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# "IMAGE OF A HAUSDORFF ARC" IS CYCLICALLY EXTENSIBLE AND REDUCIBLE

BY

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ABSTRACT. It is shown that a Hausdorff continuum $S$ is the continuous image of an arc (respectively arcwise connected) if and only if each cyclic element of $S$ is the continuous image of an arc (respectively, arcwise connected). Also, there is given an analogue to the metric space cyclic chain approximation theorem of G. T. Whyburn which applies to locally connected Hausdorff continua.

1. **Introduction.** Metric locally connected continua are continuous images of metric arcs and are metric arcwise connected. However, in 1960, S. Mardešić [3] gave an example of a Hausdorff locally connected continuum which is not the continuous image of a Hausdorff arc and which is not Hausdorff arcwise connected. A more simple example with the same properties is described in [2]. A very important concept in studying the structure of metric locally connected continua is the cyclic element theory of G. T. Whyburn [4], and with some differences, most of this theory can be carried over to and is useful in the study of Hausdorff locally connected continua (see, for example, [5]). The primary purpose of this paper is to present a proof of

THEOREM 1. *The Hausdorff locally connected continuum $S$ is the continuous image of an arc if and only if each cyclic element of $S$ is the continuous image of an arc.*

We also give the comparatively easy proof of

THEOREM 2. *The Hausdorff locally connected continuum $S$ is arcwise connected if and only if each cyclic element of $S$ is arcwise connected.*

Theorem 1 has an important corollary, Corollary 1.1, and it may be useful to the reader to consider the special case of this corollary when first reading the proof of Theorem 1.

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COROLLARY 1.1. *Every dendron is the continuous image of an arc.*

An *arc* is here meant to be a Hausdorff continuum which has only two non-cut points. Equivalently, an arc is a linearly ordered set which, with the order topology, is compact and connected. A dendron is a Hausdorff continuum $M$ such that each two points of $M$ are separated in $M$ by a point of $M$. Each cyclic element of a dendron is degenerate and is therefore trivially the continuous image of an arc. Hence, Corollary 1.1 is an immediate consequence of Theorem 1.

We use generally, the definitions, notation and statements of theorems as in [4, Chapter 4, §§1–6, 8]. With standard modifications, these can also be used in the context of locally connected Hausdorff continua, as initiated by Professor Whyburn in [5]. In §3 there is suggested an analog for compact Hausdorff spaces to the cyclic chain approximation theorem of §7, Chapter 4 of [4].

Henceforth it is assumed that $S$ is a locally connected Hausdorff continuum.

2. **Proof of Theorem 2.** If $S$ is arcwise connected and $a$ and $b$ belong to the cyclic element $E$ of $S$, then there is an arc $[a, b]$ in $S$ with endpoints $a$ and $b$. Now $[a, b] \cap E$ is connected and contains $a$ and $b$, so $[a, b]$ is a subset of $E$.

Suppose each cyclic element of $S$ is arcwise connected and $a$ and $b$ are points of $S$. Then $C(a, b) = E(a, b) + a + b + C$ where $C$ is the union of the collection $H$ of true cyclic elements $h$ of $S$ which contain two and only two points of $E(a, b) + a + b$. For $h$ in $H$, let $x_h$ and $y_h$ be the two points of $h$ in $E(a, b) + a + b$, denoted so that $x_h$ precedes $y_h$ in $E(a, b) + a + b$, and let $[x_h, y_h]$ be an arc in $h$ with endpoints $x_h$ and $y_h$. We show that

$$M = E(a, b) + a + b + \bigcup \{[x_h, y_h] \mid h \in H\}$$

is an arc in $S$ with endpoints $a$ and $b$.

(1) $\overline{M} \subset C(a, b)$, so if $P \in \overline{M} - M$, $P$ belongs to an element $h$ of $H$. But $[x_h, y_h]$ is closed and the only possible limit points of $C(a, b) - h$, which contains $M - h$, are $x_h$ and $y_h$ which belong to $M$. $M$ is compact.

(2) If $M = M_1 + M_2$ is a separation, each of $M_1$ and $M_2$ intersects $E(a, b) + a + b$, and if $a \in M_1$, there is a first point $d$ of $M_2$ in $E(a, b) + a + b$ and a last point $c$ of $M_1$ in $E(a, b) + a + b$ that precedes $d$. Then $c$ and $d$ are respectively $x_h$ and $y_h$ of some member $h$ of $H$, and $M_1 \cap [x_h, y_h] + M_2 \cap [x_h, y_h]$ is a separation of $[x_h, y_h]$ which is a contradiction. $M$ is a continuum.

(3) Suppose $x \in M - a - b$. If $x \in E(a, b)$, $x$ separates $a$ from $b$ in $S$, so $x$ separates $a$ from $b$ in $M$. If for some $h \in H$, $x$ is a cutpoint of $[x_h, y_h]$,

$$\{[M \cap C(a, x_h)] + [x_h, x)\} + \{(x, y_h] + [M \cap C(y_h, b)]\}$$

is a separation of $M - x$. $M$ is an arc.

**3. Retracts and inverse limits.** We use the following lemma.

LEMMA 1. *Suppose $X$ is a compact Hausdorff space, $D_0$ is a collection of retracts of $X$, for $\alpha \in D_0$, $\rho_\alpha: X \to \alpha$ is a retraction, and*

(1) *with the inclusion order ($\alpha \leqslant \beta$ if and only if $\alpha \subset \beta$), $D_0$ is a directed set;*

(2) $\bigcup D_0 = X$;

(3) *if $\alpha, \beta \in D_0$, $\alpha \leqslant \beta$, then $\rho_\alpha \circ \rho_\beta = \rho_\alpha$.*

*For $\alpha, \beta \in D_0$ with $\alpha \leqslant \beta$, let $\rho_{\alpha,\beta} = \rho_\alpha|\beta$ ($\rho_\alpha$ restricted to $\beta$). Then $\{\alpha, \rho_{\alpha,\beta}, D_0\}$ is an inverse limit system whose limit space $\alpha_\infty$ is homeomorphic to $X$.*

PROOF. Suppose $\alpha, \beta, \gamma \in D$ and $\alpha \leqslant \beta \leqslant \gamma$ and $x \in \gamma$. Then

$$\rho_{\alpha,\beta}(\rho_{\beta,\gamma}(x)) = \rho_\alpha(\rho_\beta(x)) = \rho_\alpha(x) = \rho_{\alpha,\gamma}(x).$$

$\{\alpha, \rho_{\alpha,\beta}, D_0\}$ is an inverse limit system. Define $H: X \to \alpha_\infty$ by

$$H(x) = (\rho_\alpha(x))_{\alpha \in D_0}.$$

For $\lambda \in D_0$, let $P_\lambda$ be the projection of the product space $\times\{\alpha: \alpha \in D_0\}$ onto $\lambda$; then $P_\lambda \circ H = \rho_\lambda$ and therefore $H$ is continuous. If $x$ and $y$ are distinct points of $X$, there are members $\zeta$ and $\eta$ of $D_0$ such that $x \in \zeta$ and $y \in \eta$, and there is $\alpha$ in $D_0$ such that $\xi \leqslant \alpha$ and $\eta \leqslant \alpha$. Then both $x$ and $y$ belong to $\alpha$ and $\rho_\alpha(x) = x \neq y = \rho_\alpha(y)$. Consequently $H(x)$ and $H(y)$ differ in the $\alpha$th coordinate and $H$ is one to one.

The sets $\{P_\alpha^{-1}(O_\alpha): \alpha \in D_0, O_\alpha$ open in $\alpha\}$ form a base for the topology of $\alpha_\infty$. Suppose $\alpha \in D_0$, $O_\alpha$ is an open set in $\alpha$ and $x \in O_\alpha$. Then $H(x) \in P_\alpha^{-1}(O_\alpha)$. Consequently $H(X)$ is dense in $\alpha_\infty$ and since $X$ is compact, $H$ is onto. $H$ is the desired homeomorphism.

For any $A$-set $A$ of $S$, we define the continuous retraction $\rho_A: S \to A$ by

$$\rho_A(x) = x \quad \text{if } x \in A,$$

$$= \text{boundary point of the component of } S - A$$

$$\text{that contains } x \text{ if } x \in S - A.$$

We let $e_0$ denote a specific point of a specific node of $S$ which is a non-cut point of $S$. We let $D_0$ denote the collection of $A$-sets of $S$ which have

only a finite number of nodes (of themselves) and which contain $e_0$. With routine arguments of cyclic element theory, one can establish that, for $X = S$, $D_0$ and $\rho_\alpha$, $\alpha \in D_0$ as just defined, the hypotheses of Lemma 1 are satisfied, and we see that $S$ is homeomorphic to the inverse limit space of $\{\alpha, \rho_{\alpha,\beta}, D_0\}$.

REMARK. It may be observed that this provides an analogue in Hausdorff spaces to Whyburn's cyclic chain approximation theorem. The analogy can be even more closely drawn by observing Lemma 2, which is not used otherwise in this paper.

LEMMA 2. *If $G$ is an open cover of $S$, there is a member $\alpha$ of $D_0$ such that every component of $S - \alpha$ is contained in a member of $G$.*

**4. The steps in the proof of Theorem 1.** If $\pi$ is a map of an arc $I$ onto $S$ and $E$ is a cyclic element of $S$, then $\rho_E \circ \pi$ is a map of $I$ onto $E$. This "half" of Theorem 1 is complete.

With the notation of §3, the strategy in the proof of the other "half" of Theorem 1 is

I. For each $\alpha \in D_0$, define an arc $I_\alpha$ and a map $\pi_\alpha$ of $I_\alpha$ onto $\alpha$.

II. For $\alpha, \beta \in D_0$, define a monotone map $\phi_{\alpha,\beta}$ of $I_\beta$ onto $I_\alpha$ such that $\pi_\alpha \circ \phi_{\alpha,\beta} = \rho_{\alpha,\beta} \circ \pi_\beta$.

III. Show that for $\alpha, \beta, \gamma \in D_0$, with $\alpha \leqslant \beta \leqslant \gamma$, $\phi_{\alpha,\gamma} = \phi_{\alpha,\beta} \circ \phi_{\beta,\gamma}$.

It will follow then, that $\{I_\alpha, \phi_{\alpha,\beta}, D_0\}$ is an inverse limit system whose inverse limit space $I_\infty$ is an arc and that the inverse limit map $\pi_\infty$ maps $I_\infty$ onto $S$. That $I_\infty$ is an arc follows from Theorem 4.4 and Lemma 4.7 of [1]. The arguments for steps I, II and III follow a common pattern and are induction arguments.

**5. Some notation.** It is necessary to consider a class $D$ which properly contains $D_0$. We let $D$ denote the collection of *all* $A$-sets in $S$ which have only a finite number of nodes. For $\alpha \in D$, we let $\alpha^*$ denote $e_0$ if $e_0 \in \alpha$, and if $e_0 \notin \alpha$, let $\alpha^*$ denote the boundary point of the component of $S - \alpha$ that contains $e_0$. If $E$ is a cyclic element of $\alpha \in D$, *the order of $E$ in $\alpha$* is (a) the number of boundary points of $E$ if $E$ is a true cyclic element of $\alpha$, and (b) the number of components of $\alpha - E$ if $E$ is a degenerate cyclic element of $\alpha$. Then $E$ will be called a *vertex of* $\alpha$ if (a) the order of $E$ in $\alpha$ is not two, or (b) $\alpha^*$ is a cutpoint of $\alpha$ of order two and $E$ is $\{\alpha^*\}$, or (c) $\alpha^*$ is a noncutpoint of $\alpha$ and $E$ is the unique cyclic element of $\alpha$ which contains $\alpha^*$.

If $\alpha \in D$, a *type*-0 *edge of* $\alpha$ is a true cyclic element of $\alpha$ which is a vertex of $\alpha$, and a *type*-1 *edge of* $\alpha$ is cyclic chain $C(a, b)$ of $\alpha$ such that $a$ and $b$ belong to nonintersecting vertices and no point of $C(a, b) - a - b$

belongs to a vertex of $\alpha$. Note that each type-0 edge of $\alpha$ may be written as $C(a, b)$ for any two distinct points $a$ and $b$ belonging to it. If $\alpha \in D$ and $n$ is a positive integer, the *height of* $\alpha$ *is* $\leqslant n$ if and only if for each point $x$ of $\alpha$, there is a sequence $C(a_0, a_1), C(a_1, a_2), \cdots, C(a_{k-1}, a_k)$ of $k \leqslant n$ edges (type-0 or type-1) of $\alpha$ such that $a_0$ is $\alpha^*$ and $x$ belongs to $C(a_{k-1}, a_k)$. Using standard arguments from cyclic element theory, one can prove the following:

PROPERTIES OF VERTICES AND EDGES. *Suppose $\alpha \in D$. Each cyclic element of $\alpha$ is of finite order. The cyclic element $E$ of $\alpha$ has order one if and only if $E$ is a node of $\alpha$. All but a finite number of cyclic elements of $\alpha$ have order two. If two edges of $\alpha$ intersect, their intersection is a cut point of $\alpha$. Every point of $\alpha$ belongs to an edge of $\alpha$ and $\alpha$ has only a finite number of edges. There is a positive integer $n$ such that the height of $\alpha$ is $\leqslant n$. Each edge of $\alpha$ belongs to $D$ and if $C(a, b)$ is a type-1 edge of $\alpha$, $a$ and $b$ are the boundary points in $\alpha$ of $\alpha - C(a, b)$, and if $C(a, b) = \gamma$, $\gamma^*$ is $a$ or $b$; further if $\gamma^* = a$ and $\beta \in D$ and $\beta \subset \gamma$ and $\beta^* = \gamma^*$, there is a member $c$ of $E(a, b) + a + b$ such that $\beta = C(a, c)$.*

Let $W$ denote the collection to which $w$ belongs if and only if for some cyclic element $E$ of $S$, $w$ is a component of $S - E$. *We assume henceforth that there is a prescribed well-order on $W$.* If $\alpha \in D$ and $E$ is a cyclic element of $\alpha$, then $E$ is a cyclic element of $S$, each component of $\alpha - E$ is contained in a component of $S - E$, and no component of $S - E$ contains two components of $\alpha - E$. Consequently, the well-order of $W$ induces a unique well-order on the components of $\alpha - E$ (the component $U$ of $\alpha - E$ precedes the component $V$ of $\alpha - E$ if and only if the component of $S - E$ that contains $U$ precedes the component of $S - E$ that contains $V$). This unique order on the components of $\alpha - E$ then induces a unique order on the closures of those components ($\bar{U}$ precedes $\bar{V}$ if and only if $U$ precedes $V$). We say then that the closures of the components of $\alpha - E$ are *ordered in the order relative to* $W$. Furthermore, if $\alpha$ is a subset of a member $\beta$ of $D$, the well-order on $W$ induces an order on the components of $\beta - E$ which in turn induces the same order on the components of $\alpha - E$ as does the well-order of $W$.

If $[P_1, Q_1], \cdots, [P_n, Q_n]$ are oriented arcs, then

$$[P_1, Q_1] \leftrightarrow [P_2, Q_2] \leftrightarrow \cdots \leftrightarrow [P_n, Q_n]$$

($\leftrightarrow$ is read "joined to") will denote the oriented arc $[P, Q]$ obtained from the disjoint union of $\{[P_i, Q_i]\}_{i=1}^n$ by the equivalence $P_i \cong Q_{i+1}$, $i = 1, \cdots, n-1$, and with $P = P_1$, $Q = Q_n$. If, further, there are maps $\pi_i$ with domain

$[P_i, Q_i]$, $i = 1, \cdots, n$, and $\pi_i(Q_i) = \pi_{i+1}(P_{i+1})$, $i = 1, \cdots, n-1$, then

$$([P_1, Q_1], \pi_1) \leftrightarrow ([P_2, Q_2], \pi_2) \leftrightarrow \cdots \leftrightarrow ([P_n, Q_n], \pi_n)$$

will denote the pair $([P, Q], \pi)$ such that $[P, Q]$ is as above and $\pi$ is the map with domain $[P, Q]$ such that $\pi(x) = \pi_i(x)$ if $x \in [P_i, Q_i], i = 1, \cdots, n$. Note that degenerate intervals $[P_j, Q_j]$ and pairs $([P_j, Q_j], \pi_j)$ can be inserted in or deleted from $\{[P_i, Q_i]\}_{i=1}^n$ and $\{[P_i, Q_i], \pi_i\}_{i=1}^n$ without changing $[P, Q]$ or $([P, Q], \pi)$.

By hypothesis of this "half" of Theorem 1, every cyclic element of $S$ is the continuous image of an arc. We assume that for each cyclic element $E$ of $S$, there is selected a specific pair $(I_E, \pi_E)$ such that $I_E$ is an arc with a fixed orientation $(I_E = [P_E, Q_E])$ and that $\pi_E$ is a map of $I_E$ onto $C$ such that $\pi_E(P_E) = \pi_E(Q_E) = E^*$. If $E$ is degenerate, it is assumed that $I_E$ is degenerate. If $a$ is a point of $E$, we let $\langle a \rangle$ (or $\langle a \rangle_E$ where necessary) denote the first point of $\pi_E^{-1}(a)$ in $I_E$.

**6. Step I, the pairs $(I_\alpha, \pi_\alpha)$.**

6.1. *The pairs $(I_\alpha, \pi_\alpha)$ for $\alpha$ an edge.* If $\alpha$ is a type-0 edge of a member of $D$, $\alpha$ is a true cyclic element of $S$ and the pair $(I_\alpha, \pi_\alpha)$ is already determined.

Suppose $\alpha$ is a type-1 edge $C(a, b)$ of a member of $D$; then $\alpha \in D$, $\alpha^*$ is either $a$ or $b$ and we assume $\alpha^* = a$. Now

$$\alpha = C(a, b) = E(a, b) + a + b + C$$

where $C$ is the union of the collection $H$ of true cyclic elements of $S$ which contain two and only two points of $E(a, b) + a + b$. If $h \in H$, $h$ also belongs to $D$ and $h^*$ is one of the two points of $h$ that belong to $E(a, b) + a + b$; we let $h^{**}$ denote the other one. Note that $h^*$ precedes $h^{**}$ in $E(a, b) + a + b$. We let $J_\circ$ (respectively $K_\circ$) denote the disjoint union of the *open* intervals $\{(P_h, \langle h^{**} \rangle_h) \mid h \in H\}$ (respectively, $\{(\langle h^{**} \rangle_h, Q_h) \mid h \in H\}$) and let

$$J_\alpha = E(a, b) + a + b + J_\circ, \quad \text{and} \quad K_\alpha = E(a, b) + a + b + K_\circ.$$

Define the order $<$ on $J_\alpha$ (respectively $K_\alpha$) by $x < y$ if and only if

(1) $x, y \in E(a, b) + a + b$ and $x$ precedes (respectively, follows) $y$ in that set, or

(2) for some $h \in H$, $x \in (P_h, \langle h^{**} \rangle)$ (respectively, $x \in (\langle h^{**} \rangle, Q_h)$) and $y \in E(a, b) + a + b$ and $h^{**}$ is $y$ or precedes $y$ in $E(a, b) + a + b$ ($h^*$ is $y$ or follows $y$ in $E(a, b) + a + b$), or

(3) $x \in E(a, b) + a + b$ and for some $h \in H$, $y \in (P_h, \langle h^{**}\rangle)$ ($y \in (\langle h^{**}\rangle, Q_h)$) and $x$ is $h^*$ or precedes $h^*$ in $E(a, b) + a + b$ ($x$ is $h^{**}$ or follows $h^{**}$ in $E(a, b) + a + b$), or

(4) for some $h_1, h_2 \in H$, $x \in (P_{h_1}, \langle h_1^{**}\rangle)$ and $y \in (P_{h_2}, h_2^{**})$ ($x \in (\langle h_1^{**}\rangle, Q_{h_1})$ and $y \in (\langle h_2^{**}\rangle, Q_{h_2})$) and either $h_1 = h_2$ and $x$ precedes $y$ on $(P_{h_1}, \langle h_1^{**}\rangle)$ ($x$ precedes $y$ on $(\langle h_1^{**}\rangle, Q_{h_1})$) or $h_1 \neq h_2$ and $h_1^*$ precedes $h_2^*$ on $E(a, b) + a + b$ ($h_2^*$ precedes $h_1^*$ on $E(a, b) + a + b$).

With these orders, and the topologies induced by these orders, $J_\alpha$ and $K_\alpha$ are oriented arcs. We let $I_\alpha = J_\alpha \longleftrightarrow K_\alpha = [P_\alpha, Q_\alpha]$ where $P_\alpha$ is the first point of $J_\alpha$ and $Q_\alpha$ is the last point of $K_\alpha$, and define $\pi_\alpha\colon I_\alpha \to \alpha = C(a, b)$ by

$$\begin{aligned} \pi_\alpha(x) &= x \quad \text{if } x \in E(a, b) + a + b, \\ &= \pi_h(x) \quad \text{if } x \in (P_h, \langle h^{**}\rangle) + (\langle h^{**}\rangle, Q_h). \end{aligned}$$

It is fairly routine to check that $\pi_\alpha$ is a map of $I_\alpha$ onto $\alpha$ and that $\pi_\alpha(P_\alpha) = \pi_\alpha(Q_\alpha) = a = \alpha^*$.

6.2. *The pair* $(I_\alpha, \pi_\alpha)$ *for* $\alpha$ *of height* $\leqslant 1$. Suppose $\alpha$ is a member of $D$ of height $\leqslant 1$. If $\alpha - \alpha^*$ is connected, $\alpha$ must be its own only edge and $(I_\alpha, \pi_\alpha)$ is already defined. If $\alpha - \alpha^*$ is not connected, we let $\alpha_1, \cdots, \alpha_p$ denote the closures of the components of $\alpha - \alpha^*$, ordered in the order relative to $W$. Then $\alpha_i$, $1 \leqslant i \leqslant p$, is an edge of $\alpha$, $\alpha_i^* = \alpha^*$ and the pair $(I_{\alpha_i}, \pi_{\alpha_i})$ is already defined. We define $(I_\alpha, \pi_\alpha)$ to be

$$(I_{\alpha_1}, \pi_{\alpha_1}) \leftrightarrow (I_{\alpha_2}, \pi_{\alpha_2}) \leftrightarrow \cdots \leftrightarrow (I_{\alpha_p}, \pi_{\alpha_p})$$

and let $P_\alpha$ and $Q_\alpha$ be respectively the first and last points of $I_\alpha$; then $\pi_\alpha(P_\alpha) = \alpha_1^* = \alpha^* = \alpha_p^* = \pi_\alpha(Q_\alpha)$.

6.3. *The induction step for the pairs* $(I_\alpha, \pi_\alpha)$. Suppose $n$ is a positive integer and the pairs $(I_\alpha, \pi_\alpha)$ have been defined so that $I_\alpha = [P_\alpha, Q_\alpha]$ and $\pi_\alpha(P_\alpha) = \pi_\alpha(Q_\alpha) = \alpha^*$ for every member $\alpha$ of $D$ of height $\leqslant n$. Let $\alpha$ be a member of $D$ of height $\leqslant n + 1$.

Suppose first that $\alpha - \alpha^*$ is connected. Then there is a unique edge $e$ of $\alpha$ that contains $\alpha^*$ (since each two edges of $\alpha$ intersect at only a cut point of $\alpha$).

(i) Suppose first $e$ is a type-1 edge of $\alpha$. Then $e = C(a^*, b)$ for some point $b$ of $\alpha$, and $b$ is the only boundary point in $\alpha$ of $\alpha - e$. Let $\alpha_1, \cdots, \alpha_p$ be the closures of the components of $\alpha - e$ (which are also components of $\alpha - b$), ordered in the order relative to $W$. Then $\alpha_i$, $1 \leqslant i \leqslant p$, is a member of $D$, $\alpha_i^* = b$ and the height of $\alpha_i$ is $\leqslant n$; the pair $(I_{\alpha_i}, \pi_{\alpha_i})$ has

been defined and $\pi_{\alpha_i}(P_{\alpha_i}) = \pi_{\alpha_i}(Q_{\alpha_i}) = b$. Recall $J_e$, $K_e$ and $\pi_e$ from §6.1 and define $(\pi_\alpha, I_\alpha)$ to be

$$(J_e, \pi_e|J_e) \leftrightarrow (I_{\alpha_1}, \pi_{\alpha_1}) \leftrightarrow \cdots \leftrightarrow (I_{\alpha_p}, \pi_{\alpha_p}) \leftrightarrow (K_e, \pi_e|K_e).$$

(ii) Suppose now $e$ is a type-0 edge of $\alpha$. Let $x_1, \cdots, x_q$ denote the boundary points of $e$ in $\alpha$, ordered so that on $I_e$, $\langle x_1\rangle < \langle x_2\rangle < \cdots < \langle x_q\rangle$, and for $1 \leqslant i \leqslant q$, let $\alpha_{i,1}, \cdots, \alpha_{i,p(i)}$ denote the closures of the components of $\alpha - e$ which have $x_i$ as their boundary point, ordered in the order relative to $W$. Then for $1 \leqslant i \leqslant q$, $1 \leqslant j \leqslant p(i)$, $\alpha_{i,j}$ is a member of $D$, $\alpha^*_{i,j} = x_i$ and the height of $\alpha_{i,j}$ is $\leqslant n$. Let $I_0$ be the interval $[P_e, \langle x_1\rangle]$ of $I_e$, $I_i$ be the interval $[\langle x_i\rangle, \langle x_{i+1}\rangle]$ of $I_e$, $i = 1, q-1$, and $I_q$ be the interval $[\langle x_q\rangle, Q_e]$ of $I_e$, and let $\pi_i = \pi_e|I_i$, $0 \leqslant i \leqslant q$. We define $(I_\alpha, \pi_\alpha)$ to be

$$\begin{aligned}(I_0, \pi_0) \leftrightarrow (I_{\alpha_{1,1}}, \pi_{\alpha_{1,1}}) \leftrightarrow \cdots \leftrightarrow (I_{\alpha_{1,p(1)}}, \pi_{\alpha_{1,p(1)}}) \leftrightarrow (I_1, \pi_1)\\ \leftrightarrow \cdots \leftrightarrow (I_{q-1}, \pi_{q-1}) \leftrightarrow (I_{\alpha_{q,1}}, \pi_{\alpha_{q,1}})\\ \leftrightarrow \cdots \leftrightarrow (I_{\alpha_{q,p(q)}}, \pi_{\alpha_{q,p(q)}}) \leftrightarrow (I_q, \pi_q).\end{aligned}$$

For $I_\alpha = [P_\alpha, Q_\alpha]$, $\pi_\alpha(P_\alpha) = \pi_e(P_e) = \alpha^* = \pi_e(Q_e) = \pi_\alpha(Q_\alpha)$.

For $\alpha$ any member of $D$ of height $\leqslant n+1$ such that $\alpha - \alpha^*$ is connected, we have defined $(I_\alpha, \pi_\alpha)$ and have that $\pi_\alpha(P_\alpha) = \pi_\alpha(Q_\alpha) = \alpha^*$. Suppose now $\alpha \in D$ of height $\leqslant n+1$ and $\alpha - \alpha^*$ is not connected. Let $\alpha_1, \cdots, \alpha_r$ denote the closures of the components of $\alpha - \alpha^*$, ordered in the order relative to $W$. Then for $1 \leqslant k \leqslant r$, $\alpha_k$ is a member of $D$ of height $\leqslant n+1$, $\alpha^*_k$ is $\alpha^*$ and $\alpha_k - \alpha^*_k$ is connected. The pairs $(I_{\alpha_k}, \pi_{\alpha_k})$ have been defined and $\pi_{\alpha_k}(P_{\alpha_k}) = \pi_{\alpha_k}(Q_{\alpha_k}) = \alpha^*_k = \alpha^*$, $1 \leqslant k \leqslant r$; we define $(I_\alpha, \pi_\alpha)$ to be

$$(I_{\alpha_1}, \pi_{\alpha_1}) \leftrightarrow (I_{\alpha_2}, \pi_{\alpha_2}) \leftrightarrow \cdots \leftrightarrow (I_{\alpha_r}, \pi_{\alpha_r}).$$

For $I_\alpha = [P_\alpha, Q_\alpha]$, $\pi_\alpha(P_\alpha) = \pi_{\alpha_1}(P_{\alpha_1}) = \alpha^* = \pi_{\alpha_r}(Q_{\alpha_r}) = \pi_\alpha(Q_\alpha)$.

This completes the induction step and the descriptions of the pairs $(I_\alpha, \pi_\alpha)$.

7. **Step II, the maps $\phi_{\alpha,\beta}$.** We define in this section, for any two members $\alpha$ and $\beta$ of $D$ such that $\alpha \subset \beta$ and $\alpha^* = \beta^*$, a monotone map $\phi_{\alpha,\beta}$ of $I_\beta$ onto $I_\alpha$ and show that $\pi_\alpha \circ \phi_{\alpha,\beta} = \rho_{\alpha,\beta} \circ \pi_\beta$. There are two degenerate cases: $\alpha = \beta$ and $\alpha = \{\beta^*\}$. In the former, $\phi_{\alpha,\beta}$ is the identity map and in the latter $\phi_{\alpha,\beta}$ is the only possible map ($I_\alpha$ is degenerate). In either case, $\phi_{\alpha,\beta}$ is monotone and onto and $\pi_\alpha \circ \phi_{\alpha,\beta} = \rho_{\alpha,\beta} \circ \pi_\beta$. Henceforth it is assumed that $\alpha, \beta \in D$ and $\alpha$ is a nondegenerate proper subset of $\beta$, and $\alpha^* = \beta^*$.

7.1. *The map $\phi_{\alpha,\beta}$ for $\beta$ an edge.* If $\beta$ is a type-0 edge of a member of $D$, the two degenerate cases are the only possibility for $\alpha$.

Suppose $\beta$ is a type-1 edge $C(a, b)$ of some member of $D$. Then $\beta \in D$ and $\beta^*$ is $a$ or $b$; we assume $\beta^*$ is $a$ so that $\beta$ is $C(\beta^*, b)$. There is a point $c$ of $E(\beta^*, b)$ such that $\alpha$ is $C(\beta^*, c) = C(\alpha^*, c)$. Referring now to the procedure for constructing $(I_\beta, \pi_\beta)$, one sees that the point $c$ belongs to $J_\beta$ and to $K_\beta$, and hence occurs exactly twice in $I_\beta$; we let $c_1$ and $c_2$ denote those occurrences of $c$ in $I_\beta$, in the order of $I_\beta$. It can further be seen that $J_\alpha$ is identical to the interval $[P_\beta, c_1]$ of $I_\beta$ and $K_\alpha$ is identical to the interval $[c_2, Q_\beta]$ of $I_\beta$, and that

$$(I_\alpha, \pi_\alpha) = ([P_\beta, c_1], \pi_\beta|[P_\beta, c_1]) \leftrightarrow ([c_2, Q_\beta], \pi_\beta|[c_2, Q_\beta]).$$

We define $\phi_{\alpha,\beta}$ by

$$\begin{aligned} \phi_{\alpha,\beta}(x) &= x \qquad \text{if } x \in [P_\beta, c_1] \text{ or } x \in [c_2, Q_\beta], \\ &= \{c_1, c_2\} \text{ if } x \in [c_1, c_2]. \end{aligned}$$

Each point inverse of $\phi_{\alpha,\beta}$ is either degenerate or the interval $[c_1, c_2]$; $\phi_{\alpha,\beta}$ is monotone and also onto. There is only one component of $\beta - \alpha$ (and it is $C(c, h) - c$) and that component has $c$ as its only boundary point in $\alpha$. Therefore, $\rho_{\alpha,\beta}(x) = c$ if $x \in \beta - \alpha$. If $x \in [P_\beta, c_1]$ or $x \in [c_2, Q_\beta]$, then $\pi_\beta(x) = \pi_\alpha(x)$ is a point of $\alpha$ and $\phi_{\alpha,\beta}(x) = x$, and

$$\pi_\alpha(\phi_{\alpha,\beta}(x)) = \pi_\alpha(x) = \pi_\beta(x) = \rho_{\alpha,\beta}(\pi_\beta(x)).$$

If $x \in [c_1, c_2]$,

$$\pi_\beta(x) \in \beta - \alpha, \qquad \phi_{\alpha,\beta}(x) = \{c_1, c_2\}$$

and

$$\pi_\alpha(\phi_{\alpha,\beta}(x)) = \pi_\alpha(\{c_1, c_2\}) = c = \rho_{\alpha,\beta}(\beta - \alpha) = \rho_{\alpha,\beta}(\pi_\beta(x)).$$

We conclude that $\pi_\alpha \circ \phi_{\alpha,\beta} = \rho_{\alpha,\beta} \circ \pi_\beta$.

7.2. *The map $\phi_{\alpha,\beta}$ for $\beta$ of height* $\leqslant 1$. Suppose $\beta$ is a member of $D$ of height $\leqslant 1$. If $\beta - \beta^*$ is connected, $\beta$ must be its own only edge and $\phi_{\alpha,\beta}$ is already defined. If $\beta - \beta^*$ is not connected, let $\beta_1, \cdots, \beta_p$ denote the closures of the components of $\beta - \beta^*$, ordered in the order relative to $W$. Then $\beta_i$, $1 \leqslant i \leqslant p$, is an edge of $\beta$ and we let $\alpha_i = \alpha \cap \beta_i$ and observe that $\alpha_i \in D$, $\alpha_i^* = \beta_i^* = \beta^*$ and the maps $\phi_{\alpha_i,\beta_i}$ have been defined. It may be that $\alpha_i$ is degenerate for some or several $i$'s, $1 \leqslant i \leqslant p$, and that they consequently were not considered as part of $(I_\alpha, \pi_\alpha)$; however for such $\alpha_i$, $I_{\alpha_i}$ is degenerate and we may write

$$(I_\alpha, \pi_\alpha) = (I_{\alpha_1}, \pi_{\alpha_1}) \leftrightarrow \cdots \leftrightarrow (I_{\alpha_p}, \pi_{\alpha_p}),$$

$$(I_\beta, \pi_\beta) = (I_{\beta_1}, \pi_{\beta_1}) \leftrightarrow \cdots \leftrightarrow (I_{\beta_p}, \pi_{\beta_p}).$$

It is important to note that, excluding the degenerate cases mentioned, $\alpha_1, \cdots, \alpha_p$ is ordered in the order relative to $C$. We define $\phi_{\alpha,\beta}$ by

$$\phi_{\alpha,\beta}(x) = \phi_{\alpha_i,\beta_i}(x) \quad \text{if } x \in I_{\beta_i}.$$

Then $\phi_{\alpha,\beta}$ is monotone and onto because $\phi_{\alpha_i,\beta_i}$, $1 \leqslant i \leqslant p$, is monotone and onto. If $x \in \beta$, there is an integer $i$ such that $x \in \beta_i$, and $\rho_{\alpha,\beta}(x) = \rho_{\alpha_i,\beta_i}(x)$. Because $\pi_{\alpha_i} \circ \phi_{\alpha_i,\beta_i} = \rho_{\alpha_i,\beta_i} \circ \pi_{\beta_i}$, $1 \leqslant i \leqslant p$, we have $\pi_\alpha \circ \phi_{\alpha,\beta} = \rho_{\alpha,\beta} \circ \pi_\beta$.

7.3. *The induction step for* $\phi_{\alpha,\beta}$. Suppose $n$ is a positive integer and $\phi_{\alpha,\beta}$ has been defined, is monotone, onto and satisfies $\pi_\alpha \circ \phi_{\alpha,\beta} = \rho_{\alpha,\beta} \circ \pi_\beta$, for all $\alpha$, $\beta$ in $D$ such that $\alpha \subset \beta$, $\alpha^* = \beta^*$, and $\beta$ has height $\leqslant n$. Let $\beta$ be a member of $D$ of height $\leqslant n + 1$.

Suppose first that $\beta - \beta^*$ is connected; then there is a unique edge $e$ of $\beta$ that contains $\beta^*$.

(i) Suppose first that $e$ is a type-1 edge of $\beta$. We further suppose that $e \subset \alpha$. Then $e$ is $C(\beta^*, b)$ where $b$ is the only boundary point in $\beta$ of $\beta - e$. Let $\beta_1, \cdots, \beta_p$ be the closures of the components of $\beta - e$, ordered in the order relative to $W$, and let $\alpha_i = \alpha \cap \beta_i$, $i = 1, \cdots, p$. Then for $1 \leqslant i \leqslant p$, $\alpha_i \in D$, $\beta_i \in D$, $\alpha_i \subset \beta_i$, $\alpha_i^* = \beta_i^* = b$, the height of $\beta_i \leqslant n$ and the map $\phi_{\alpha_i,\beta_i}$ is already defined. If $z$ is $\alpha$ or $\beta$, the pair $(I_z, \pi_z)$ is defined to be

$$(I_z, \pi_z) = (J_e, \pi_e|J_e) \leftrightarrow (I_{z_1}, \pi_{z_1}) \leftrightarrow \cdots \leftrightarrow (I_{z_p}, \pi_{z_p}) \leftrightarrow (K_e, \pi_e|K_e).$$

We define $\phi_{\alpha,\beta}$ by

$$\begin{aligned} \phi_{\alpha,\beta}(x) &= x && \text{if } x \in J_e \text{ or } x \in K_e, \\ &= \phi_{\alpha_i,\beta_i}(x) && \text{if } x \in I_{\beta_i}. \end{aligned}$$

Then, $\phi_{\alpha,\beta}$ is monotone and onto because $\phi_{\alpha_i,\beta_i}$ is monotone and onto. If $x \in \beta$, there is an integer $i$ such that $x \in \beta_i$, and $\rho_{\alpha,\beta}(x) = \rho_{\alpha_i,\beta_i}(x)$. Because $\pi_{\alpha_i} \circ \phi_{\alpha_i,\beta_i} = \rho_{\alpha_i,\beta_i} \circ \pi_{\beta_i}$, $i = 1, \cdots, p$, we have $\pi_\alpha \circ \phi_{\alpha,\beta} = \rho_{\alpha,\beta} \circ \pi_\beta$.

If $\alpha$ is a proper subset of $e$, there is a point $c$ of $E(\beta^*, b)$ such that $\alpha$ is $C(\beta^*, c)$. The preceding paragraph defines $\phi_{e,\beta}\colon I_\beta \to I_e$ and since $e$ is an edge, $\phi_{\alpha,e}\colon I_e \to I_\alpha$ was defined in §7.1. We define $\phi_{\alpha,\beta}$ to be $\phi_{\alpha,e} \circ \phi_{e,\beta}$. Since $\phi_{\alpha,e}$ and $\phi_{e,\beta}$ are monotone and onto, $\phi_{\alpha,\beta}$ is monotone and onto. Since $\rho_{\alpha,\beta} = \rho_{\alpha,e} \circ \rho_{e,\beta}$ and $\pi_e \circ \phi_{e,\beta} = \rho_{e,\beta} \circ \pi_\beta$ and $\pi_\alpha \circ \phi_{\alpha,e} = \rho_{\alpha,e} \circ \pi_e$, we conclude that $\pi_\alpha \circ \phi_{\alpha,\beta} = \rho_{\alpha,\beta} \circ \pi_\beta$.

(ii) Suppose now $e$ is a type-0 edge of $\beta$. Since we are considering $\alpha$ to be nondegenerate, it must be that $e \subset \alpha$. Let $x_1, \cdots, x_q$ denote the boundary points of $e$ in $\beta$, ordered so that on $I_e$, $\langle x_1 \rangle < \langle x_2 \rangle < \cdots < \langle x_q \rangle$, and for $1 \leqslant i \leqslant q$,

let $\beta_{i,1}, \cdots, \beta_{i,p(i)}$ denote the closures of the components of $\beta - e$ which have $x_i$ as their boundary point, ordered in the order relative to $W$. For $1 \leqslant i \leqslant q$, $1 \leqslant j \leqslant p(i)$, let $\alpha_{i,j} = \alpha \cap \beta_{i,j}$; then $\alpha_{i,j}$ and $\beta_{i,j}$ belong to $D$, $\alpha^*_{i,j} = \beta^*_{i,j} = x_i$ and the height of $\beta_{i,j}$ is $\leqslant n$ and the map $\phi_{\alpha_{i,j},\beta_{i,j}}$ is already defined. Let $I_0$ be the interval $[P_e, \langle x_1 \rangle]$ of $I_e$, $I_i$ be the interval $[\langle x_i \rangle, \langle x_{i+1} \rangle]$ of $I_e$, $i = 1, q-1$, and let $I_q$ be the interval $[\langle x_q \rangle, Q_e]$ of $I_e$, and let $\pi_i = \pi_e | I_i$, $i = 0, \cdots, q$. Then if $z$ is either $\alpha$ or $\beta$, $(I_z, \pi_z)$ is

$$(I_0, \pi_0) \leftrightarrow (I_{z_{1,1}}, \pi_{z_{1,1}}) \leftrightarrow \cdots \leftrightarrow (I_{z_{1,p(1)}}, \pi_{z_{1,p(1)}}) \leftrightarrow (I_1, \pi_1)$$
$$\leftrightarrow \cdots \leftrightarrow (I_{q-1}, \pi_{q-1}) \leftrightarrow (I_{z_{q,1}}, \pi_{z_{q,1}})$$
$$\leftrightarrow \cdots \leftrightarrow (I_{z_{q,p(q)}}, \pi_{z_{q,p(q)}}) \leftrightarrow (I_q, \pi_q).$$

Then $\phi_{\alpha,\beta}$ is defined by

$$\phi_{\alpha,\beta}(x) = x \qquad \text{if } x \in I_i,\ i = 0, \cdots, q,$$
$$= \phi_{\alpha_{i,j},\beta_{i,j}}(x) \quad \text{if } x \in I_{\beta_{i,j}},\ 1 \leqslant i \leqslant q,\ 1 \leqslant j \leqslant p(i).$$

That $\rho_{\alpha,\beta}$ is monotone, onto and satisfies $\pi_\alpha \circ \phi_{\alpha,\beta} = \rho_{\alpha,\beta} \circ \pi_\beta$ follows from the analogous properties of $\phi_{\alpha_{i,j},\beta_{i,j}}$, $1 \leqslant i \leqslant q$, $1 \leqslant j \leqslant p(i)$

This completes the definition of $\phi_{\alpha,\beta}$ for $\beta - \beta^*$ connected. Now suppose $\beta - \beta^*$ is not connected. Let $\beta_1, \cdots, \beta_r$ denote the closures of the components of $\beta - \beta^*$, ordered in the order relative to $W$, and for $1 \leqslant k \leqslant r$, let $\alpha_k = \alpha \cap \beta_k$. Then for $1 \leqslant k \leqslant r$, $\alpha_k$ and $\beta_k$ belong to $D$, $\alpha^*_k = \beta^*_k = \beta^*$, $\beta_k$ is of height $\leqslant n + 1$ and $\beta_k - \beta^*_k$ is connected and the map $\phi_{\alpha_k,\beta_k}$ has been defined. If $z$ is $\alpha$ or $\beta$, $(I_z, \pi_z)$ is

$$(I_{z_1}, \pi_{z_1}) \leftrightarrow (I_{z_2}, \pi_{z_2}) \leftrightarrow \cdots \leftrightarrow (I_{z_r}, \pi_{z_r})$$

We define $\phi_{\alpha,\beta}$ by

$$\phi_{\alpha,\beta}(x) = \phi_{\alpha_k,\beta_k}(x) \quad \text{if } x \in I_{\beta_k},\ 1 \leqslant k \leqslant r.$$

As before, $\phi_{\alpha,\beta}$ is monotone, onto and satisfies $\pi_\alpha \circ \phi_{\alpha,\beta} = \rho_{\alpha,\beta} \circ \pi_\beta$ because of the analogous properties of $\phi_{\alpha_k,\beta_k}$, $1 \leqslant k \leqslant r$.

This completes the induction step, and step II, the definition of the maps $\phi_{\alpha,\beta}$.

8. **Step III, the transitivity** $\phi_{\alpha,\beta} \circ \phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$. The last property to be established is to show that for $\alpha, \beta, \gamma \in D$, $\alpha \leqslant \beta \leqslant \gamma$ and $\alpha^* = \beta^* = \gamma^*$,

that $\phi_{\alpha,\beta} \circ \phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$. In the event that $\alpha = \{\alpha^*\}$ or $\alpha = \beta$ or $\beta = \gamma$, the result is immediate, and henceforth we assume that $\alpha, \beta, \gamma \in D$, $\alpha$ is a nondegenerate proper subset of $\beta$ and $\beta$ is a proper subset of $\gamma$, and that $\alpha^* = \beta^* = \gamma^*$.

8.1. $\phi_{\alpha,\beta} \circ \phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$ *for* $\gamma$ *an edge.* If $\gamma$ is a type-0 edge of a member of $D$, $\gamma$ is a true cyclic element of $S$, and we must have one of the degenerate cases $\alpha = \beta$ or $\beta = \gamma$.

Suppose $\gamma$ is a type-1 edge, $C(a, b)$ of some member of $D$. Then $\gamma \in D$ and $\gamma^*$ is either $a$ or $b$; assume $\gamma^* = a$ so that $\gamma$ is $C(\gamma^*, b)$. Then there exist points $c$ and $d$ of $E(\gamma^*, b)$ such that $c$ precedes $d$ in $E(\gamma^*, b)$ and $\alpha$ is $C(\gamma^*, c)$ and $\beta$ is $C(\gamma^*, d)$. Each of $c$ and $d$ occurs twice on $I_\gamma$, and we let $c_1$ and $c_2$ and $d_1$ and $d_2$ denote those occurrences, in the order of $I_\gamma$. Then $I_\gamma$, $I_\alpha$ and $I_\beta$ can be written

$$I_\gamma = [P_\gamma, c_1] \leftrightarrow [c_1, d_1] \leftrightarrow [d_1, d_2] \leftrightarrow [d_2, c_2] \leftrightarrow [c_2, Q_\gamma]$$

$$I_\alpha = [P_\gamma, c_1] \leftrightarrow [c_2, Q_\gamma]$$

$$I_\beta = [P_\gamma, d_1] \leftrightarrow [d_2, Q_\gamma]$$

$$= [P_\gamma, c_1] \leftrightarrow [c_1, d_1] \leftrightarrow [d_2, c_2] \leftrightarrow [c_2, Q_\gamma].$$

If $x \in [P_\gamma, c_1]$ or $x \in [c_2, Q_\gamma]$,

$$\phi_{\alpha,\gamma}(x) = x = \phi_{\beta,\gamma}(x) = \phi_{\alpha,\beta}(\phi_{\beta,\gamma}(x)).$$

If $x \in [c_1, d_1]$ or $x \in [d_2, c_2]$,

$$\phi_{\alpha,\gamma}(x) = \{c_1, c_2\} = \phi_{\alpha,\beta}(x) = \phi_{\alpha,\beta}(\phi_{\beta,\gamma}(x)).$$

If $x \in [d_1, d_2]$,

$$\phi_{\alpha,\gamma}(x) = \{c_1, c_2\} = \phi_{\alpha,\beta}(\{d_1, d_2\}) = \phi_{\alpha,\beta}(\phi_{\beta,\gamma}(x)).$$

We conclude that $\phi_{\alpha,\gamma} = \phi_{\alpha,\beta} \circ \phi_{\beta,\gamma}$.

8.2. $\phi_{\alpha,\beta} \circ \phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$ *for* $\gamma$ *of height* $\leqslant 1$. Suppose $\gamma$ is a member of $D$ of height $\leqslant 1$. If $\gamma - \gamma^*$ is connected, $\gamma$ is its only edge and the result is established in §8.1. If $\gamma - \gamma^*$ is not connected, let $\gamma_1, \cdots, \gamma_p$ denote the closures of the components of $\gamma - \gamma^*$, ordered in the order relative to $W$. Then $\gamma_i$, $1 \leqslant i \leqslant p$, is an edge of $\gamma$, and we let $\alpha_i = \alpha \cap \gamma_i$ and $\beta_i = \beta \cap \gamma_i$. Now if $z$ is $\alpha$, $\beta$ or $\gamma$,

$$I_z = I_{z_1} \leftrightarrow I_{z_2} \leftrightarrow \cdots \leftrightarrow I_{z_p}.$$

For $1 \leqslant i \leqslant p$, $\phi_{\alpha_i,\gamma_i} = \phi_{\alpha_i,\beta_i} \circ \phi_{\beta_i,\gamma_i}$ and if $(y, z)$ is $(\phi, \beta)$, $(\beta, \gamma)$ or $(\alpha, \gamma)$, $\phi_{\gamma,z}|I_i = \phi_{y_i,z_i}$. From these equations, we conclude that $\phi_{\alpha,\gamma} = \phi_{\alpha,\beta} \circ \phi_{\beta,\gamma}$.

8.3. *The induction step for* $\phi_{\alpha,\gamma} = \phi_{\alpha,\beta} \circ \phi_{\beta,\gamma}$. Suppose $n$ is a positive integer and $\phi_{\alpha,\gamma} = \phi_{\alpha,\beta} \circ \phi_{\beta,\gamma}$ has been established for all $\alpha, \beta, \gamma \in D$ such that $\alpha \leqslant \beta \leqslant \gamma$, $\alpha^* = \beta^* = \gamma^*$ and the height of $\gamma$ is $\leqslant n$. Let $\gamma$ be a member of $D$ of height $\leqslant n + 1$.

Suppose first that $\gamma - \gamma^*$ is connected. Then there is a unique edge $e$ of $\gamma$ that contains $\gamma^*$.

(i) Assume $e$ is a type-1 edge of $\gamma$ and that $e \subset \alpha$. Then $e$ is $C(\gamma^*, b)$ where $b$ is the only boundary point in $\gamma$ of $\gamma - e$. Let $\gamma_1, \cdots, \gamma_p$ be the closures of the components of $\gamma - e$, ordered in the order relative to $W$, and let $\alpha_i = \alpha \cap \gamma_i$ and $\beta_i = \beta \cap \gamma_i$, $1 \leqslant i \leqslant p$. Then, for $1 \leqslant i \leqslant p$, $\alpha_i$, $\beta_i$ and $\gamma_i$ belong to $D$, $\alpha_i^* = \beta_i^* = \gamma_i^* = b$ and the height of $\gamma_i$ is $\leqslant n$; therefore $\phi_{\alpha_i,\gamma_i} = \phi_{\alpha_i,\beta_i} \circ \phi_{\beta_i,\gamma_i}$. If $z$ is $\alpha$, $\beta$ or $\gamma$,

$$I_z = J_e \leftrightarrow I_{z_1} \leftrightarrow \cdots \leftrightarrow I_{z_p} \leftrightarrow K_e.$$

If $x \in J_e$ or $x \in K_e$

$$\phi_{\alpha,\gamma}(x) = x = \phi_{\alpha,\beta}(x) = \phi_{\alpha,\beta}(\phi_{\beta,\gamma}(x)).$$

If $x \in I_{\gamma_i}$, $1 \leqslant i \leqslant p$,

$$\phi_{\alpha,\gamma}(x) = \phi_{\alpha_i,\gamma_i}(x) = \phi_{\alpha_i,\beta_i}(\phi_{\beta_i,\gamma_i}(x)) = \phi_{\alpha,\beta}(\phi_{\beta,\gamma}(x)).$$

We conclude that $\phi_{\alpha,\gamma} = \phi_{\alpha,\beta} \circ \phi_{\beta,\gamma}$.

Now suppose $\alpha \subset e$ and suppose first that $\beta \subset e$ also. Then from the last paragraph of §7.3(i), $\phi_{\alpha,\gamma} = \phi_{\alpha,e} \circ \phi_{e,\gamma}$ and $\phi_{\beta,\gamma} = \phi_{\beta,e} \circ \phi_{e,\gamma}$. Since $e$ is an edge, from §8.1 $\phi_{\alpha,e} = \phi_{\alpha,\beta} \circ \phi_{\beta,e}$. Consequently,

$$\phi_{\alpha,\gamma} = \phi_{\alpha,e} \circ \phi_{e,\gamma} = \phi_{\alpha,\beta} \circ \phi_{\beta,e} \circ \phi_{e,\gamma} = \phi_{\alpha,\beta} \circ \phi_{\beta,\gamma}.$$

Now suppose $\alpha \subset e$ and $e \subset \beta$ and that $e$ is also an edge of $\beta$. From §7.3(i), $\phi_{\alpha,\gamma} = \phi_{\alpha,e} \circ \phi_{e,\gamma}$ and $\phi_{\alpha,\beta} = \phi_{\alpha,e} \circ \phi_{e,\beta}$. From the first paragraph of this section (§8.3(i)), $\phi_{e,\gamma} = \phi_{e,\beta} \circ \phi_{\beta,\gamma}$. Consequently,

$$\phi_{\alpha,\gamma} = \phi_{\alpha,e} \circ \phi_{e,\gamma} = \phi_{\alpha,e} \circ \phi_{e,\beta} \circ \phi_{\beta,\gamma} = \phi_{\alpha,\beta} \circ \phi_{\beta,\gamma}.$$

Finally, suppose $\alpha \subset e$ and $e \subset \beta$ and that $e$ is not an edge of $\beta$. In this event there is a type-1 edge $f$ of $\beta$ which contains $e$. Now, from §7.3(i), $\phi_{\alpha,\gamma} = \phi_{\alpha,e} \circ \phi_{e,\gamma}$ and $\phi_{e,\beta} = \phi_{e,f} \circ \phi_{f,\beta}$, from the first paragraph of this section (§8.3(i)), $\phi_{e,\gamma} = \phi_{e,f} \circ \phi_{f,\gamma}$ and $\phi_{f,\gamma} = \phi_{f,\beta} \circ \phi_{\beta,\gamma}$, and from the second paragraph of this section (§8.3(i)), $\phi_{\alpha,\beta} = \phi_{\alpha,e} \circ \phi_{e,\beta}$. These five equations yield

$$\begin{aligned}\phi_{\alpha,\gamma} &= \phi_{\alpha,e} \circ \phi_{e,\gamma} = \phi_{\alpha,e} \circ \phi_{e,f} \circ \phi_{f,\gamma} \\ &= \phi_{\alpha,e} \circ \phi_{e,f} \circ \phi_{f,\beta} \circ \phi_{\beta,\gamma} \\ &= \phi_{\alpha,e} \circ \phi_{e,\beta} \circ \phi_{\beta,\gamma} = \phi_{\alpha,\beta} \circ \phi_{\beta,\gamma}.\end{aligned}$$

(ii) Suppose now $e$ is a type-0 edge of $\gamma$. Since we are considering $\alpha$ to be nondegenerate, $e \subset \alpha$. Let $x_1, \cdots, x_q$ denote the boundary points of $e$, ordered so that on $I_e$, $\langle x_1 \rangle < \langle x_2 \rangle < \cdots < \langle x_q \rangle$, and for $1 \leqslant i \leqslant q$, let $\gamma_{i,1}, \cdots, \gamma_{i,p(i)}$ denote the closures of the components of $\gamma - e$ which have $x_i$ as their boundary point, ordered in the order relative to $W$. For $1 \leqslant i \leqslant q$, $1 \leqslant j \leqslant p(i)$ let $\alpha_{i,j} = \alpha \cap \gamma_{i,j}$ and $\beta_{i,j} = \beta \cap \gamma_{i,j}$; then $\alpha_{i,j}$, $\beta_{i,j}$ and $\gamma_{i,j}$ belong to $D$, $\alpha^*_{i,j} = \beta^*_{i,j} = \gamma^*_{i,j} = x_i$, the height of $\gamma_{i,j}$ is $\leqslant n$, and it has been established that

$$\phi_{\alpha_{i,j},\gamma_{i,j}} = \phi_{\alpha_{i,j},\beta_{i,j}} \circ \phi_{\beta_{i,j},\gamma_{i,j}}.$$

As before, we let $I_0$ be the interval $[P_e, \langle x_1 \rangle]$ of $I_e$, $I_i$ be the interval $[\langle x_i \rangle, \langle x_{i+1} \rangle]$ of $I_e$, $1 \leqslant i \leqslant q-1$, and $I_q$ be the interval $[\langle x_q \rangle, Q_e]$ of $I_e$, and let $\pi_i = \pi_e | I_i$, $0 \leqslant i \leqslant q$. Then if $z$ is $\alpha$, $\beta$ or $\gamma$

$$I_z = I_0 \leftrightarrow I_{z_{1,1}} \leftrightarrow \cdots \leftrightarrow I_{z_{1,p(1)}} \leftrightarrow I_1 \leftrightarrow \cdots$$

$$\leftrightarrow I_{q-1} \leftrightarrow I_{z_{q,1}} \leftrightarrow \cdots \leftrightarrow I_{z_{q,p(q)}} \leftrightarrow I_q.$$

If $(y, z)$ is one of $(\alpha, \beta)$, $(\beta, \gamma)$ or $(\alpha, \gamma)$,

$$\begin{aligned} \phi_{y,z}(t) &= t, && \text{if } t \in I_i,\ 0 \leqslant i \leqslant q, \\ &= \phi_{y_{i,j},z_{i,j}}(t), && \text{if } t \in I_{i,j},\ 1 \leqslant i \leqslant q,\ 1 \leqslant j \leqslant p(i). \end{aligned}$$

The result $\phi_{\alpha,\gamma} = \phi_{\alpha,\beta} \circ \phi_{\beta,\gamma}$ follows from the analogous result on the subintervals of $I_\gamma$.

The argument for the case $\gamma - \gamma^*$ connected is complete. Now suppose $\gamma - \gamma^*$ is not connected, and let $\gamma_1, \cdots, \gamma_r$ denote the closures of the components of $\gamma - \gamma^*$, ordered in the order relative to $W$, and for $1 \leqslant k \leqslant \gamma$, let $\alpha_k = \alpha \cap \gamma_k$ and $\beta_k = \beta \cap \gamma_k$. Then for $1 \leqslant k \leqslant r$, $\alpha_k$, $\beta_k$ and $\gamma_k$ belong to $D$, $\alpha^*_k = \beta^*_k = \gamma^*_k = \gamma^*$, $\gamma_k$ is of height $\leqslant n+1$, $\gamma_k - \gamma^*_k$ is connected, and it has been established that $\phi_{\alpha_k,\gamma_k} = \phi_{\alpha_k,\beta_k} \circ \phi_{\beta_k,\gamma_k}$. If $z$ is $\alpha$, $\beta$ or $\gamma$,

$$I_z = I_{z_1} \leftrightarrow I_{z_2} \leftrightarrow \cdots \leftrightarrow I_{z_r}$$

and if $(y, z)$ is one of $(\alpha, \beta)$, $(\beta, \gamma)$ or $(\alpha, \gamma)$,

$$\phi_{y,z}(x) = \phi_{y_k,z_k}(x) \quad \text{if } x \in I_{z_k},\ 1 \leqslant k \leqslant r.$$

Then $\phi_{\alpha,\gamma} = \phi_{\alpha,\beta} \circ \phi_{\beta,\gamma}$ follows from the analogous results on the subintervals of $I_\gamma$.

This completes the induction step, and step III, the transitivity $\phi_{\alpha,\beta} \circ \phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$, and the argument for Theorem 1.

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