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THE p-CLASS IN A DUAL B^*-ALGEBRA

BY

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ABSTRACT. In this paper, we introduce and study the class A_p $(0 < p \leqslant \infty)$ in a dual B^*-algebra A. We show that, for $1 \leqslant p \leqslant \infty$, A_p is a dual A^*-algebra which is a dense two-sided ideal of A. If $1 < p < \infty$, we obtain that A_p is uniformly convex and hence reflexive. We also identify the conjugate space of A_p $(1 \leqslant p < \infty)$. This is a generalization of the class C_p of compact operators on a Hilbert space.

1. **Introduction.** Let H be a Hilbert space and $LC(H)$ the algebra of all compact operators on H. Then $LC(H)$ is a simple dual B^*-algebra and every simple dual B^*-algebra is of this form. The class C_p of compact operators in $LC(H)$ has many interesting properties and has been studied in various articles (e.g., see [2], [3] and [4]). The present work is an attempt to introduce a similar class of spaces in an arbitrary dual B^*-algebra.

Let A be a dual B^*-algebra. The class A_p $(0 < p \leqslant \infty)$ is introduced in §3. After establishing some crucial inequalities, we show that A_p $(1 \leqslant p \leqslant \infty)$ is a dual A^*-algebra which is a dense two-sided ideal of A. In §4, we study the algebras A_1 and A_2. We obtain that every proper H^*-algebra is of the form A_2 and $A_1 = \{xy\colon x, y \in A_2\}$. §5 is devoted to showing the uniform convexity in A_p $(1 < p < \infty)$. Finally we identify the conjugate space of A_p $(1 \leqslant p < \infty)$ in §6.

In this paper, our approach is elementary and the techniques are not new. In fact, they are borrowed from [3], [4], [10] and [11]. The author is grateful for these invaluable references.

2. **Notation and preliminaries.** Definitions not explicitly given are taken from Rickart's book [7].

For any set E in a Banach algebra A, let $l(E)$ and $r(E)$ denote the left and right annihilators of E, respectively. Then A is called a dual algebra

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if for every closed right ideal R and every closed left ideal I, we have $r(l(R)) = R$ and $l(r(I)) = I$. See [5] and [7] for some of its properties.

An idempotent e in a Banach algebra A is said to be minimal if eAe is a division algebra. In case A is semisimple, this is equivalent to saying that Ae (eA) is a minimal left (right) ideal of A.

Let A be a Banach algebra. A bounded linear operator S on A is called a right centralizer if $S(xy) = (Sx)y$ for all x, y in A. For each a in A, the operator $L_a: x \to ax$ $(x \in A)$ is a right centralizer on A.

Let H be a Hilbert space with an inner product $(\ ,\ )$. If x and y are elements in H, then $x \otimes y$ will denote the operator on H defined by $(x \otimes y)(h) = (h, y)x$ for all h in H.

In this paper, all algebras and linear spaces under consideration are over the field of complex numbers.

NOTATION. In this paper, A will denote a dual B^*-algebra with norm $\|\cdot\|$.

We shall ofter use, without explicitly mentioning, the following fact: For any orthogonal family $\{e_\alpha\}$ of hermitian idempotents of A, $\Sigma_\alpha e_\alpha x$ is summable in A, and especially when $\{e_\alpha\}$ is a maximal family, $x = \Sigma_\alpha e_\alpha x$ for all x in A (see [5, p. 30, Theorem 16]).

Let B be a closed commutative *-subalgebra of A and e a minimal idempotent in B. It follows easily from [7, p. 261, Lemma (4.10.1)] that e is hermitian. Also if f is any other minimal idempotent in B, then $fe = ef = 0$. If B is maximal, then e is a minimal idempotent in A.

LEMMA 2.1. *Let e be a hermitian minimal idempotent in A, $a \in A$, and $\{f_\beta\}$ a maximal orthogonal family of hermitian minimal idempotents in A. Then $\|ae\|^2 = \Sigma_\beta \|f_\beta ae\|^2$.*

PROOF. Since A is a dual B^*-algebra, it follows from [7, p. 259, Theorem (4.9.24)] and [7, p. 269, Corollary (4.10.20)] that $A = (\Sigma_\lambda LC(H_\lambda))_0$, where $LC(H_\lambda)$ is the algebra of all compact operators on a Hilbert space H_λ. It is easy to see that $e \in LC(H_{\lambda_0})$ for some λ_0. Let $\{f_\gamma\} = \{f_\beta\} \cap LC(H_{\lambda_0})$. Then we can write $f_\gamma = x_\gamma \otimes x_\gamma$ with $x_\gamma \in H_{\lambda_0}$ and $\|x_\gamma\| = 1$. Similarly $e = y \otimes y$ with $y \in H_{\lambda_0}$ and $\|y\| = 1$. Since $\{f_\gamma\}$ is a maximal orthogonal family of hermitian minimal idempotents in $LC(H_{\lambda_0})$, it follows easily that $\{x_\gamma\}$ is a complete orthonormal set in H_{λ_0}. Put $b = ae$. Then $b \in LC(H_{\lambda_0})$ and $be = ae$. Hence

$$\|ae\|^2 = \|eb^*be\| = \|(y \otimes y)b^*b(y \otimes y)\| = \|by\|^2.$$

Similarly $\|f_\gamma be\| = |(by, x_\gamma)|$. Since $f_\beta ae = 0$ if $f_\beta \notin \{f_\gamma\}$, by Parseval's identity we have

$$\sum_\beta \|f_\beta ae\|^2 = \sum_\gamma \|f_\gamma be\|^2 = \sum_\gamma |(by, x_\gamma)|^2 = \|by\|^2 = \|ae\|^2.$$

This completes the proof.

The following lemma is useful in this paper and it is similar to [10, p. 29, Lemma 1].

LEMMA 2.2. *Let $a \in A$ and $\{e_\alpha\}$, $\{f_\beta\}$ any two maximal orthogonal families of hermitian minimal idempotents in A. Then*

$$\sum_\alpha \|ae_\alpha\|^2 = \sum_\beta \|af_\beta\|^2 = \sum_\beta \|a^*f_\beta\|^2.$$

PROOF. We note first that $\|f_\beta ae_\alpha\| = \|e_\alpha a^* f_\beta\|$. If $\Sigma_\alpha \|ae_\alpha\|^2$ is summable, then by Lemma 2.1, we have

$$\sum_\alpha \|ae_\alpha\|^2 = \sum_\alpha \sum_\beta \|f_\beta ae_\alpha\|^2 = \sum_\beta \sum_\alpha \|e_\alpha a^* f_\beta\|^2 = \sum_\beta \|a^* f_\beta\|^2. \tag{2.1}$$

Hence, in particular, $\Sigma_\beta \|af_\beta\|^2 = \Sigma_\beta \|a^*f_\beta\|^2$. The lemma now follows from (2.1).

Suppose b is a normal element in A. Let B (resp. B') be a maximal commutative *-subalgebra of A containing b and $\{e_\alpha\}$ (resp. $\{e_\omega\}$) the maximal orthogonal family of hermitian minimal idempotents in B (resp. B'). Then $be_\alpha = e_\alpha be_\alpha = k_\alpha e_\alpha$ for some constant k_α. Similarly $be_\omega = k_\omega e_\omega$ for some constant k_ω. Let K (resp. K') be the set of all nonzero k_α (resp. k_ω). We note that k_{α_1} may be equal to k_{α_2} for some $\alpha_1 \neq \alpha_2$. However we consider them as different elements in K.

LEMMA 2.3. *The set K is either finite or countable and $K = K'$. The set of all distinct constants in K is precisely the set of all nonzero constants in the spectrum of b.*

PROOF. Let B_0 be the intersection of all maximal commutative *-subalgebras of A containing b. Let $\{f_\beta\}$ be the maximal orthogonal family of hermitian minimal idempotents in B_0. Since B_0 is a dual B^*-algebra, $b = \Sigma_\beta bf_\beta = \Sigma_\beta \lambda_\beta f_\beta$, where λ_β are constants. Therefore there exists only a countable number of f_β for which $bf_\beta \neq 0$. Also, for each nonzero λ_{β_0}, the set $\{\lambda_\beta\colon \lambda_\beta = \lambda_{\beta_0}\}$ if finite. It is now easy for us to write $b = \Sigma_{n=1}^\infty \lambda_n f_n$, where λ_n are distinct nonzero constants and $\{f_n\}$ is an orthogonal family of hermitian idempotents in B_0 such that $\lambda_n f_n = bf_n$. Note that f_n is not necessarily minimal. Since B is dual and $f_n \in B$, it is well known that

$$f_n = e_{\alpha_{n_1}} + \cdots + e_{\alpha_{n_p}}, \text{ where } e_{\alpha_{n_i}} \in \{e_\alpha\} \ (i = 1, 2, \cdots, p).$$

Considering the right ideal f_nA of A, by [1, p. 497, Theorem 2.2], the number n_p is independent of the choice of B. Since $bf_n = \lambda_n f_n$, we see easily that

$$be_{\alpha_{n_i}} = \lambda_n e_{\alpha_{n_i}} \qquad (i = 1, 2, \cdots, p). \tag{2.2}$$

If $f_m = e_{\alpha_{m_1}} + \cdots + e_{\alpha_{m_q}}$ $(m \neq n)$, then it follows from (2.2) that

$$\{e_{\alpha_{m_1}}, \cdots, e_{\alpha_{m_q}}\} \cap \{e_{\alpha_{n_1}}, \cdots, e_{\alpha_{n_p}}\} = \emptyset,$$

because $\lambda_n \neq \lambda_m$. Also

$$b = \sum_n \lambda_n f_n = \sum \lambda_n (e_{\alpha_{n_1}} + \cdots + e_{\alpha_{n_p}}).$$

Let E be the set of all such $e_{\alpha_{n_p}}$. Then E is countable. For simplicity, we write $E = \{e_1, e_2, \cdots\}$ and $b = \Sigma_{n=1}^{\infty} k_n e_n$, where $k_n e_n = be_n$ and $k_n \neq 0$ (because $\lambda_n \neq 0$). Let $\{e_\gamma\} = \{e_\alpha\} - E$. We show that $be_\gamma = 0$ for all γ. In fact, since $b = \Sigma_\alpha be_\alpha = \Sigma_n be_n$, it follows that $\Sigma_\gamma be_\gamma = 0$. Let F_α be the multiplicative linear functional on B corresponding to the maximal modular ideal $B(1 - e_\alpha)$ of B. For any fixed γ_0, we have $be_{\gamma_0} = k_{\gamma_0} e_{\gamma_0}$, for some constant k_{γ_0}. Then

$$k_{\gamma_0} = F_{\gamma_0}(be_{\gamma_0}) = \sum_\gamma F_{\gamma_0}(be_\gamma) = F_{\gamma_0}\left(\sum_\gamma be_\gamma\right) = 0.$$

Hence it follows that $be_\gamma = 0$ for all γ. Consequently $K = \{k_n\}$. Similarly we can show that $K' = \{k_n\}$. Therefore $K = K'$. Now the last part of the lemma follows easily from [7, p. 111, Theorem (3.1.6)]. This completes the proof.

Let b, $\{e_\alpha\}$ and $\{e_n\}$ be as in the proof of Lemma 2.3. Then $b = \Sigma_\alpha k_\alpha e_\alpha = \Sigma_n k_n e_n$ is called a spectral representation of b. By Lemma 2.3, $\{k_n\}$ is independent of $\{e_n\}$. Also if $k_\alpha \neq k_n$ for all n, then $k_\alpha = 0$.

Suppose a is a nonzero element in A. Let $a^*a = \Sigma_n r_n e_n$ be a spectral representation of a^*a. We claim that

$$a = \sum_n ae_n. \tag{2.3}$$

In fact, since $\Sigma_n ae_n$ is summable and $a^*a = \Sigma_n a^*ae_n = \Sigma_n e_n a^*ae_n = \Sigma_n r_n e_n$, it follows that $(a - \Sigma_n ae_n)^*(a - \Sigma_n ae_n) = 0$. Hence $a = \Sigma_n ae_n$. We note that $ae_n \neq 0$; for otherwise $r_n e_n = a^*ae_n = 0$.

Since a^*a is a positive element, $r_n > 0$ for all n. Put $k_n = \sqrt{r_n} > 0$. We show that $\Sigma_n k_n e_n$ is summable in A. In fact, for any two positive integers m, n $(m < n)$, $\|\Sigma_{i=m}^n k_i e_i\|^2 = \|\Sigma_{i=m}^n r_i e_i\|$. Since $\Sigma_n r_n e_n$ is summable, so is $\Sigma_n k_n e_n$. Put

$$[a] = \sum_n k_n e_n. \tag{2.4}$$

Then $[a]^* = [a]$ and $[a]^2 = a^*a$. Hence $[a] = (a^*a)^{1/2}$. For each x in A,

$$\begin{aligned} \left\| \sum_{i=m}^n k_i^{-1} a e_i x \right\|^2 &= \left\| \left(\sum_{i=m}^n k_i^{-1} a e_i x \right)^* \left(\sum_{i=m}^n k_i^{-1} a e_i x \right) \right\| \\ &= \left\| \sum_{i=m}^n x^* e_i x \right\| = \left\| \sum_{i=m}^n e_i x \right\|^2 \leqslant \|x\|^2. \end{aligned} \tag{2.5}$$

Since $\Sigma_n e_n x$ is summable in A, so is $\Sigma_n k_n^{-1} a e_n x$. Define a mapping W on A by

$$Wx = \sum_n k_n^{-1} a e_n x \qquad (x \in A). \tag{2.6}$$

Then it follows from (2.3), (2.4) and (2.5) that $W[a] = a$ and $\|W\| = 1$.

We note that $a e_n a^* \neq 0$; for otherwise $r_n^2 e_n = a^* a e_n a^* a = 0$. Put $f_n = k_n^{-2} a e_n a^*$. Since $(0) \neq f_n A \subset a e_n A$ and $a e_n \neq 0$, it follows from [7, p. 45, Lemma (2.1.8)] that $f_n A = a e_n A$ is a minimal right ideal of A. Hence we see that $\{f_n\}$ is an orthogonal family of hermitian minimal idempotents in A. By (2.3), $aa^* = \Sigma_n a e_n a^* = \Sigma_n k_n^2 f_n$ and so it is a spectral representation of aa^* by the proof of Lemma 2.3. For each x in A, by a similar argument in (2.5), we have

$$\left\| \sum_{i=m}^n k_n^{-1} e_n a^* x \right\|^2 = \left\| \sum_{i=m}^n f_i x \right\|^2 \leqslant \|x\|^2. \tag{2.7}$$

Since $\Sigma_n f_n x$ is summable, so is $\Sigma_n k_n^{-1} e_n a^* x$. Therefore we can define a mapping W^* on A by

$$W^* x = \sum_n k_n^{-1} e_n a^* x \qquad (x \in A). \tag{2.8}$$

It follows easily from (2.4) and (2.7) that $W^* a = [a]$ and $\|W^*\| = 1$. Also both W and W^* are right centralizers on A. We shall refer to the operator W as the partial isometry associated with a.

We remark that similar concepts were introduced in [9].

3. The p-class in A. As before, A will be a dual B^*-algebra with norm $\|\cdot\|$. Suppose a is a nonzero element in A. Let $a^*a = \Sigma_n r_e e_n$ be a spectral representation of a^*a and $k_n = \sqrt{r_n}$. Since a^*a is a positive element in A, $r_n > 0$ and so $k_n > 0$. We define

$$|a|_p = \left(\sum_n k_n^p\right)^{1/p} \quad (0 < p < \infty), \tag{3.1}$$

$$|a|_\infty = \max \{k_n : n = 1, 2, \cdots\}.$$

For $a = 0$, we define $|a|_p = 0$ $(0 < p \leqslant \infty)$.

REMARK. By Lemma 2.3, $|a|_p$ is well defined.

DEFINITION. For $0 < p \leqslant \infty$, let $A_p = \{a \in A: |a|_p < \infty\}$.

REMARK. For $0 < p \leqslant \infty$, $|a|_p \geqslant 0$ and $|a|_p = 0$ if and only if $a = 0$. Also $|ka|_p = |k|\,|a|_p$ for any constant k.

We now have some elementary properties of $|a|_p$.

LEMMA 3.1. *Let a be an element in A and $0 < p \leqslant \infty$. Then*

(i) $\|a\| = |a|_\infty \leqslant |a|_p$. *Thus* $A_\infty = A$.

(ii) $|a|_p = |[a]|_p$. *Hence* $a \in A_p$ *if and only if* $[a] \in A_p$.

(iii) *If* $p \leqslant q$, *then* $|a|_p \geqslant |a|_q$ *and so* $A_p \subset A_q$.

(iv) *If e is a hermitian minimal idempotent in A, then $|e|_p = 1$ and so $e \in A_p$.*

(v) $|a|_p = |a^*|_p$. *Hence* $a \in A_p$ *if and only if* $a^* \in A_p$.

PROOF. Let $a^*a = \Sigma_n r_n e_n$ be a spectral representation of a^*a and $[a] = \Sigma_n k_n e_n$ with $k_n = \sqrt{r_n}$.

(i) This follows from $\|a\|^2 = \|a^*a\|$ and [7, p. 112, Corollary (3.1.7)].

(ii) This follows from $[a] = [[a]] = \Sigma_n k_n e_n$.

(iii) and (iv). This is clear.

(v) We can assume that $a \neq 0$. Put $f_n = k_n^{-2} a e_n a^*$. Then $aa^* = \Sigma_n k_n^2 f_n$ is a spectral representation of aa^* (see §2). Therefore it follows that $|a^*|_p = |a|_p$. This completes the proof of the lemma.

Let a be a positive element in A and B_0 the intersection of all maximal commutative *-subalgebras of A containing a. If $\{f_\beta\}$ is the maximal orthogonal family of hermitian minimal idempotents in B_0, then $a = \Sigma_\beta af_\beta = \Sigma_\beta \lambda_\beta f_\beta$, where λ_β are nonnegative constants.

DEFINITION. For $0 < p < \infty$, we define $a^p = \Sigma_\beta \lambda_\beta^p f_\beta$.

REMARK. Let $a = \Sigma_\alpha k_\alpha e_\alpha = \Sigma_n k_n e_n$ be a spectral representation of a. If a^p exists, then by the proof of Lemma 2.3 $a^p = \Sigma_\alpha k_\alpha^p e_\alpha = \Sigma_n k_n^p e_n$ is a spectral representation of a^p.

LEMMA 3.2. *Let a be a positive element in A and $0 < p$, $q < \infty$. If a^q exists, then $|a^q|_{p/q} = |a|_p^q$.*

PROOF. This is clear.

LEMMA 3.3. *Let* $a \in A$ *and* $0 < p < \infty$. *Then the following statements are equivalent*:

(i) $a \in A_p$.

(ii) $[a]^p \in A_1$.

(iii) $[a]^{p/2} \in A_2$.

If any of these conditions holds, then $|a|_p^p = \Sigma_\beta \|f_\beta [a]^p f_\beta\|$, *where* $\{f_\beta\}$ *is a maximal orthogonal family of hermitian minimal idempotents in* A.

PROOF. Let $[a] = \Sigma_\alpha k_\alpha e_\alpha = \Sigma_n k_n e_n$ be a spectral representation of $[a]$.

(i) $\Longleftrightarrow$ (ii) This follows from the equality $|a|_p^p = \Sigma_n k_n^p = |[a]^p|_1$.

(ii) $\Longleftrightarrow$ (iii) This follows from the equality $|[a]^p|_1 = \Sigma_n k_n^p = |[a]^{p/2}|_2^2$.

If any of these conditions holds, then by Lemma 2.2, we have

$$|a|_p^p = \sum_\alpha k_\alpha^p = \sum_\alpha \|[a]^{p/2} e_\alpha\|^2$$
$$= \sum_\beta \|[a]^{p/2} f_\beta\|^2 = \sum_\beta \|f_\beta [a]^p f_\beta\|.$$

This completes the proof.

LEMMA 3.4. *Let* a *be a positive element in* A *and* f *a hermitian minimal idempotent in* A. *Then*

(i) $\|fa^pf\| \geqslant \|faf\|^p$ $(1 \leqslant p < \infty)$.

(ii) $\|fa^pf\| \leqslant \|faf\|^p$ $(0 < p \leqslant 1)$.

PROOF. Let $a = \Sigma_\alpha k_\alpha e_\alpha$ be a spectral representation of a.

(i) Clearly we can assume that $1 < p < \infty$. Then by Hölder's inequality and Lemma 2.1, we have

$$\|faf\| = \|a^{1/2} f\|^2 = \sum_\alpha \|e_\alpha a^{1/2} f\|^2 = \sum_\alpha k_\alpha \|e_\alpha f\|^2$$
$$\leqslant \Big(\sum_\alpha k_\alpha^p \|e_\alpha f\|^2\Big)^{1/p} \Big(\sum_\alpha \|e_\alpha f\|^2\Big)^{(p-1)/p}$$
$$= \Big(\sum_\alpha \|e_\alpha a^{p/2} f\|^2\Big)^{1/p} (\|f\|^2)^{(p-1)/p} = \|fa^pf\|^{1/p}.$$

(ii) Replacing a by a^p and p by $1/p$ in (i), we get (ii).

LEMMA 3.5. *Let* $a \in A_p$ *and* $\{f_\beta\}$ *be a maximal orthogonal family of hermitian minimal idempotents in* A. *Then*

(i) $|a|_p^p \leqslant \Sigma_\beta \|af_\beta\|^p$ $(1 \leqslant p \leqslant 2)$.

(ii) $|a|_p^p \geqslant \Sigma_\beta \|af_\beta\|^p$ $(2 \leqslant p < \infty)$.

If $[a] = \Sigma_n k_n e_n$ *is a spectral representation of* $[a]$, *then* $|a|_p^p = \Sigma_n \|ae_n\|^p$ $(0 < p < \infty)$.

PROOF. (i) If $1 \leqslant p \leqslant 2$, then by Lemma 3.4(ii), we have $\|f_\beta[a]^p f_\beta\| \leqslant \|f_\beta[a]^2 f_\beta\|^{p/2} = \|af_\beta\|^p$. Therefore (i) follows now from Lemma 3.3.

(ii) This can be proved similarly.

If $[a] = \Sigma_n k_n e_n$, then $\|ae_n\| = \|e_n a^* a e_n\|^{1/2} = k_n$. Therefore $|a|_p^p = \Sigma_n \|ae_n\|^p$ $(0 < p < \infty)$. This completes the proof.

LEMMA 3.6. *Suppose $a, b \in A$ and $1 \leqslant p \leqslant \infty$, then the following statements hold*:

(i) *If $a \in A_p$ and S is a right centralizer on A, then $Sa \in A_p$ and $|Sa|_p \leqslant \|S\|\ |a|_p$.*

(ii) *If $a \in A_p$ and $b \in A$, then $|ab|_p \leqslant \|b\|\ |a|_p$ and $|ba|_p \leqslant \|b\|\ |a|_p$. Hence ab and ba are in A_p.*

(iii) *If a, b are in A_p, then $|ab|_p \leqslant |a|_p\ |b|_p$.*

PROOF. Clearly we can assume that $1 \leqslant p < \infty$.

(i) Suppose $1 \leqslant p \leqslant 2$. Let $[a] = \Sigma_\alpha k_\alpha e_\alpha$ be a spectral representation of $[a]$. Then by Lemma 3.5, we have

$$|Sa|_p^p \leqslant \sum_\alpha \|(Sa)e_\alpha\|^p \leqslant \|S\|^p \sum_\alpha \|ae_\alpha\|^p = \|S\|^p |a|_p^p.$$

If $2 \leqslant p < \infty$, let $[Sa] = \Sigma_\alpha k_\alpha e_\alpha$ be a spectral representation of $[SA]$. Then by a similar argument, we have $|Sa|_p \leqslant \|S\|\ |a|_p$. This proves (i).

(ii) This follows easily from (i) and Lemma 3.1(v).

(iii) This follows from (ii) and Lemma 3.1(i).

LEMMA 3.7. *Let $a \in A_p$ and $\{f_\beta\}$ a maximal orthogonal family of hermitian minimal idempotents in A. Then*

$$\sum_\beta \|f_\beta a f_\beta\|^p \leqslant |a|_p^p \qquad (1 \leqslant p < \infty). \tag{3.2}$$

PROOF. Let W be the partial isometry associated with a and $b = W[a]^{1/2}$. Then $a = W[a] = b[a]^{1/2}$. It follows from Cauchy's inequality that

$$\sum_\beta \|f_\beta a f_\beta\|^p \leqslant \Big(\sum_\beta \|f_\beta b\|^{2p}\Big)^{1/2} \Big(\sum_\beta \|[a]^{1/2} f_\beta\|^{2p}\Big)^{1/2}. \tag{3.3}$$

By Lemma 3.3 and Lemma 3.4, we have

$$\sum_\beta \|[a]^{1/2} f_\beta\|^{2p} = \sum_\beta \|f_\beta[a]f_\beta\|^p \leqslant \sum_\beta \|f_\beta[a]^p f_\beta\| = |a|_p^p. \tag{3.4}$$

By Lemma 3.2, Lemma 3.3 and Lemma 3.4, we have

$$\begin{aligned} \sum_\beta \|f_\beta b\|^{2p} &\leqslant \sum_\beta \|f_\beta (bb^*)^p f_\beta\| = |b^*|_{2p}^{2p} = |b|_{2p}^{2p} \\ &\leqslant |[a]^{1/2}|_{2p}^{2p} = |a|_p^p. \end{aligned} \tag{3.5}$$

Substituting (3.4) and (3.5) into (3.3), we get (3.2). This completes the proof.

In order that $|\cdot|_p$ be a norm on A_p $(1 \leqslant p \leqslant \infty)$, it is sufficient now to show the triangle inequality.

LEMMA 3.8. *Let* $a, b \in A_p$, *then* $|a+b|_p \leqslant |a|_p + |b|_p$ $(1 \leqslant p \leqslant \infty)$. *Hence* $a + b \in A_p$.

PROOF. We can assume $1 \leqslant p < \infty$. Write $[a+b] = \Sigma_\alpha k_\alpha e_\alpha$ and $[a+b] = W^*(a+b)$ (see (2.8)). Then by Lemma 3.5, Lemma 3.7 and Minkowski's inequality, we have

$$\begin{aligned} |a+b|_p &= \Big(\sum_\alpha \|e_\alpha[a+b]\,e_\alpha\|^p\Big)^{1/p} \\ &\leqslant \Big(\sum_\alpha \|e_\alpha W^* a e_\alpha\|^p\Big)^{1/p} + \Big(\sum_\alpha \|e_\alpha W^* b e_\alpha\|^p\Big)^{1/p} \\ &\leqslant |W^*a|_p + |W^*b|_p \leqslant |a|_p + |b|_p. \end{aligned}$$

This completes the proof.

Now we have the main result of this section.

THEOREM 3.9. *For* $1 \leqslant p \leqslant \infty$, A_p *is a dual* A^**-algebra which is a dense two-sided ideal of* A.

PROOF. By a similar argument in the proof of [4, p. 265, Corollary 3.2], we can show that A_p is complete. (We use maximal orthogonal families of hermitian minimal idempotents instead of orthonormal bases.) Hence A_p is an A^*-algebra which is a two-sided ideal of A. It follows from Lemma 3.1(iv) that A_p contains the socle S of A. Since S is dense in A, so is A_p. We claim that, for each a in A_p, a belongs to the closure of aA_p in A_p. In fact, let $[a] = \Sigma_{i=1}^\infty k_i e_i$ be a spectral representation of $[a]$ and W the partial isometry associated with a. Put $f_n = \Sigma_{i=1}^n e_i$ $(n = 1, 2, \cdots)$. Then

$$|a - af_n|_p \leqslant |[a] - [a]f_n|_p = \Big|\sum_{i=n+1}^\infty k_i e_i\Big|_p = \Big(\sum_{i=n+1}^\infty k_i^p\Big)^{1/p}.$$

Since $a \in A_p$, it follows that $|a - af_n|_p \to 0$ as $n \to \infty$. Hence by [5, p. 29, Lemma 8 (3)], A_p is a dual algebra. This completes the proof.

We shall need the following result.

COROLLARY 3.10. *Let* $\{e_\gamma\}$ *be any orthogonal family of hermitian minimal idempotents of* A *and* $x \in A_p$ $(1 \leqslant p \leqslant \infty)$, *then* $\Sigma_\gamma e_\gamma x$ *is summable in* $|\cdot|_p$ *and especially when* $\{e_\gamma\}$ *is a maximal family* $x = \Sigma_\gamma e_\gamma x$ *in* A_p.

PROOF. This follows from Theorem 3.9 and Theorem 5.2 in [12].

Finally we remark that many statements and proofs in this section are similar to those given in [4] and [11].

4. **The algebras** A_1 **and** A_2. We have a characterization of a proper H^*-algebra.

THEOREM 4.1. *The algebra* A_2 *is a proper* H^**-algebra. Conversely, every proper* H^**-algebra is of the form* A_2 *for some dual* B^**-algebra* A.

PROOF. Let $a, b \in A_2$ and $\{f_\beta\}$ a maximal orthogonal family of hermitian minimal idempotents in A. Then $f_\beta b^* a f_\beta = \lambda_\beta f_\beta$ for some constant λ_β. We claim that $\Sigma_\beta \lambda_\beta$ is absolutely summable and independent of the choice of $\{f_\beta\}$. In fact, let $x, y \in Af_\beta$. Then $y^*x = \langle x, y\rangle_\beta f_\beta$ for some constant $\langle x, y\rangle_\beta$. It follows from [7, p. 261, Theorem (4.10.3)] and [7, p. 263, Theorem (4.10.6)] that $\langle x, y\rangle_\beta$ defines a complete inner product on Af_β such that $\langle x, x\rangle_\beta = \|x\|^2$. Now by Lemma 2.2 and the proof of [10, p. 30, Lemma 4], we can show that $\Sigma_\beta \lambda_\beta$ is absolutely summable and independent of $\{f_\beta\}$. Define

$$(a, b) = \sum_\beta \lambda_\beta \qquad (a, b \in A_2). \tag{4.1}$$

Then by the proof of [10, p. 31, Lemma 5], (,) is an inner product on A_2 such that $(xa, b) = (a, x^*b)$ and $(ax, b) = (a, bx^*)$ for all x in A. Also $|a|_2^2 = (a, a)$. Therefore A_2 is a proper H^*-algebra.

Conversely, let B be a proper H^*-algebra. Then B is a dense two-sided ideal of some dual B^*-algebra A. We can show that $B = A_2$ and this completes the proof.

LEMMA 4.2. *Let* $1/p + 1/q = 1$, *where* $1 \leqslant p, q \leqslant \infty$. *If* $a \in A_p$ *and* $b \in A_q$, *then* $ab \in A_1$ *and* $|ab|_1 \leqslant |a|_p |b|_q$.

PROOF. Suppose first that $2 \leqslant p < \infty$, $1 < q \leqslant 2$. Let $[b] = \Sigma_\alpha k_\alpha e_\alpha$ be a spectral representation of $[b]$. Also write $[ab] = W^*ab$. Then by Lemma 3.3, Lemma 3.5 and Hölder's inequality, we have

$$\begin{aligned} |ab|_1 &= \sum_\alpha \|e_\alpha [ab]\, e_\alpha\| = \sum_\alpha \|e_\alpha W^* ab e_\alpha\| \\ &\leqslant |W^*a|_p |b|_q \leqslant |a|_p |b|_q. \end{aligned} \tag{4.2}$$

By a similar argument, we can show that (4.2) holds for $1 < p \leqslant 2$, $2 \leqslant q < \infty$.

We now identify A_1.

THEOREM 4.3. $A_1 = \{xy\colon x, y \in A_2\}$.

PROOF. If $a \in A_1$, then by Lemma 3.3, $[a]^{1/2} \in A_2$. Let W be the partial isometry associated with a. Then $a = W[a] = (W[a]^{1/2})([a]^{1/2}) \in \{xy : x, y \in A_2\}$. The converse follows from Lemma 4.2 and this completes the proof.

Let $a \in A_1$. Then by Theorem 4.3, $a = c^*b$ for some b, c in A_2. Define

$$\operatorname{tr} a = (b, c) \qquad (a \in A_1), \tag{4.3}$$

where (b, c) is given by (4.1).

LEMMA 4.4. *Let $a \in A_1$, $\{f_\beta\}$ a maximal orthogonal family of hermitian minimal idempotents in A and $\lambda_\beta f_\beta = f_\beta a f_\beta$. Then $\operatorname{tr} a$ is well defined, $\operatorname{tr} a = \Sigma_\beta \lambda_\beta = \Sigma_\beta (af_\beta, f_\beta)$ and $|\operatorname{tr} a| \leqslant |a|_1$.*

PROOF. By the proof of Theorem 4.1, $\Sigma_\beta \lambda_\beta$ is absolutely summable and independent of $\{f_\beta\}$. It is clear that $\operatorname{tr} a = \Sigma_\beta \lambda_\beta = \Sigma_\beta (af_\beta, f_\beta)$. Therefore $\operatorname{tr} a$ is well defined. By Lemma 3.7, $|\operatorname{tr} a| \leqslant \Sigma_\beta \|f_\beta a f_\beta\| \leqslant |a|_1$.

5. **The uniform convexity of A_p $(1 < p < \infty)$.** For each a in A, we define a linear operator L_a on A_2 by

$$L_a(x) = ax \qquad (x \in A_2). \tag{5.1}$$

Since $|ax|_2 \leqslant \|a\| \, |x|_2$, it follows that L_a is bounded on A_2. Let $(\,,\,)$ be the given inner product on A_2.

LEMMA 5.1. *Let a be a positive element in A. Then L_a is positive and $L_{a^r} = (L_a)^r$ $(0 < r < \infty)$.*

PROOF. This is clear.

We now establish [4, p. 260, Lemma 2.6] for A_p.

LEMMA 5.2. *Let a and b be two positive elements in A and $0 < r < \infty$. If $(a + b)^r, a^r$ and b^r are in A_1, then*

(i) $\operatorname{tr} (a + b)^r \leqslant \operatorname{tr} a^r + \operatorname{tr} b^r$ $(0 < r \leqslant 1)$.

(ii) $\operatorname{tr} (a + b)^r \geqslant \operatorname{tr} a^r + \operatorname{tr} b^r$ $(1 \leqslant r < \infty)$.

PROOF. We assume first that $0 < r \leqslant 1$. Let $S = L_a$, $T = L_b$ and $U = L_{a+b}$. Then by the proof of [4, p. 260, Lemma 2.6], there exist operators C and D on A_2 such that

$$\|C\| \leqslant 1, \qquad \|D\| \leqslant 1, \qquad CU^{1/2} = S^{1/2}, \qquad DU^{1/2} = T^{1/2},$$
$$U^r = U^{r/2}C^*CU^{r/2} + U^{r/2}D^*DU^{r/2}.$$

Let $\{f_\beta\}$ be a maximal orthogonal family of hermitian minimal idempotents in A. Then by Lemma 5.1, we have

$$
(5.2)\qquad \begin{aligned} \operatorname{tr}(a+b)^r &= \sum_\beta ((a+b)^r f_\beta, f_\beta) = \sum_\beta (U^r f_\beta, f_\beta) \\ &= \sum_\beta (CU^{r/2} f_\beta, CU^{r/2} f_\beta) + \sum_\beta (DU^{r/2} f_\beta, DU^{r/2} f_\beta). \end{aligned}
$$

Since $C(a+b)^{r/2} \in A_2$ and $CU^{r/2} f_\beta = C(a+b)^{r/2} f_\beta$, it follows from (5.2) that

$$
(5.3)\qquad \begin{aligned} \operatorname{tr}(a+b)^r &= |C(a+b)^{r/2}|_2^2 + |D(a+b)^{r/2}|_2^2 \\ &= |(C(a+b)^{r/2})^*|_2^2 + |(D(a+b)^{r/2})^*|_2^2. \end{aligned}
$$

Let $a = \Sigma_\alpha k_\alpha e_\alpha$ be a spectral representation of a. Since $(C(a+b)^{r/2})^* e_\alpha = (CU^{r/2})^* e_\alpha = (a+b)^{r/2} C^* e_\alpha$, it follows from [4, p. 252, Lemma 2,1] that

$$
\begin{aligned} ((C(a+b)^{r/2})^* e_\alpha, (C(a+b)^{r/2})^* e_\alpha) &= ((a+b)^r C^* e_\alpha, C^* e_\alpha) \\ &\leqslant ((a+b) C^* e_\alpha, C^* e_\alpha)^r = (a e_\alpha, e_\alpha)^r = k_\alpha^r = (a^r e_\alpha, e_\alpha). \end{aligned}
$$

Therefore $|(C(a+b)^{r/2})^*|_2^2 \leqslant \operatorname{tr} a^r$. Similarly $|(D(a+b)^{r/2})^*|_2^2 \leqslant \operatorname{tr} b^r$. Hence by (5.3), we have $\operatorname{tr}(a+b)^r \leqslant \operatorname{tr} a^r + \operatorname{tr} b^r$. The case $1 \leqslant r < \infty$ can be proved in a similar way and the proof is complete.

By using maximal orthogonal families of hermitian minimal idempotents and a similar argument in the proof of [4, p. 259, Lemma 2.5], we have:

LEMMA 5.3. *Let a be a positive element in A and b a hermitian element in A such that $a+b$ and $a-b$ are positive. Suppose $(a+b)^r$, $(a-b)^r$ and a^r are in A_1. Then*

(i) $\operatorname{tr}(a+b)^r + \operatorname{tr}(a-b)^r \leqslant \operatorname{tr} a^r$ $(0 < r \leqslant 1)$.

(ii) $\operatorname{tr}(a+b)^r + \operatorname{tr}(a-b)^r \geqslant \operatorname{tr} a^r$ $(1 \leqslant r < \infty)$.

Now we have the following result.

THEOREM 5.4. *Let a and b be two elements in A_p and $1/p + 1/q = 1$. Then*

(i) $\quad 2^{p-1}(|a|_p^p + |b|_p^p) \leqslant |a+b|_p^p + |a-b|_p^p \leqslant 2(|a|_p^p + |b|_p^p) \qquad (0 < p \leqslant 2)$

(ii) $\quad |a+b|_p^q + |a-b|_p^q \leqslant 2(|a|_p^p + |b|_p^p)^{q/p} \qquad (1 < p \leqslant 2)$

(iii) $\quad 2(|a|_p^p + |b|_p^p) \leqslant |a+b|_p^p + |a-b|_p^p \leqslant 2^{p-1}(|a|_p^p + |b|_p^p) \qquad (2 \leqslant p < \infty)$

(iv) $\quad 2(|a|_p^p + |b|_p^p)^{q/p} \leqslant |a+b|_p^q + |a-b|_p^q \qquad (2 \leqslant p < \infty).$

PROOF. This can be proved by using Lemma 5.2, Lemma 5.3 and the proof of [4, p. 261, Theorem 2.7]. We omit the details.

As observed in [4], we have:

COROLLARY 5.5. *For* $1 < p < \infty$, A_p *is uniformly convex and reflexive.*

6. **The conjugate space of** A_p. In this section, we always assume that $1 \leq p < \infty$ and $1/p + 1/q = 1$. Let A_p^* be the conjugate space of A_p. We shall show that $A_p = A_q^*$ in a natural way.

For each a in A_p $(1 \leq p < \infty)$, we define

$$F_a(x) = \operatorname{tr} ax \qquad (x \in A_q). \tag{6.1}$$

THEOREM 6.1. *For each* a *in* A_p $(1 \leq p < \infty)$, $F_a \in A_q^*$ *and* $\|F_a\| = |a|_p$.

PROOF. By Lemma 4.2, F_a is well defined. It is clear that $F_a \in A_q^*$ and $\|F_a\| \leq |a|_p$. By a similar argument in the proof of [11, p. 786, Proposition 3.26], we can show that $\|F_a\| \geq |a|_p$. This completes the proof.

We now establish a converse of Theorem 6.1.

THEOREM 6.2. *For* $1 \leq p < \infty$, *every continuous linear functional* F *on* A_q *is of the form* F_a *for some* a *in* A_p, *where* F_a *is defined in* (6.1).

PROOF. We assume first that $p = 1$ and $F \in A_\infty^* = A^*$. Then it is clear that $F \in A_2^*$. Since A_2 is a Hilbert space, by the Riesz representation theorem, there exists some a in A_2 such that $F(x) = (x, a^*) = \operatorname{tr} ax$ for all x in A_2. By the proof of Theorem 6.1, we can show that $a \in A_1$ and so $F = F_a$.

Now we consider the case $1 < p < \infty$ and assume $F \in A_q^*$. Then $F \in A_1^*$. Hence by the proof of [8, p. 103, Theorem 2], there exists a right centralizer S on A_2 such that

$$F(y) = \operatorname{tr} Sy \qquad (y \in A_1). \tag{6.2}$$

Let $\{e_\alpha\}$ be a maximal orthogonal family of hermitian minimal idempotents in A and $\{E_\gamma\}$ the direct set of all finite sums $e_{\alpha_1} + e_{\alpha_2} + \cdots + e_{\alpha_n}$. Define F_γ on A_q by

$$F_\gamma(x) = F(E_\gamma x) \qquad (x \in A_q). \tag{6.3}$$

Since $S(E_\gamma x) = (SE_\gamma)(E_\gamma x) = ((SE_\gamma)E_\gamma)x = (SE_\gamma)x$ and $E_\gamma x \in A_1$, by (6.2) and (6.3) we have

$$F_\gamma(x) = \operatorname{tr} S(E_\gamma x) = \operatorname{tr} (SE_\gamma)x \qquad (x \in A_q). \tag{6.4}$$

Since $SE_\gamma = (SE_\gamma)E_\gamma \in A_p$, by (6.4) and Theorem 6.1, $|SE_\gamma|_p = \|F_\gamma\| \leq \|F\|$. Therefore $\{SE_\gamma\}$ is a bounded set in A_p. Since A_p is reflexive (Corollary 5.5), we can assume that $SE_\gamma \to a$ weakly for some a in A_p. Hence $ae_\alpha = Se_\alpha$ for all α. Therefore by (6.2), $F(e_\alpha x) = \operatorname{tr} ae_\alpha x$. For each x in A_q, by

Corollary 3.10, $x = \Sigma_\alpha e_\alpha x$ in $|\cdot|_q$. Hence it follows that $F(x) = \operatorname{tr} ax$ $(x \in A_q)$. This completes the proof.

REMARK. Some arguments in the above proof are similar to those in the proof of [3, p. 130, Theorem III. 12.2].

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