d+1})$$

of the type $(u_{ih}^)(1 \leq i \leq d+1, 1 \leq h \leq m)$, all of whose vectors α lie in S and have $0 < \mu(\alpha) < 1$, and there are neighborhoods $N(i_1, \dots, i_{d+1})$ of $\mu(\alpha(i_1, \dots, i_{d+1}))$, such that*

$$(35) \quad N(i_1, \dots, i_d, i_{d+1}) \subseteq N(i_1, \dots, i_d) \quad (1 \leq i_1 \leq l_1, \dots, 1 \leq i_{d+1} \leq l_{d+1}).$$

Also, if $\beta \in N(i_1, \dots, i_d, 2j-1)$, $\beta' \in N(i_1, \dots, i_d, 2j)$ with $1 \leq i_1 \leq l_1, \dots, 1 \leq i_d \leq l_d$ and with $1 \leq j \leq l$, then

$$(36) \quad |\mu(\beta) - \mu(\beta')| < \varepsilon/4r.$$

Finally, there is an integer p such that if N, N' are ranges with $|N| \geq p, |N'| \geq p$, and if vectors $\beta(i_1, \dots, i_d, i_{d+1})$ have $\mu(\beta(i_1, \dots, i_{d+1})) \in N(i_1, \dots, i_{d+1})$ for $1 \leq i_1 \leq l_1, \dots, 1 \leq i_{d+1} \leq l_{d+1}$, then there are subranges $L \subseteq N, L' \subseteq N'$ with

$$(37) \quad |L| = |L'| = r$$

and with

$$(38) \quad f^\square(L, L', \beta(i_1, \dots, i_d, 2j-1), \beta(i_1, \dots, i_d, 2j)) > \frac{1}{2} - \varepsilon$$

$$(1 \leq i_1 \leq l_1, \dots, 1 \leq i_d \leq l_d, 1 \leq j \leq l).$$

We shall prove this lemma by constructing partial directed systems of order $d+1$ defined on segments, and by gradually increasing these segments.

Lemma 12. *Suppose $l_{d+1} = 2l > 0$, $v > 0$, $\varepsilon > 0$. Let H be a nonempty segment. There is a partial directed system*

$$\alpha(i_1, \dots, i_d, i_{d+1}) \quad ((i_1, \dots, i_d) \in H, 1 \leq i_{d+1} \leq l_{d+1})$$

of the type (u_{ih}^*) with all the vectors α in S and satisfying $0 < \mu(\alpha) < 1$, and there are neighborhoods $N(i_1, \dots, i_d, i_{d+1})$ of $\mu(\alpha(i_1, \dots, i_d, i_{d+1}))$, defined for $(i_1, \dots, i_d) \in H$, $1 \leq i_{d+1} \leq l_{d+1}$, such that

$$(39) \quad N(i_1, \dots, i_d, i_{d+1}) \subseteq N(i_1, \dots, i_d) \quad ((i_1, \dots, i_d) \in H, 1 \leq i_{d+1} \leq l_{d+1}),$$

and such that

$$(40) \quad |\mu(\beta) - \mu(\beta')| < \varepsilon/4v$$

if $\beta \in N(i_1, \dots, i_d, 2j-1)$, $\beta' \in N(i_1, \dots, i_d, 2j)$ for some $(i_1, \dots, i_d) \in H$, $1 \leq j \leq l$. There is an integer $p = p(H)$ such that if N, N' are ranges with $|N| \geq p(H)$, $|N'| \geq p(H)$, and if $\beta(i_1, \dots, i_d, i_{d+1}) \in N(i_1, \dots, i_d, i_{d+1})$ $((i_1, \dots, i_d) \in H, 1 \leq i_{d+1} \leq l_{d+1})$, then there are subranges $L \subseteq N$, $L' \subseteq N'$ with

$$(41) \quad |L| = |L'| = v$$

and with

$$(42) \quad f^\square(L, L', \beta(i_1, \dots, i_d, 2j-1), \beta(i_1, \dots, i_d, 2j)) > \frac{1}{2} - \varepsilon$$

$$((i_1, \dots, i_d) \in H, 1 \leq j \leq l).$$

The case when $H = K$, the set of all d -tuples $(i_1, \dots, i_d)$, is Lemma 11.

Proof of Lemma 12. We shall proceed by "induction on H ". We shall assume that H is a segment properly contained in K , and that either H is empty, or H is nonempty and Lemma 12 is true for H . There is a unique segment H^* which consists of H and a single further d -tuple $(t_1, \dots, t_d)$. We shall now prove Lemma 12 for H^* . We shall tacitly assume that H is nonempty; the necessary modifications in the argument when H is empty are trivial.

Suppose $\alpha(i_1, \dots, i_d, i_{d+1})$ defined for $(i_1, \dots, i_d) \in H$, $1 \leq i_{d+1} \leq l_{d+1}$, and respective neighborhoods $N(i_1, \dots, i_d, i_{d+1})$ and the number $p = p(H)$ have the

desired properties as enunciated in Lemma 12 with respect to H .

Since $\alpha(t_1, \dots, t_d)$ lies in $S^{(1)}(u_1, \dots, u_m)$, there is a monotonic sequence $\alpha_s(t_1, \dots, t_d) (s = 1, 2, \dots)$ of the type $(u_1, \dots, u_m)$ which tends to $\alpha(t_1, \dots, t_d)$, and is such that the numbers $\mu(\alpha_s(t_1, \dots, t_d)) (s = 1, 2, \dots)$ are all distinct. We now put $r = p(H) + \nu$ and apply Lemma 7 to l, r, ε and to the sequence $\alpha_1(t_1, \dots, t_d), \alpha_2(t_1, \dots, t_d), \dots$. There is a finite subsequence

$$(43) \quad \alpha(t_1, \dots, t_d, 1), \dots, \alpha(t_1, \dots, t_d, 2l),$$

and there are neighborhoods $N(t_1, \dots, t_d, i)$ of $\mu(\alpha(t_1, \dots, t_d, i)) (1 \leq i \leq 2l)$, and there is an integer q , with the following properties. For any $\beta(t_1, \dots, t_d, 1), \dots, \beta(t_1, \dots, t_d, 2l)$ with $\mu(\beta(t_1, \dots, t_d, i)) \in N(t_1, \dots, t_d, i) (1 \leq i \leq 2l)$, we have

$$(44) \quad |\mu(\beta(t_1, \dots, t_d, 2j-1)) - \mu(\beta(t_1, \dots, t_d, 2j))| < \varepsilon/4r < \varepsilon/4\nu$$

$$(1 \leq j \leq l).$$

Furthermore, if N, N' are ranges with $|N| \geq q, |N'| \geq q$, there are subranges $N_H \subseteq N, N'_H \subseteq N'$ with

$$(45) \quad |N_H| = |N'_H| = r > p(H)$$

and with

$$(46) \quad f^\square(N_H, N'_H, \beta(t_1, \dots, t_d, 2j-1), \beta(t_1, \dots, t_d, 2j)) > \frac{1}{2} - \varepsilon \quad (1 \leq j \leq l).$$

Now the sequence $\alpha_s(t_1, \dots, t_d) (s = 1, 2, \dots)$ may be replaced by a subsequence with $\mu(\alpha_s(\dots))$ contained in $N(t_1, \dots, t_d)$, and hence the sequence (43) can be chosen so that all its measures μ lie in $N(t_1, \dots, t_d)$. The neighborhoods $N(t_1, \dots, t_d, i)$ can be chosen so small that $N(t_1, \dots, t_d, i) \subseteq N(t_1, \dots, t_d) (1 \leq i \leq 2l)$. This, together with the "inductive" assumption (39) yields

$$(47) \quad N(i_1, \dots, i_d, i_{d+1}) \subseteq N(i_1, \dots, i_d) \quad ((i_1, \dots, i_d) \in H^*, 1 \leq i_{d+1} \leq l_{d+1}).$$

The "inductive" assumption (40) for H together with (44) gives a condition like (40) for H^* . Lemma 10 shows that, moreover, the sequence (43) can be chosen so that $\alpha(i_1, \dots, i_d, i_{d+1})$ with $(i_1, \dots, i_d) \in H^*, 1 \leq i_{d+1} \leq l_{d+1}$, is a partial directed system on H^* .

Now suppose $\beta(i_1, \dots, i_d, i_{d+1}) \subseteq N(i_1, \dots, i_d, i_{d+1}) ((i_1, \dots, i_d) \in H^*, 1 \leq i_{d+1} \leq l_{d+1})$, $|N| \geq q, |N'| \geq q$, and suppose N_H, N'_H are chosen as above with (45) and (46). By the "inductive" assumption on H , there are subranges $L \subseteq N_H, L' \subseteq N'_H$ with (41), (42). Since $L \subseteq N_H, L' \subseteq N'_H$, the relations (42) together with (46) yield

$$f^\square(L, L', \beta(i_1, \dots, i_d, 2j-1), \beta(i_1, \dots, i_d, 2j)) > \frac{1}{2} - \varepsilon$$

$$((i_1, \dots, i_d) \in H^*, 1 \leq i_{d+1} \leq l_{d+1}).$$

Hence Lemma 12 is true for H^* with $p(H^*) = q$.

11. Proof of the proposition completed. We saw in §10 that if $M^{(d+1)}(S)$ contains an element μ in $0 < \mu < 1$, then $\mu \in M^{(d)}(S^{(1)}(u_1, \dots, u_m))$ for some m -tuple $(u_1, \dots, u_m)$. We applied the inductive hypothesis to $S^{(1)}(u_1, \dots, u_m)$. There is a directed system $\alpha(i_1, \dots, i_d)$ of order d , and there are neighborhoods $N(i_1, \dots, i_d)$ and an integer $r = r^{(d)}$ with the properties stated in the proposition. Lemma 11 asserted the existence of a directed system $\alpha(i_1, \dots, i_d, i_{d+1})$ of order $d + 1$, and neighborhoods $N(i_1, \dots, i_d, i_{d+1})$ and a number p which have certain properties in relation to the given directed system of order d .

Lemma 13. *Suppose we have the same hypotheses as in Lemma 11, and let $\alpha(i_1, \dots, i_{d+1})$, $N(i_1, \dots, i_{d+1})$, p be as in Lemma 11. Suppose*

$$(48) \quad \beta(i_1, \dots, i_{d+1}) \quad (1 \leq i_1 \leq l_1, \dots, 1 \leq i_{d+1} \leq l_{d+1})$$

is a directed system with

$$(49) \quad \mu(\beta(i_1, \dots, i_{d+1})) \subseteq N(i_1, \dots, i_{d+1}) \quad (1 \leq i_1 \leq l_1, \dots, 1 \leq i_{d+1} \leq l_{d+1}),$$

but not necessarily of the same type as $\alpha(i_1, \dots, i_{d+1})$. We know from Lemma 11 that if $|N| \geq p$, $|N'| \geq p$, then there are subranges $L \subseteq N$, $L' \subseteq N'$ with (37) and (38).

We now claim that for every $(i_1, \dots, i_d)$ with $1 \leq i_1 \leq l_1, \dots, 1 \leq i_d \leq l_d$, we have

$$(50) \quad \sum_{j=1}^l f^\nabla(L, L', \beta(i_1, \dots, i_d, 2j-1), \beta(i_1, \dots, i_d, 2j)) > l(\frac{1}{2} - 2\varepsilon) - 2mr.$$

Proof. Throughout, we keep $i_1, \dots, i_d$ fixed. Since (48) is a directed system, the sequence $\beta(i_1, \dots, i_d, i)$ with $i = 1, 2, \dots, 2l$ is monotonic. Hence, by Lemma 1, a point x_n can lie in at most m of the l sets

$$A(\beta(i_1, \dots, i_d, 2j-1)) \wedge A(\beta(i_1, \dots, i_d, 2j)) \quad (j = 1, 2, \dots, l).$$

It follows from Lemma 8 that at most m of the l functions

$$(51) \quad f(n, \beta(i_1, \dots, i_d, 2j-1), \beta(i_1, \dots, i_d, 2j)) \quad (j = 1, 2, \dots, l)$$

can have a jump at n . Hence in view of (37), at most $2mr$ of these functions can have a jump at any $n \in L$ or any $n' \in L'$. It will suffice to show that if a function (51) has no jump in L or in L' , then this function satisfies

$$(52) \quad f^\nabla(L, L', \beta(i_1, \dots, i_d, 2j-1), \beta(i_1, \dots, i_d, 2j)) > \frac{1}{2} - 2\varepsilon.$$

Suppose $n_0 \in L$, $n'_0 \in L'$. By (38), the values of our function (51) at $n = n_0$ and at $n = n'_0$ differ by at least $\frac{1}{2} - \varepsilon$. Without loss of generality we may assume that

$$f(n_0, \beta(i_1, \dots, i_d, 2j-1), \beta(i_1, \dots, i_d, 2j)) \\ - f(n'_0, \beta(i_1, \dots, i_d, 2j-1), \beta(i_1, \dots, i_d, 2j)) > \frac{1}{2} - \varepsilon.$$

Now $f(n, \beta(i_1, \dots, i_d, 2j-1), \beta(i_1, \dots, i_d, 2j))$ is of the type $\mu(\beta(i_1, \dots, i_d, 2j-1)) - \mu(\beta(i_1, \dots, i_d, 2j))$ and has no jump in L . Hence for every $n \in L$,

$$|f(n, \beta(\dots, 2j-1), \beta(\dots, 2j)) - f(n_0, \beta(\dots, 2j-1), \beta(\dots, 2j))| \\ = |n - n_0| |\mu(\beta(\dots, 2j-1)) - \mu(\beta(\dots, 2j))| \\ \leq r(\varepsilon/4r) = \varepsilon/4,$$

by virtue of (36) and (49). A similar inequality holds for every $n' \in L'$, and hence we have for every $n \in L$ and every $n' \in L'$,

$$f(n, \beta(\dots, 2j-1), \beta(\dots, 2j)) - f(n', \beta(\dots, 2j-1), \beta(\dots, 2j)) \\ > \frac{1}{2} - \varepsilon - 2(\varepsilon/4) > \frac{1}{2} - 2\varepsilon.$$

Therefore (52) holds, and Lemma 13 is true.

The proof of the proposition is now completed as follows. In Lemma 11 and in Lemma 13, the number l is still at our disposal. We now choose it so large that $le > 2mr$. Then the right-hand side of (50) may be replaced by $l(\frac{1}{2} - 3\varepsilon)$. Since $f(n, \alpha, \beta) = f(n, \alpha) - f(n, \beta)$, we may rewrite (50) as

$$(53) \quad \sum_{j=1}^l (f(\dots, \beta(i_1, \dots, i_d, 2j-1)) - f(\dots, \beta(i_1, \dots, i_d, 2j)))^\nabla(L, L') \\ > l(\frac{1}{2} - 3\varepsilon).$$

For every directed system $\beta(i_1, \dots, i_{d+1})$ with (49) and every pair N, N' with $|N| \geq p$, $|N'| \geq p$, there are subranges $L \subseteq N$, $L' \subseteq N'$ with $|L| = |L'| = r$ and with (53) for arbitrary $i_1, \dots, i_d$. In particular, in every single N with $|N| \geq p$, there are two subranges $L \subseteq N$, $L' \subseteq N$ with $|L| = |L'| = r$ and with (53). By Lemma 3 applied to $f(n) = f(n, \beta(i_1, \dots, i_d, 2j-1))$ and $g(n) = f(n, \beta(i_1, \dots, i_d, 2j))$ we have

$$f^*(N, \beta(i_1, \dots, i_d, 2j-1)) + f^*(N, \beta(i_1, \dots, i_d, 2j)) \\ \geq (f(\dots, \beta(i_1, \dots, i_d, 2j-1)) - f(\dots, \beta(i_1, \dots, i_d, 2j)))^\nabla(L, L') \\ + \frac{1}{2}(f^*(L, \beta(i_1, \dots, i_d, 2j-1)) + f^*(L, \beta(i_1, \dots, i_d, 2j)) \\ + f^*(L', \beta(i_1, \dots, i_d, 2j-1)) + f^*(L', \beta(i_1, \dots, i_d, 2j))).$$

We now take the sum over $j = 1, 2, \dots, l$ and use (53) to obtain

$$\sum_{i_{d+1}=1}^{i_{d+1}} f^*(N, \beta(i_1, \dots, i_d, i_{d+1})) > l\left(\frac{1}{2} - 3\varepsilon\right) \\ + \frac{1}{2} \sum_{i_{d+1}=1}^{i_{d+1}} (f^*(L, \beta(i_1, \dots, i_d, i_{d+1})) + f^*(L', \beta(i_1, \dots, i_d, i_{d+1}))).$$

This holds for arbitrary $i_1, \dots, i_d$ with $1 \leq i_1 \leq l_1, \dots, 1 \leq i_d \leq l_d$. We now take the sum over all d -tuples $(i_1, \dots, i_d)$ and divide by $l_1 \dots l_d l_{d+1} = 2l_1 \dots l_d l$. We obtain

$$\begin{aligned} (l_1 \dots l_{d+1})^{-1} \sum_{i_1=1}^{l_1} \dots \sum_{i_{d+1}=1}^{l_{d+1}} f^*(N, \beta(i_1, \dots, i_{d+1})) &> \frac{1}{4} - \frac{3}{2}\epsilon \\ &+ l_{d+1}^{-1} \sum_{i_{d+1}=1}^{l_{d+1}} \left(\frac{1}{2} (l_1 \dots l_d)^{-1} \sum_{i_1=1}^{l_1} \dots \sum_{i_d=1}^{l_d} (f^*(L, \beta(i_1, \dots, i_{d+1})) \right. \\ &\quad \left. + f^*(L', \beta(i_1, \dots, i_{d+1}))) \right) \\ &= \frac{1}{4} - \frac{3}{2}\epsilon + l_{d+1}^{-1} \sum_{i_{d+1}=1}^{l_{d+1}} z(i_{d+1}), \end{aligned}$$

say. Now for fixed i_{d+1} , $\beta(i_1, \dots, i_d, i_{d+1})$ with $1 \leq i_1 \leq l_1, \dots, 1 \leq i_d \leq l_d$ is a directed system of order d . We have

$$\mu(\beta(i_1, \dots, i_d, i_{d+1})) \in N(i_1, \dots, i_d, i_{d+1}) \subseteq N(i_1, \dots, i_d)$$

by (35), (49), and both L, L' have length $r = r^{(d)}$. Hence by our inductive assumption, i.e. by the case d of the proposition, we have

$$z(i_{d+1}) \geq \frac{1}{4}(d+1) + \frac{1}{12} - \epsilon,$$

whence

$$(54) \quad (l_1 \dots l_{d+1})^{-1} \sum_{i_1=1}^{l_1} \dots \sum_{i_{d+1}=1}^{l_{d+1}} f^*(N, \beta(i_1, \dots, i_d, i_{d+1})) > \frac{1}{4}(d+2) + \frac{1}{12} - \frac{5}{2}\epsilon.$$

This holds for every range N with $|N| \geq p = r^{(d+1)}(5\epsilon/2)$, say. The whole construction could be carried out with $2\epsilon/5$ in place of ϵ , and then our inequality (54) would become (8) with $d+1$ in place of d .

This finishes our inductive proof of the proposition.

12. Proof of Theorem 2. It is easily seen that the general case of Theorem 2 follows from the 2-dimensional case, so that we may restrict ourselves to this case.

Pick numbers $t_1 > t_2 > \dots$ with

$$(55) \quad 0 < t_j < 1/(8j) \quad (j = 1, 2, \dots),$$

and let $\mathbf{x}_j = (x_j, y_j)$ be the point $(1 - \cos t_j, \sin t_j)$. Then the points $\mathbf{x}_1, \mathbf{x}_2, \dots$ lie on the circle $(x-1)^2 + y^2 = 1$, and they satisfy

$$(56) \quad \sqrt{(x_j^2 + y_j^2)} < 1/(4j) \quad (j = 1, 2, \dots).$$

For every μ in $0 \leq \mu \leq 1$, we are going to construct sets $F(n, \mu)$ ($n = 1, 2, \dots$) as follows. If $\mu < \frac{1}{2}$, let $F(1, \mu)$ be empty, and if $\mu \geq \frac{1}{2}$, let $F(1, \mu)$ consist of $\mathbf{x}_1$.

Then always

$$(57) \quad ||F(1, \mu)| - \mu| \leq \frac{1}{2},$$

where $|F|$ denotes the number of elements of a finite set F . Now suppose $F(n, \mu)$ has already been chosen and is a subset of $\{x_1, \dots, x_n\}$ with

$$(58) \quad ||F(n, \mu)| - n\mu| \leq \frac{1}{2}.$$

Then $-(3/2) \leq |F(n, \mu)| - (n+1)\mu \leq \frac{1}{2}$, so that either $||F(n, \mu)| - (n+1)\mu| \leq \frac{1}{2}$ or $||F(n, \mu)| + 1 - (n+1)\mu| \leq \frac{1}{2}$. If the first inequality holds, put $F(n+1, \mu) = F(n, \mu)$; otherwise let $F(n+1, \mu)$ consist of $F(n, \mu)$ and of x_{n+1} . In either case we have $||F(n+1, \mu)| - (n+1)\mu| \leq \frac{1}{2}$. Continuing in this way we obtain sets $F(1, \mu), F(2, \mu), \dots$ which satisfy (58) for $n = 1, 2, \dots$.

Let $G_1(\mu)$ be the convex hull of the sets $F(1, \mu), F(2, \mu), \dots$. Then $G_1(\mu)$ is the convex hull of certain points among $x_1, x_2, \dots$. If $\mu = 0$, the sets $F(n, \mu)$ are empty, and hence so is $G_1(\mu)$. If $0 < \mu \leq \frac{1}{3}$, put $n_0 = [1/(3\mu)]$ and apply (58) with $n = n_0$. Since $n_0\mu \leq \frac{1}{3}$, we obtain $|F(n_0, \mu)| = 0$, and hence $x_1, \dots, x_{n_0}$ do not lie in $G_1(\mu)$. Thus $G_1(\mu)$ is the convex hull of certain points among $x_{n_0+1}, x_{n_0+2}, \dots$, and in view of (56) we obtain

$$\mu(G_1(\mu)) \leq \frac{\pi}{4}(1/(4(n_0+1)))^2 \leq \left(\frac{\pi}{64}\right)(3\mu)^2 < \mu.$$

If $\frac{1}{3} < \mu \leq 1$, we have $\mu(G_1(\mu)) \leq \pi/4 - 1/2 < 1/3 < \mu$. Hence always $\mu(G_1(\mu)) \leq \mu$.

If $0 \leq \mu \leq \frac{1}{2}$, let $G_2(\mu)$ be the convex hull of $F(1, \mu), F(2, \mu), \dots$ and of the triangle $0 \leq x < 1, 0 \leq y < 1, y < x$. If $\mu = 1$, let $G_2(\mu)$ be U^2 . In these cases we have $\mu(G_2(\mu)) \geq \mu$. If $\frac{1}{2} < \mu < 1$, we have $x_1 \in F(1, \mu)$ by (57), and there is a smallest integer n_1 such that $x_{n_1+1} \notin F(n_1+1, \mu)$. In this case let $G_2(\mu)$ be the convex hull of $F(1, \mu), F(2, \mu), \dots$ and the open quadrilateral with vertices $(1, 0), (1, 1), x_{n_1}, x^*$, where $x^* = (x^*, 1)$ is the intersection of the line $y = 1$ and the tangent to the circle $(x-1)^2 + y^2 = 1$ at x_{n_1} . The quadrilateral contains the open rectangle with vertices $(x^*, y_{n_1}), (1, y_{n_1}), (x^*, 1), (1, 1)$ of area $(1-x^*) \cdot (1-y_{n_1}) = ((1-y_{n_1})/(1-x_{n_1}))(1-y_{n_1}) > 1-2y_{n_1} > 1-2t_{n_1}$. Since $x_1, \dots, x_{n_1}$ are in $F(n_1+1, \mu)$, but x_{n_1+1} is not, (58) yields $|n_1 - (n_1+1)\mu| \leq \frac{1}{2}$, whence $n_1 \geq (\mu - \frac{1}{2})/(1-\mu)$, whence, by (55),

$$\mu(G_2(\mu)) > 1 - 2t_{n_1} > 1 - (1/4n_1) \geq$$

$$3/4 \geq \mu \quad \text{if} \quad 1/2 < \mu < 3/4,$$

$$1 - (1 - \mu) = \mu \quad \text{if} \quad 3/4 \leq \mu < 1.$$

Since $G_1(\mu), G_2(\mu)$ are convex sets with $\mu(G_1(\mu)) \leq \mu \leq \mu(G_2(\mu))$, there is a convex set $S(\mu)$ with $G_1(\mu) \subseteq S(\mu) \subseteq G_2(\mu)$ and with $\mu(S(\mu)) = \mu$. The set $S(\mu)$ lies in U^2 , since $G_2(\mu)$ does. The intersection of $G_1(\mu)$ with $\{x_1, \dots, x_n\}$ is $F(n, \mu)$, and the intersection of $G_2(\mu)$ with $\{x_1, \dots, x_n\}$ is $F(n, \mu)$, so that also the intersection of $S(\mu)$ with $\{x_1, \dots, x_n\}$ is $F(n, \mu)$. Therefore, by (58),

$$D(n, S(\mu)) = |Z(n, S(\mu)) - n\mu(S(\mu))| = ||F(n, \mu) - n\mu| \leq \frac{1}{2}$$

for $n = 1, 2, \dots$, so that $E(S(\mu)) \leq \frac{1}{2}$. Since μ was arbitrary in $0 \leq \mu \leq 1$, Theorem 2 is proved.

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