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TOPOLOGICAL SEMIGROUPS AND REPRESENTATIONS

BY

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ABSTRACT. Let S be a topological semigroup (separately continuous multiplication) with identity and $W(S)$ the Banach space of all weakly almost periodic functions on S. It is well known that if $S = G$ is a locally compact group, then $W(G)$ always has a (unique) invariant mean. In other words, there exists $m \in W(G)^*$ such that $\|m\| = m(1) = 1$ and $m(l_s f) = m(r_s f) = m(f)$ for any $s \in G$, $f \in W(G)$ where $l_s f(t) = f(st)$ and $r_s f(t) = f(ts)$, $t \in S$. The main purpose of this paper is to present several characterisations (functional analytic and algebraic) of the existence of a left (right) invariant mean on $W(S)$. In particular, we prove that $W(S)$ has a left (right) invariant mean iff a certain compact topological semigroup $p(S)^\omega$ (to be defined) associated with S contains a right (left) zero. Other results in this direction are also obtained.

1. **Notations and terminologies.** Let S be a topological semigroup (separately continuous multiplication) with identity e (see L. deLeeuw and I. Glicksberg [3] (1) for definition), $l_1(S)$ the Banach algebra of all absolutely summable functions on S with usual l_1-norm and convolution as multiplication ($\theta_1 * \theta_2(s) = \Sigma_{s_1 s_2 = s}\, \theta_1(s_1)\theta_2(s_2)$, for any $\theta_1, \theta_2 \in l_1(S)$ and $s \in S$, see M. Day [2, p. 521]) and let $p(S)$ be the set of all $\theta \in l_1(S)$ such that $\theta \geq 0$, $\|\theta\|_1 = 1$ and $\{s \in S\colon \theta(s) > 0\}$ is finite. Each $\theta \in p(S)$ is called a finite mean (Day [2, p. 513]). With convolution as multiplication $p(S)$ is a semigroup.

Consider the Banach space $W(S)$ of all weakly almost periodic functions on S with supremum norm [3, §5, p. 80]. An element $m \in W(S)^*$ is called a mean if $\|m\| = m(1) = 1$. m is (left) [right] invariant if $(m(l_s f) = m(f))$ $[m(r_s f) = m(f)]$ $m(l_s f) = m(r_s f) = m(f)$ for any $s \in S$, $f \in W(S)$ where $l_s f(t) = f(st)$, $r_s f(t) = f(ts)$, $t \in S$. We shall often use the notation LIM (RIM) for left invariant mean (right invariant mean).

If B is a Banach space and $\mathcal{B}(B)$ the algebra of all bounded linear opera-

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(1) Terminologies, techniques and results from this particular reference will be used very often later. If only the readers could feel how much the present author is indebted to their work.

tors in B, then $\mathcal{B}(B)$ is a topological semigroup under operator multiplication and the weak operator topology (WO) [3, §3, p. 71]. Any subsemigroup of $\mathcal{B}(B)$ containing the identity operator is called a semigroup of operators.

Let S be a semigroup of operators in B, S is called weakly almost periodic if for each $x \in B$, the set $\{Tx: T \in S\}$ has compact closure in the weak topology of B. It is known that if S is weakly almost periodic, then the weak operator closure, $\bar{S}$ of S in $\mathcal{B}(B)$ is a compact topological semigroup with weak operator topology [3, Theorem 3.1, p. 72].

2. Main theorems.

THEOREM 2.1. *Let S be a topological semigroup, with identity e, for each $\theta \in p(S)$, define $T_\theta = \Sigma_s \theta(s) r_s$ (the sum is actually finite). Then the map $T: p(S) \times W(S) \to W(S)$ defined by $T(\theta, f) = T_\theta f$ is a representation of the semigroup $p(S)$ as a weakly almost periodic convex semigroup of operators in $W(S)$. The closure $p(S)^\omega$ of $\{T_\theta: \theta \in p(S)\}$ in the weak operator topology is a compact convex topological semigroup in the same topology.*(2) *Moreover, the following statements are equivalent.*

(1) *$W(S)$ has a left invariant mean.*

(2) *The semigroup $p(S)^\omega$ has a right zero; that is there exists some $V \in p(S)^\omega$ such that $UV = V$ for any $U \in p(S)^\omega$.*

(3) *The continuous bounded functions $C(p(S)^\omega)$ on $p(S)^\omega$ has a multiplicative left invariant mean (see Mitchell [14, §1, p. 117] for definition).*

(4) *$C(p(S)^\omega)$ has a left invariant mean.*

PROOF. Clearly, each $T_\theta: W(S) \to W(S)$ is a bounded linear operator in $W(S)$. Also if $\theta_1, \theta_2 \in p(S)$,

$$\begin{aligned} T_{\theta_1 * \theta_2} &= \sum_s \theta_1 * \theta_2(s) r_s = \sum_s \sum_{s_1 s_2 = s} \theta_1(s_1)\theta_2(s_2) r_s \\ &= \sum_{s_1} \sum_{s_2} \theta_1(s_1)\theta_2(s_2) r_{s_1} \circ r_{s_2} \\ &= \sum_{s_1} \theta_1(s_1) r_{s_1} \circ \sum_{s_2} \theta_2(s_2) r_{s_2} = T_{\theta_1} \circ T_{\theta_2}. \end{aligned}$$

Hence T is indeed a representation. Since $p(S)$ is convex in $l_1(S)$, $\{T_\theta: \theta \in p(S)\}$ is convex in $\mathcal{B}(W(S))$. By definition of T_θ, $\{T_\theta f: \theta \in p(S)\}$ is contained in $\mathrm{CO}\{r_s f: s \in S\}$ for each $f \in W(S)$ (CO A denotes the convex

(2) Unfortunately the notation $p(S)^\omega$ is a little ambiguous since $p(S)$ can also be considered as a topological semigroup (say under the weak topology of $l_1(S)$) and $p(S)^\omega$ the weakly almost periodic compactification of $p(S)$. However we shall have no occasion in this paper to consider the above situation and no confusion will arise.

hull of A in a vector space). But $\{r_s f\colon s \in S\}$ and hence $\mathrm{CO}\{r_s f\colon s \in S\}$ has compact closure in the weak topology of $W(S)$ (Krein-Šmulian theorem [4, p. 434]). It follows that $\{T_\theta f\colon \theta \in p(S)\}$ also has compact closure in the same topology or $\{T_\theta\colon \theta \in p(S)\}$ is weakly almost periodic. By [3. Theorem 3.1, p. 72], $p(S)^\omega$ is a compact topological semigroup in the weak operator topology.

Next assume (1) holds and let m be a left invariant mean on $W(S)$. There exists a net of finite means $\theta_\alpha \in p(S)$ such that $\theta_\alpha \to m$ in weak* topology of $W(S)^*$ (the finite means are weak* dense in the set of means; see for example [2, §3] or apply the Hahn-Banach theorem). Consider the net T_{θ_α} in $p(S)^\omega$ which is WO compact, we can assume $T_{\theta_\alpha} \to V$ (WO), using a subnet if necessary. We claim that V a right zero of $p(S)^\omega$. We first show that $r_s V = V$ for any $s \in S$. Since $T_{\theta_\alpha} \to V$ (WO), $r_s T_{\theta_\alpha} \to r_s V$ (WO). Hence for each $s \in S$, $f \in W(S)$, $(r_s T_{\theta_\alpha})f \to (r_s V)f$ weakly and a fortiori pointwise. But for each $t \in S$,

$$(r_s T_{\theta_\alpha})f(t) = T_{\theta_\alpha} f(ts) = \sum_\sigma \theta_\alpha(\sigma) r_\sigma f(ts) = \sum_\sigma \theta_\alpha(\sigma) f(ts\sigma)$$

$$= \theta_\alpha(l_{ts} f) \to m(l_{ts} f) = m(f).$$

In other words $(r_s T_{\theta_\alpha})f \to m(f)\cdot 1$ pointwise. Hence $(r_s V)f = m(f)\cdot 1$ for any $s \in S$, $f \in W(S)$. Consequently $(r_s V)f = Vf = m(f)\cdot 1$ for any $s \in S$, $f \in W(S)$ (putting $s = e$). Therefore $r_s V = V$ for any $s \in S$. (Note that the use of the identity e can be avoided, the same arguments as above applied to $T_{\theta_\alpha} \to V$ (WO) ensure that $Vf = m(f)\cdot 1$.) It follows that $T_\theta V = V$ for any $\theta \in p(S)$. Since $\{T_\theta\colon \theta \in p(S)\}$ is WO dense in $p(S)^\omega$ we must have $UV = V$ for any $U \in p(S)^\omega$ and (1) implies (2).

Incidentally Vf is the constant function $m(f)\cdot 1$. Conversely assume (2) and let V be a right zero of $p(S)^\omega$, then $r_s V = V$ for any $s \in S$. There is a net $\theta_\alpha \in p(S)$ such that $T_{\theta_\alpha} \to V$ (WO). We can assume θ_α (or a subnet of θ_α) $\to m$ in weak* topology of $W(S)^*$ for some mean m on $W(S)$. We set out to prove that m is a left invariant mean. As above, for each $s \in S$, $f \in W(S)$, $(r_s T_{\theta_\alpha})f \to (r_s V)f = Vf$ weakly and hence pointwise. In particular we have $(r_s T_{\theta_\alpha})f(e) \to Vf(e)$. But

$$(r_s T_{\theta_\alpha})f(e) = \sum_\sigma \theta_\alpha(\sigma) f(s\sigma) = \theta_\alpha(l_s f) \to m(l_s f)$$

for any s. Hence $m(l_s f) = m(f) = Vf(e)$ and m is left invariant. Therefore (1) is equivalent to (2).

(2) and (3) are equivalent by a result of Mitchell [15, Corollary 2, p. 121]. Note that the definition of topological semigroup in [15] requires jointly con-

tinuous multiplication. However his result [15, Corollary 2, p. 121] remains valid for separately continuous topological semigroups and separately continuous representations. (See footnote (2) in [15, p. 120].)

Clearly (3) implies (4). On the other hand, let $C(p(S)^\omega)$ have a left invariant mean, then the pair $p(S)^\omega$, $C(p(S)^\omega)$ has the common fixed point property on *convex* compacta with respect to A-representations by affine maps, by [1, Theorem 1, p. 128] (see [1, §2] for these terminologies). Now consider $p(S)^\omega$ acting on itself on the left by left multiplication $(U, V) \to UV$. $Y = p(S)^\omega$ is a compact convex subset of a separated locally convex space ($\mathcal{B}(W(S))$ with WO topology). For each U, the map $V \to UV$ is clearly continuous and affine. Moreover if h is an affine continuous function on Y, then the function $U \to h(UV)$ belongs to $C(p(S)^\omega)$ for any $V \in p(S)^\omega$. Consequently the map $(U, V) \to UV$ of $p(S)^\omega \times p(S)^\omega \to p(S)^\omega$ is an A-representation. By [1, Theorem 1, p. 128] there exists some $V \in p(S)^\omega$ such that $UV = V$ for any $U \in p(S)^\omega$. Thus $p(S)^\omega$ has a right zero which in turn implies (3) by what we have proved above. This completes the proof of the theorem.

REMARK. In the proof of (2) implies (1) above, if we do not use the identity e, we can only prove that $m(l_{ts}f) = m(l_t f)$ for any $s, t \in S$, $f \in W(S)$. That is m is left invariant on all functions of the form $l_t f$, $t \in S$, $f \in W(S)$ since $l_{ts} = l_s \circ l_t$.

There is a one-to-one correspondence between the right zeros of $p(S)^\omega$ and the left invariant means on $W(S)$ (if any of them exists at all). Let Φ: $W(S)^* \to \mathcal{B}(W(S))$ be defined by $\Phi(m)(f) = m(f) \cdot 1$. Clearly Φ is a bounded linear operator and an isometry (into). In fact we have the following theorem.

THEOREM 2.2. *m is a left invariant mean on $W(S)$ iff $\Phi(m)$ is a right zero of $p(S)^\omega$. Moreover, the restriction of Φ to the convex set of left invariant means on $W(S)$ is an affine homeomorphism onto the convex set of all right zeros of $p(S)^\omega$ when they are endowed with the weak* topology of $W(S)^*$ and the weak operator topology of $\mathcal{B}(W(S))$ respectively.*

PROOF. Referring to the proof of the preceding theorem, if m is a LIM on $W(S)$, there is a right zero V of $p(S)^\omega$ such that $Vf = m(f) \cdot 1$ or $Vf = \Phi(m)f$ for any $f \in W(S)$. Thus $\Phi(m) = V$ is a right zero of $p(S)^\omega$. Conversely, if V is a right zero of $p(S)^\omega$, there is a LIM m on $W(S)$ such that $m(l_t f) = Vf(t)$ for any $t \in S$, $f \in W(S)$ (because $T_{\theta_\alpha} f(t) \to Vf(t)$, with θ_α as in the proof of (2) implies (1) in Theorem 2.1). Hence $Vf(t) = m(l_t f) = m(f)$ since m is a LIM. Thus $Vf = m(f) \cdot 1 = \Phi(m)f$ or $V = \Phi(m)$. Finally $m_\alpha \to m$ weak* in $W(S)^*$ iff $m_\alpha(f) \to m(f)$ $\forall f \in W(S)$ iff $m_\alpha(f) \cdot 1 \to m(f) \cdot 1$ weakly in $W(S)$ $\forall f \in W(S)$ or $\Phi(m_\alpha) \to \Phi(m)$ (WO) in $B(W(S))$. Since the left invariant means and right zeros of $p(S)^\omega$ are clearly convex, the theorem follows immediately.

It is interesting to observe that Theorem 2.1 remains valid if we interchange the words "left" and "right" throughout. Surprisingly, the proof is analogous but not symmetrical due to the asymmetry (between left and right translations) in a semigroup.

THEOREM 2.3. *Let S be a topological semigroup with identity e; then the following statements are equivalent.*

(1) *$W(S)$ has a right invariant mean.*

(2) *The semigroup has a left zero, that is, there exists some $V \in p(S)^\omega$ such that $VU = V$ for any $U \in p(S)^\omega$.*

(3) *$C(p(S)^\omega)$ has a multiplicative right invariant mean.*

(4) *$C(p(S)^\omega)$ has a right invariant mean.*

PROOF. In view of the above remark, we give the proof here for comparison. Assume that $W(S)$ has a RIM m. Let $\theta_\alpha \in p(S)$ with $\theta_\alpha \to m$ in weak* topology of $W(S)^*$ and let T_{θ_α} (or a subnet of T_{θ_α}) $\to V$ (WO) for some $V \in p(S)^\omega$. Then for each $s \in S$, $T_{\theta_\alpha}(r_s f) \to V(r_s f)$ weakly hence pointwise. But

$$T_{\theta_\alpha}(r_s f)(t) = \sum_\sigma \theta_\alpha(\sigma) f(t\sigma s) = \sum_\sigma \theta_\alpha(\sigma) r_s(l_t f)(\sigma)$$

$$= \theta_\alpha(r_s(l_t f)) \to m(r_s(l_t f)) = m(l_t f) = m_l(f)(t)$$

($m_l(f)$ thus defined is called the left introversion of f by m; it is known that $m_l(f) \in W(S)$ if $f \in W(S)$ and $m \in W(S)^*$. See for example [3, Lemma 5.13, p. 87]). Hence $V(r_s f) = m_l(f)$ for any $s \in S$, $f \in W(S)$. In particular $V(r_s f) = Vf = m_l(f)$ or $Vr_s = V$ for any $s \in S$. Consequently $VU = V$ for any $U \in p(S)^\omega$. Note that m and V are connected by the relation $Vf = m_l f$ *which is not necessarily a constant function.* On the other hand if V is a left zero of $p(S)^\omega$, then $Vr_s f = Vf$ for any $s \in S$, $f \in W(S)$. Let μ be *any* mean in $W(S)^*$ and let $m = V^*\mu$. Since $Vf \geqslant 0$ if $f \geqslant 0$, $V(1) = 1$ (which is true for any operator in $p(S)^\omega$), $V^*\mu = \mu \circ V$ is always a mean on $W(S)$. But $m(r_s f) = \mu(Vr_s f) = \mu(Vf) = m(f)$ or m is a RIM on $W(S)$. Hence (1) and (2) are equivalent.

The equivalence of (2) and (3) follows from a version of Mitchell's result for topological semigroups with separately continuous multiplication namely [15, Corollary 2, p. 121] where the words "left" and "right" are interchanged and the phrase "continuous representation" is replaced by "separately continuous antirepresentation".

The last equivalence can be handled in the same way as in Theorem 2.1, using right multiplication in $p(S)^\omega$ which is an antirepresentation.

The analogue of Theorem 2.2 is now easily obtained with the help of the map Ψ: $W(S)^* \to B(W(S))$ defined by $\Psi(m)f = m_l(f)$. Again it is easy to verify that Ψ is bounded linear. In fact, we have

THEOREM 2.4. *m is a right invariant mean on $W(S)$ iff $\Psi(m)$ is a left zero of $p(S)^\omega$. Moreover, the restriction of Ψ on the convex set of right invariant means on $W(S)$ is an affine homeomorphism onto the convex set of all left zeros of $p(S)^\omega$ with respect to the weak* topology of $W(S)^*$ and the weak operator topology of $B(W(S))$.*

PROOF. If m is a RIM on $W(S)$, then there is a left zero V of $p(S)^\omega$ such that $Vf = m_l(f) = \Psi(m)f$ for any $f \in W(S)$. Hence $\Psi(m) = V$. Conversely, if V is a left zero of $p(S)^\omega$, let $m = V^*m_e = m_e \circ V$ where m_e is the evaluation at the identity e. Then m is a RIM on $W(S)$. Now

$$\Psi(m)f(t) = m_l(f)(t) = m(l_t f) = V(l_t f)(e) = \lim_\alpha T_{\theta_\alpha}(l_t f)(e)$$

$$= \lim_\alpha \sum_s \theta_\alpha(s) r_s l_t f(e) = \lim_\alpha \sum_s \theta_\alpha(s) r_s f(t) = \lim_\alpha T_{\theta_\alpha} f(t) = Vf(t)$$

where T_{θ_α} is any net such that $T_{\theta_\alpha} \to V$ (WO). Hence $\Psi(m) = V$. Now the map Ψ is one-to-one (although not necessarily an isometry) because $\Psi(m) = \Psi(n)$ implies $m_l(f) = n_l(f)$ and in particular $m(f) = m_l(f)(e) = n_l(f)(e) = n(f)$ for any $f \in W(S)$ or $m = n$. Since both the right invariant means on $W(S)$ and the left zeros of $p(S)^\omega$ are convex, the proof will be complete if we can show that for any *means* m_α, m on $W(S)$, $m_\alpha \to m$ weak* in $W(S)^*$ iff $\Psi(m_\alpha) \to \Psi(m)$ (WO) in $B(W(S))$. However, this follows from the observations that (1) for each $f \in W(S)$, the weakly closed convex hull $\omega\,\mathrm{CLCO}\{r_s f\colon s \in S\}$ of $\{r_s f\colon s \in S\}$ is weakly, hence pointwise, compact and the two topologies agree on $\omega\,\mathrm{CLCO}\{r_s f\colon s \in S\}$. (2) $m_l(f) \in \omega\,\mathrm{CLCO}\{r_s f\colon s \in S\}$ for any *mean* m on $W(S)$. This is clear if $m = \theta \in p(S)$. In general, let $\theta_\alpha \to m$ ω^* in $W(S)^*$ then $(\theta_\alpha)_l f \to m_l f$ pointwise and hence $m_l f \in \omega\,\mathrm{CLCO}\{r_s f\colon s \in S\}$. (3) $m_\alpha \to m$ weak* in $W(S)^*$ iff $(m_\alpha)_l f \to m_l f$ pointwise, hence weakly by (1).

As a simple application of the preceding theorems, we prove the following result the first part of which is well known (see R. Holmes and A. Lau [10, §4, Lemmas 6 and 7]).

THEOREM 2.5. *If $W(S)$ has a left invariant mean m and a right invariant mean n, then $m = n$ and it is the unique* (*left*) [*right*] *invariant mean on*

$W(S)$. The semigroup $p(S)^\omega$ contains a unique (left) [right] zero V. Moreover the kernel $K(p(S)^\omega)$ is $\{V\}$.

PROOF. Since $\Phi(m)$ is a right zero of $p(S)^\omega$ and $\Psi(n)$ is a left zero of $p(S)^\omega$, we have $\Phi(m) = \Psi(n)\Phi(m) = \Psi(n)$ or $m(f)\cdot 1 = n_l(f)$ for any $f \in W(S)$. In particular $n(f) = n_l(f)(e) = m(f)$ or $m = n$. It is clear that this is the unique left or right or two-sided invariant mean on $W(S)$. Let $V = \Phi(m)$, then V is a right zero and a left zero of $p(S)^\omega$ since $\Phi(m) = \Psi(n)$. Therefore V is the unique left or right or two-sided zero of $p(S)^\omega$. Now $\{V\}$ is a (minimal) two-sided ideal in $p(S)^\omega$. Hence $K(p(S)^\omega) \subset \{V\}$. Since $p(S)^\omega$ is compact, $K(p(S)^\omega)$ is nonempty [3, Theorem 2.3, p. 66]. Hence $K(p(S)^\omega) = \{V\}$.

3. **Multiplicative invariant means on** $W(S)$. Since $W(S)$ is an algebra under pointwise multiplication [3, Theorem 5.3, p. 82], we can also consider multiplicative invariant means (MLIM) on $W(S)$ (a mean m on $W(S)$ is multiplicative if $m(fg) = m(f)m(g)$ for any $f, g \in W(S)$). The natural compact topological semigroup to work with is S^ω, the weakly almost periodic compactification of S [3, p. 82]. The results in §2 all have their analogues in this new situation. In particular we state the following theorem.

THEOREM 3.1. *Let S be a topological semigroup with identity; then the following statements are equivalent.*

(1) *$W(S)$ has a multiplicative left (right) invariant mean.*

(2) *S^ω has a right (left) zero.*

(3) *$C(S^\omega)$ has a multiplicative left (right) invariant mean.*

The proof of Theorem 3.1 is similar to that of Theorem 2.1 (and Theorem 2.3) using the fact that the evaluation functionals $\{m_s: s \in S\}$ is weak* dense in the set of multiplicative means on $W(S)$ where m_s is defined by $m_s(f) = f(s)$, $f \in W(S)$, (see Mitchell [15, p. 119]) and the fact that each operator V in S^ω is multiplicative (as a result, V^* maps the set of multiplicative means in $W(S)^*$ into itself).

It is interesting to observe that the equivalence of (1) and (3) can also be proved by applying the *algebra* isomorphism $\tilde{\gamma}$: $C(S^\omega) \to W(S)$ of [3, Theorem 13] (where it is denoted by $\tilde{R}$) and using [3, Lemma 2.10] for the continuous homomorphism r: $S \to S^\omega$ such that $r(s) = r_s$.

However, it should be noted that since S^ω is not convex, the existence of a LIM on $C(S^\omega)$ is not enough to ensure the existence of a multiplicative LIM on $C(S^\omega)$.

4. The convex semigroup $p(S)^\omega$. In this section, we shall study in detail the semigroup $p(S)^\omega$. Since $p(S)^\omega$ is compact and convex, we already know a lot about its structure. For example, the kernel $K(p(S)^\omega)$ is nonempty and consists entirely of projections [3, Theorem 7.2]. Let S^ω be the weakly almost periodic compactification of S and $r\colon S \to S^\omega$ be defined by $r(s) = r_s$ and $\tilde{r}\colon C(S^\omega) \to W(S)$ the induced isomorphism [3, Theorem 5.31] such that $\tilde{r}h = h \circ r$, $h \in C(S^\omega)$. $\tilde{r}$ is in fact an algebra isomorphism and isometry. For the semigroup $p(S)^\omega$, we cannot expect an isomorphism between $C(p(S)^\omega)$ and $W(S)$ because in general the semigroup $p(S)^\omega$ is too "large" for this to happen. While S is dense in S^ω (which makes $\tilde{r}$ isometric, hence injective), $p(S)^\omega$ is only the closed convex hull of S (more precisely of $\{r_s\colon s \in S\}$) and hence of S^ω. However, $W(S)$ is always a homomorphic image of $C(p(S)^\omega)$ as indicated in the following theorem.

THEOREM 4.1. *Let $\iota\colon S^\omega \to p(S)^\omega$ be the inclusion map and $\eta\colon S \to p(S)^\omega$ the composition $\iota \circ r$. The induced map $\tilde{\eta}\colon C(p(S)^\omega) \to C(S)$ defined by $\tilde{\eta}F = F \circ \eta$, $F \in C(p(S)^\omega)$ is a continuous algebra homomorphism of $C(p(S)^\omega)$ onto $W(S)$. The kernel* Ker $\tilde{\eta}$ *of $\tilde{\eta}$ is precisely the ideal of all F in $C(p(S)^\omega)$ which vanish on S^ω and $\tilde{\eta}$ induces an algebra isomorphism $\bar{\tilde{\eta}}$ of $C(p(S)^\omega)/$*Ker $\tilde{\eta}$ *onto $W(S)$ such that $\tilde{\eta} = \bar{\tilde{\eta}} \circ \pi$ where π is the natural surjection of $C(p(S)^\omega)$ onto $C(p(S)^\omega)/$*Ker $\tilde{\eta}$.

$$\begin{tikzcd} S \arrow[r, "r"] \arrow[rr, bend left, "\eta"] & S^\omega \arrow[r, "\iota"] & p(S)^\omega \end{tikzcd}$$

$$\begin{tikzcd} W(S) & C(S^\omega) \arrow[l, "\tilde{r}"'] & C(p(S)^\omega) \arrow[l, "\tilde{\iota}"'] \arrow[ll, bend right, "\tilde{\eta}"'] \arrow[d, "\pi"] \\ & & C(p(S)^\omega)/\mathrm{Ker}\,\tilde{\eta} \arrow[ull, "\bar{\tilde{\eta}}"] \end{tikzcd}$$

PROOF. It is easy to see that η is a continuous homomorphism. Since $\tilde{\eta} = \widetilde{\iota \circ r} = \tilde{r} \circ \tilde{\iota}$, $\tilde{\eta}$ maps $C(p(S)^\omega)$ into $W(S)$. Let $f \in W(S)$ and define F by $F(U) = m_e(Uf) = Uf(e)$, then $F \in C(p(S)^\omega)$ and $\tilde{\eta}F = F \circ \eta = f$ since $F \circ \eta(s) = F(r_s) = (r_s f)(e) = f(s)$. Therefore $\tilde{\eta}$ maps $C(p(S)^\omega)$ onto $W(S)$. Obviously $\tilde{\eta}$ is an algebra homomorphism and Ker $\tilde{\eta}$ is an ideal. Now $F \in$ Ker $\tilde{\eta}$ iff $\tilde{\eta}F = F \circ \eta = 0$ iff $F(r_s) = 0\ \forall s \in S$ or equivalently $F = 0$

on S^ω since $\{r_s: s \in S\}$ is dense in S^ω. The last statement is trivial (in view of the Open Mapping Theorem).

Now let $m(p(S))$ be the space of all bounded functions on $p(S)$ with supremum norm. For each $f \in W(S)$, define a function $\tau f \in m(p(S))$ by $\tau f(\theta) = (f, \theta) = \Sigma_s f(s)\theta(s)$, $\theta \in p(S)$. Let W be the (norm) closed subalgebra of $m(p(S))$ generated by τf, $f \in W(S)$. In [15, §5] Mitchell showed that the algebra W is translation invariant and that W has a multiplicative left invariant mean iff $W(S)$ has a left invariant mean. In what follows, we shall unify this result of Mitchell's with the results in [3, §4] and in the present paper by showing that W is actually isomorphic to the algebra $C(p(S)^\omega)$.

As in [3, §4], if S_1 is a weakly almost periodic semigroup of operators in B, $C_B(S_1)$ is defined to be the (norm) closed subalgebra of $C(S_1)$ generated by the constants and all functions of the form:

$$F(U) = (Ux, x^*), \quad x \in B, \quad x^* \in B^*.$$

We shall be mainly concerned with the special case when $B = W(S)$ and $S_1 = \{T_\theta: \theta \in p(S)\}$. Let $j: S_1 \to \bar{S}_1 = p(S)^\omega$ be the inclusion map (closure taken in WO topology of $\mathcal{B}(W(S))$). By [3, Lemma 4.8, p. 77], the induced map $\tilde{j}: C(p(S)^\omega) \to C(S_1)$ maps the algebra $C(p(S)^\omega)$ isomorphically and isometrically onto $C_B(S_1) = C_{W(S)}(S_1)$.

Observe that $p(S)$ is a topological semigroup when $p(S)$ is endowed with the weak topology of $l_1(S)$. Moreover if we define $T: p(S) \to S_1$ by $T(\theta) = T_\theta$, $\theta \in p(S)$, then T is a continuous homomorphism with respect to the weak topology of $l_1(S)$ and the weak operator topology of $\mathcal{B}(W(S))$. Continuity follows from the fact that if $\theta_\alpha \to \theta$ weakly in $l_1(S)$, then for any $\mu \in W(S)^*$, $f \in W(S)$,

$$\mu(T_{\theta_\alpha} f) = \mu\left(\sum_s \theta_\alpha(s) r_s f\right) = \sum_s \theta_\alpha(s)\mu(r_s f) \to \sum_s \theta(s)\mu(r_s f) = \mu(T_\theta f)$$

since $s \to \mu(r_s f)$ is a bounded function on S (actually it is even a function in $W(S)$).

THEOREM 4.2. *Let* $\tilde{T}: C(S_1) \to C(p(S))$ *be the induced map such that* $\tilde{T}F = F \circ T$, $F \in C(S_1)$, *then* $\tilde{T}$ *maps the algebra* $C_{W(S)}(S_1)$ *isomorphically and isometrically onto* W.

PROOF. Observe that for each $f \in W(S)$, τf is a continuous function on $p(S)$ (with weak topology of $l_1(S)$). Therefore $W \subset C(p(S))$. Clearly $\tilde{T}$ is an algebra homomorphism. Since T is onto, $\tilde{T}$ is an isometry. In fact

$$\|\tilde{T}F\| = \sup\{|\tilde{T}F(\theta)|:\ \theta \in p(S)\} = \sup\{|F(T_\theta)|:\ \theta \in p(S)\}$$
$$= \sup\{|F(T_\theta)|:\ T_\theta \in S_1\} = \|F\|.$$

Suppose F is a function in $C(S_1)$ of the form $F(U) = \mu(Uf)$, $\mu \in W(S)^*$, $f \in W(S)$. We have for any $\theta \in p(S)$,

$$\tilde{T}F(\theta) = F(T_\theta) = \mu(T_\theta f) = \sum_s \theta(s)\mu(r_s f) = (\mu_r f, \theta) = \tau(\mu_r f)(\theta)$$

where $\mu_r f(s) = \mu(r_s f)$ and $\mu_r f \in W(S)$ by [3, Lemma 5.13, p. 87]. Hence $\tilde{T}F = \tau(\mu_r f) \in W$ or $F \in \tilde{T}^{-1}(W)$ which is a norm closed subalgebra of $C(S_1)$ containing the constants. (W always contains the constants.) Therefore $C_{W(S)}(S_1) \subset \tilde{T}^{-1}(W)$. On the other hand, for any $f \in W(S)$, we can define F in $C(S_1)$ by $F(U) = m_e(Uf) = Uf(e)$, $U \in S_1$. Then $\tilde{T}F(\theta) = F(T_\theta) = T_\theta f(e) = (f, \theta) = \tau f(\theta)$ or $\tau f = \tilde{T}F \in \tilde{T}(C_{W(S)}(S_1))$ which is a norm closed subalgebra of $C(p(S))$. Consequently $W \subset \tilde{T}(C_{W(S)}(S_1))$. This implies that W is exactly the image of $C_{W(S)}(S_1)$ under $\tilde{T}$ and the proof is complete.

Since $C(p(S)^\omega)$ is isometrically isomorphic to $C_{W(S)}(S_1)$ under $\tilde{j}$, it is also isometrically isomorphic to W under $\tilde{T} \circ \tilde{j}$. Moreover, we have

COROLLARY 4.3. *Let $p(S)$ have the weak topology and $S_1 = \{T_\theta : \theta \in p(S)\}$; then the following statements are equivalent.*

(1) *$C(p(S)^\omega)$ has a (multiplicative) left invariant mean.*

(2) *$C_{W(S)}(S_1)$ has a (multiplicative) left invariant mean.*

(3) *W has a (multiplicative) left invariant mean.*

PROOF. For left invariant means, (1) and (2) are equivalent by [3, Lemma 2.10] applied to the continuous homomorphism j: $S_1 \to p(S)^\omega$. (2) and (3) are equivalent because the map T is *onto* and $\tilde{T}$ "commutes" with left translations ($l_\theta \circ \tilde{T} = \tilde{T} \circ l_{T(\theta)}$) and so we can use the arguments of [3, Lemma 2.10] (although S_1 is not compact).

For multiplicative left invariant means, the same proof can be used because both $\tilde{j}$ and $\tilde{T}$ are algebra homomorphisms (i.e. multiplicative).

REMARK. It follows that all possible six combination statements in the corollary are equivalent by Theorem 2.1 ((3) $\Leftrightarrow$ (4)) which together with this corollary yield Mitchell's theorem [15, Theorem 3, p. 125].

5. **The semigroup of means on** $W(S)$. Let $m(S)$ be the Banach space of all bounded functions on S with supremum norm and X a linear subspace of $m(S)$ containing the constants such that (1) $l_s f \in X$ for any $s \in S$, $f \in X$ and (2) $m_l f \in X$ for any $f \in X$, $m \in X^*$ where $m_l f(s) = m(l_s f)$, $s \in S$. $m \in X^*$ is called a mean on X iff $\|m\| = m(1) = 1$.

If $m, n \in X^*$, define the Arens product $m \odot n$ by $(m \odot n)f = m(n_l f)$ $f \in X$. X^* becomes an algebra with Arens multiplication. It is easy to see that if m, n are means on X, so is $m \odot n$. Therefore the set of means M on X is a semigroup under Arens product (see [2, §6] for a detailed description of the special case $X = m(S)$). It is clear that the Arens product $m \odot n$ is continuous in the first variable with respect to the weak* topology of X^*. However, the product is not continuous in the second variable with respect to the weak* topology of X^* even if we restrict ourselves to means on X. Thus M with the weak* topology of X^* is in general not a topological semigroup. (See remark after Lemma 3 in [2, §6].)

When $X = W(S)$, the situation is more satisfactory as the following theorem shows.

THEOREM. *The set M of means in $W(S)^*$ is a compact convex semigroup under Arens product and the weak* topology of $W(S)^*$. The map Ψ: $W(S)^* \to \mathcal{B}(W(S))$ such that $\Psi(m) = m_1$ (as in Theorem 2.4) is an isometric isomorphism of the Banach algebra $W(S)^*$ (with Arens product) into $\mathcal{B}(W(S))$.*

Moreover, the restriction (also denoted by Ψ) of Ψ to M is an affine homeomorphism of M with weak topology of $W(S)^*$ onto $p(S)^\omega$ with weak operator topology of $\mathcal{B}(W(S))$. The induced map $\widetilde{\Psi}$: $C(p(S)^\omega) \to C(M)$ is an isometric isomorphism between the algebras $C(p(S)^\omega)$ and $C(M)$ which commutes with left translations ($l_m \circ \widetilde{\Psi} = \widetilde{\Psi} \circ l_{\Psi(m)}$, $m \in M$).*

PROOF. M is clearly a convex semigroup under Arens product and is weak* compact in $W(S)^*$. We first show that the product $m \odot n$ is continuous in the second variable with respect to the weak* topology of $W(S)^*$. Let $m, n_\alpha, n \in M$ and assume $n_\alpha \to n$ weak* in $W(S)^*$, then $(n_\alpha)_l f \to n_l f$ pointwise on S for any $f \in W(S)$. Now $(n_\alpha)_l f$, $n_l f$ all belong to the weakly compact set w CLCO $\{r_s f\colon s \in S\}$ on which the weak topology (of $W(S)$) and the pointwise topology coincide (see the proof of Theorem 2.4). Hence $(n_\alpha)_l f \to n_l f$ weakly in $W(S)$ and $(m \odot n_\alpha)f = m((n_\alpha)_l f) \to m(n_l f) = (m \odot n)f$ for any $f \in W(S)$ or $m \odot n_\alpha \to m \odot n$ weak* in $W(S)^*$. Therefore M is a compact topological semigroup under Arens product and weak* topology of $W(S)^*$.

Clearly $\|\Psi(m)\| = \|m_l\| \leq \|m\|$. In fact $\|\Psi(m)\| = \|m\|$, since

$$\|m\| = \sup\{|m(f)|\colon \|f\| = 1\} \leq \sup\{\|m_l f\|\colon \|f\| = 1\} = \|m_l\|$$

as S contains identity. Moreover, Ψ is an algebra homomorphism since $\Psi(m \odot n) = (m \odot n)_l = m_l \circ n_l = \Psi(m) \circ \Psi(n)$ (essentially because l_s and m_l commute). Therefore Ψ is an isomorphism into.

Let m be any mean on $W(S)$, then there is a net θ_α in $p(S)$ such that $\theta_\alpha \to m$ weak* in $W(S)^*$. Then $T_{\theta_\alpha} f \to m_l f$ pointwise on S and hence weakly in $W(S)$ for any $f \in W(S)$. (Because both $T_{\theta_\alpha} f$ and $m_l f$ belong to w CLCO$\{r_s f\colon s \in S\}$.) Hence $T_{\theta_\alpha} \to m_l$ (WO) and $m_l \in p(S)^\omega$. Conversely if $T \in p(S)^\omega$, then $T_{\theta_\alpha} \to T$ (WO) for some net $\theta_\alpha \in p(S)$. Let m be any weak* cluster point of θ_α in $W(S)^*$, then m is a mean on $W(S)$ and it is easy to verify that $m_l = T$. Consequently Ψ maps M onto $p(S)^\omega$. That Ψ is a homeomorphism with respect to the appropriate topologies can be proved by using the same arguments as in the proof of Theorem 2.4. The last statement of the theorem is now trivial.

6. **Applications to fixed point theorems.** Let K be a compact convex subset of a separated locally convex space E with continuous dual E^*. An affine action of S on K is a map $S \times K \to K$ denoted by $(s, x) \to sx$, $s \in S$, $x \in K$ such that

(1) For each $s \in S$, the map $x \to sx$ is an affine mapping of K into itself.

(2) $(st)x = s(tx)$ for any $s, t \in S$, $x \in K$.

The action is called separately (jointly) continuous when the map $S \times K \to K$ is separately (jointly) continuous. In this section, we shall consider mainly *separately continuous actions.* Let $AF(K)$ denote the space of all affine continuous functions on K, then $AF(K)$ is a norm closed linear subspace of $C(K)$ and $AF(K)$ separates points of K (see Argabright [1] or Phelps [18]). We shall need the following lemma which is also of independent interest.

LEMMA 6.1. *The functions in $AF(K)$ determine the topology of K. That is, if $x_\alpha, x \in K$ and $h(x_\alpha) \to h(x)$ for any $h \in AF(K)$, then $x_\alpha \to x$ in K.*

PROOF. For any $x^* \in E^*$, the restriction of x^* to K is in $AF(K)$. Hence $x^*(x_\alpha) \to x^*(x)$ and $x_\alpha \to x$ weakly in K which is compact. Therefore $x_\alpha \to x$ in K.

For each $x \in K$, $h \in AF(K)$, define a function $T_x h$ on S by $T_x h(s) = h(sx)$. By separate continuity of the action, $T_x h \in C(S)$. We say that the action is *right uniformly continuous* (*weakly right uniformly continuous*) iff for each $h \in AF(K)$, the map $x \to T_x h$ is continuous from K into $C(S)$ when $C(S)$ has the norm (weak) topology.

LEMMA 6.2. *Let $S \times K \to K$ be a separately continuous affine action of S on K which is right uniformly continuous (weakly right uniformly continuous), then $T_x h \in A(S)$ $(W(S))$ for any $x \in K$, $h \in AF(K)$.*

PROOF. The assertion involving right uniformly continuous actions is due to A. Lau [12, proof of Lemma 3.1]. We reproduce the proof here for completeness.

In any case $T_x h \in C(S)$. For any $s \in S$, we have

$$r_s(T_x h) = T_{sx} h \subset \{T_y h\colon y \in O(x)^-\}$$

where $O(x)^- = \{sx\colon s \in S\}^-$ which is compact.

Consequently, if the given action is right uniformly continuous (weakly right uniformly continuous), then the set $\{T_y h\colon y \in O(x)^-\}$ being the norm (weakly) continuous image of a compact set $O(x)^-$ is norm (weakly) compact in $C(S)$. Hence $T_x h \in A(S)$ $(W(S))$.

As in Lau [11], an affine action $S \times K \to K$ is called equicontinuous if for each $y \in K$ and $P \in \mathcal{U}$ where $\mathcal{U}$ is the *unique* uniformity which determines the topology of K (Kelley [11, p. 197]), there is some $Q \in \mathcal{U}$ such that $(sx, sy) \in P$ for any $s \in S$ whenever $(x, y) \in Q$. The next result exhibits some relations between various types of continuity conditions on affine actions.

LEMMA 6.3. *Let $S \times K \to K$ be a separately continuous affine action of S on K, then*

(1) *If the action is equicontinuous, it is right uniformly continuous.*

(2) *If the action is right uniformly continuous, it is jointly continuous.*

PROOF. The first assertion is due to A. Lau [12, proof of Lemma 3.1]. For convenience of the readers, we give the proof here. Let $h \in AF(K)$. Assume that the action is equicontinuous. By compactness of K, h is uniformly continuous (Kelley [11, p. 198]). For any $z \in K$, $\epsilon > 0$, there is some $P \in \mathcal{U}$ where $\mathcal{U}$ is the unique uniformity which determines the topology of K such that $|h(x) - h(y)| < \epsilon$ whenever $(x, y) \in P$. By equicontinuity, there is some $Q \in \mathcal{U}$ such that $(sx, sz) \in P$ for any $s \in S$ whenever $(x, z) \in Q$. Let $y \in Q[z] = \{y \in K\colon (y, z) \in Q\}$ which is a neighborhood of z in K, then

$$\|T_y h - T_z h\| = \sup_s |h(sy) - h(sz)| < \epsilon.$$

In other words, the action is right uniformly continuous. This established (1).

To prove (2), assume that the action is right uniformly continuous. Let $s_\alpha \to s$ in S and $y_\alpha \to y$ in K. For each $h \in AF(K)$ and $\epsilon > 0$, we can find some α_0 such that if $\alpha \geqslant \alpha_0$, then

$$\sup_t |h(ty_\alpha) - h(ty)| < \tfrac{1}{2}\epsilon \quad \text{and} \quad |h(s_\alpha y) - h(sy)| < \tfrac{1}{2}\epsilon$$

by right uniform continuity and separate continuity. Hence if $\alpha \geqslant \alpha_0$

$$|h(s_\alpha y_\alpha) - h(sy)| \leqslant |h(s_\alpha y_\alpha) - h(s_\alpha y)| + |h(s_\alpha y) - h(sy)| < \epsilon.$$

In other words, $h(s_\alpha y_\alpha) \to h(sy)$ $\forall h \in AF(K)$. By Lemma 6.1, $s_\alpha y_\alpha \to sy$ in K. This completes the proof.

Consider now the *convex* semigroup $p(S)^\omega$. An affine action $p(S)^\omega \times K \to K$ (denoted by $(U, x) \to Ux$) of $p(S)^\omega$ on K is said to be *convex* if, for any $U, V \in p(S)^\omega$, $\alpha, \beta \geq 0$, $\alpha + \beta = 1$ and $x \in K$,

$$(\alpha U + \beta V)x = \alpha Ux + \beta Vx.$$

This is the same as saying that for each $x \in K$, the map $U \to Ux$ is an affine mapping of $p(S)^\omega$ into K.

The next two theorems show that there is a natural relation between affine actions of S and convex affine actions of $p(S)^\omega$. (Examples will be given later.)

THEOREM 6.4. *Let $S \times K \to K$ be a separately continuous and weakly right uniformly continuous affine action of S on K, then there is a unique separately continuous and weakly right uniformly continuous convex affine action $p(S)^\omega \times K \to K$ of $p(S)^\omega$ on K such that $\tilde{\eta} \circ T_x^\omega = T_x$ for any $x \in K$ where $\tilde{\eta}$: $C(p(S)^\omega) \to W(S)$ is the homomorphism defined in Theorem* 4.1 *and $T_x^\omega h(U) = h(Ux)$, $h \in AF(K)$, $U \in p(S)^\omega$ and $x \in K$. (That is T_x^ω arises from the action $p(S)^\omega \times K \to K$ as T_x from $S \times K \to K$.)*

Moreover if the action $S \times K \to K$ is right uniformly continuous, so is the induced action $p(S)^\omega \times K \to K$.

PROOF. Since $S \times K \to K$ is weakly right uniformly continuous, by Lemma 6.2, $T_x h \in W(S)$ for any $h \in AF(K)$, $x \in K$. Let $U \in p(S)^\omega$, $x \in K$. There is a net $\theta_\alpha \in p(S)$ such that $T_{\theta_\alpha} \to U$ (WO). Hence $T_{\theta_\alpha}(T_x h) \to U(T_x h)$ weakly hence pointwise on S. Consider the net $y_\alpha = \Sigma_s\, \theta_\alpha(s) sx \in K$ which is compact. We can assume y_α (or some subnet) $\to x_0 \in K$. Let $t \in S$, then $T_{\theta_\alpha}(T_x h)(t) \to U(T_x h)(t)$. Using the facts that (1) $x \to tx$ is an affine transformation on K, (2) h is an affine continuous function on K and (3) the action $S \times K \to K$ is separately continuous, we have

$$T_{\theta_\alpha}(T_x h)(t) = \sum_s \theta_\alpha(s) r_s(T_x h)(t) = \sum_s \theta_\alpha(s) h(tsx) = h(ty_\alpha) \to h(tx_0)$$

and

$$\cdot\, T_{\theta_\alpha}(T_x h)(e) = \sum_s \theta_\alpha(S) h(sx) = h(y_\alpha) \to h(x_0), \quad h \in AF(K).$$

(Notice that the second convergence is *not* obtained from the first by formally putting $t = e$ because we do not assume that $x \to ex$ is the identity transformation of K. However $ex_0 = x_0$ as a consequence since $AF(K)$ separates points.)

Therefore for any $h \in AF(K)$, $U(T_x h) = T_{x_0} h$ and $U(T_x h)(e) = h(x_0)$.

Define a map $p(S)^\omega \times K \to K$ by setting $Ux = x_0$ where x_0 is the *unique* point in K determined by $h(x_0) = U(T_x h)(e)$, $h \in AF(K)$. Then

$$(*) \qquad U(T_x h) = T_{Ux} h \quad \text{and} \quad U(T_x h)(e) = h(Ux).$$

Using these equations and the fact that $AF(K)$ separates points of K, it is straightforward to verify that if $U, V \in p(S)^\omega$, $x, y \in K$, $\alpha, \beta \geqslant 0$, $\alpha + \beta = 1$, then

$$U(\alpha x + \beta y) = \alpha Ux + \beta Uy, \qquad (UV)x = U(Vx)$$

and

$$(\alpha U + \beta V)x = \alpha Ux + \beta Vx.$$

Hence the map $p(S)^\omega \times K \to K$ such that $(U, x) \to Ux$ is indeed a convex affine action of $p(S)^\omega$ on K.

Let $U_\alpha \to U$ (WO) in $p(S)^\omega$, then for any $h \in AF(K)$, $h(U_\alpha x) = U_\alpha(T_x h)(e) \to U(T_x h)(e) = h(Ux)$ and $U_\alpha x \to Ux$ by Lemma 6.1.

If $x_\alpha \to x$ in K, then $T_{x_\alpha} h \to T_x h$ weakly for any $h \in AF(K)$ by weakly right uniform continuity of $S \times K \to K$. Hence $h(Ux_\alpha) = U(T_{x_\alpha} h)(e) \to U(T_x h)(e) = h(Ux)$, for any $h \in AF(K)$. By Lemma 6.1 again, $Ux_\alpha \to Ux$.

Consequently the action $p(S)^\omega \times K \to K$ defined above is separately continuous. Let $T_x^\omega h \in C(p(S)^\omega)$ be defined by $T_x^\omega h(U) = h(Ux)$, $U \in p(S)^\omega$, $x \in K$ and $h \in AF(K)$, then $T_x^\omega h(U) = U(T_x h)(e)$ (from equations (*)). Now if $\tilde{\eta}$: $C(p(S)^\omega) \to W(S)$ is the homomorphism defined in Theorem 4.1, then for any $s \in S$, $h \in AF(K)$,

$$\tilde{\eta}(T_x^\omega h)(s) = (T_x^\omega h)(\eta(s)) = (T_x^\omega h)(r_s) = h(r_s x) = r_s(T_x h)(e) = T_x h(s) = h(sx).$$

Therefore $\tilde{\eta} \circ T_x^\omega = T_x$ for any $x \in K$, which is equivalent to the condition that $r_s x = sx$, $s \in S$, $x \in K$.

Either of these two conditions determines the action $p(S)^\omega \times K \to K$ uniquely. For if $(U, x) \to U \cdot x$ is any separately continuous convex affine action of $p(S)^\omega$ on K such that $r_s \cdot x = sx$, $s \in S$, $x \in K$, then $T_{\theta_\alpha} \cdot x \to U \cdot x$ for some net $\theta_\alpha \in p(S)$ such that $T_{\theta_\alpha} \to U$ (WO) by separate continuity. By convexity

$$T_{\theta_\alpha} \cdot x = \sum_s \theta_\alpha(s) r_s \cdot x = \sum_s \theta_\alpha(s)(r_s \cdot x) = \sum_s \theta_\alpha(s) sx = y_\alpha.$$

Since y_α (or some subnet of y_α) $\to Ux$ we have $U\cdot x = Ux$ for any $U \in p(S)^\omega$ and $x \in K$. This establishes the uniqueness.

To show that the action $p(S)^\omega \times K \to K$ is weakly right uniformly continuous, let $h \in AF(K)$ be fixed. For each $x \in K$

$$\|T_x^\omega h\| = \sup_U |T_x^\omega h(U)| = \sup_U |h(Ux)| = \sup_U |U(T_x h)(e)|$$

$$\leqslant \sup_U \|U(T_x h)\| \leqslant \|T_x h\| \quad (\text{since } \|U\| \leqslant 1)$$

$$= \sup_s |h(sx)| \leqslant \sup_{y \in K} |h(y)|;$$

therefore $\{T_x^\omega h\colon x \in K\}$ is a norm bounded subset of $C(p(S)^\omega)$. Consider the map $x \to T_x^\omega h$ of K into $C(p(S)^\omega)$. Since the action $S \times K \to K$ is weakly right uniformly continuous and

$$T_{x_\alpha}^\omega h(U) = U(T_{x_\alpha} h)(e) \to U(T_x h)(e) = T_x^\omega h(U)$$

if $U \in p(S)$ and $x_\alpha \to x$ in K, it is clear that the same map is continuous when $C(p(S)^\omega)$ has the pointwise topology. Therefore $\{T_x^\omega h\colon x \in K\}$ is pointwise compact and hence weakly compact in $C(p(S)^\omega)$ by Grothendieck's Theorem (recall that $p(S)^\omega$ is compact). As a result the map $x \to T_x^\omega h$ is weakly continuous which shows that the induced action $p(S)^\omega \times K \to K$ is weakly right uniformly continuous.

Finally if $S \times K \to K$ is right uniformly continuous, so is the induced action $p(S)^\omega \times K \to K$ because

$$\|T_x^\omega h - T_y^\omega h\| = \sup_U |U(T_x h)(e) - U(T_y h)(e)|$$

$$\leqslant \sup_U \|U(T_x h - T_y h)\| \leqslant \|T_x h - T_y h\|.$$

This completes the proof.

REMARK 6.5. It is easy to see that a point $x \in K$ is a common fixed point of action $S \times K \to K$ (i.e. $sx = x\ \forall s \in S$) iff it is a common fixed point of the induced action $p(S)^\omega \times K \to K$.

We next show that every "reasonable" action of $p(S)^\omega$ on K is induced by an action of S on K.

THEOREM 6.6. *Let $p(S)^\omega \times K \to K$ be a separately continuous weakly right uniformly continuous convex affine action of $p(S)^\omega$ on K, then there is a (necessarily unique) separately continuous weakly right uniformly continuous affine action $S \times K \to K$ of S on K which induces $p(S)^\omega \times K \to K$. Moreover if the given action of $p(S)^\omega$ is right uniformly continuous, so is $S \times K \to K$.*

PROOF. The proof is much simpler than that of Theorem 6.4. Given such an action $p(S)^\omega \times K \to K$, we define $S \times K \to K$ by $sx = \eta(s)x$ where η is the homomorphism defined in Theorem 4.1. It is straightforward to verify that this is a separately continuous affine action of S on K such that $\widetilde{\eta} \circ T_x^\omega h = T_x h$ for any $x \in K$, $h \in AF(K)$. Consider the homomorphism $\widetilde{\eta}$: $C(p(S)^\omega) \to W(S)$ of Theorem 4.1. $\widetilde{\eta}$ is norm continuous hence weakly continuous. Consequently if $x_\alpha \to x$ in K, then $T_{x_\alpha} h = \widetilde{\eta} T_{x_\alpha}^\omega h \to \widetilde{\eta} T_x^\omega h = T_x h$ weakly in $W(S)$ since the action $p(S)^\omega \times K \to K$ is weakly right uniformly continuous. Hence the action $S \times K \to K$ defined above is weakly right uniformly continuous. It is obvious that this action induces $p(S)^\omega \times K \to K$ (in the sense of Theorem 6.4). The last statement follows from the fact that $\widetilde{\eta}$ is norm continuous.

COROLLARY 6.7. *Let $K = p(S)^\omega$, define the maps $S \times p(S)^\omega \to p(S)^\omega$ and $p(S)^\omega \times p(S)^\omega \to p(S)^\omega$ by $(s, V) \to r_s V$ and $(U, V) \to UV$ where $s \in S$, $U, V \in p(S)^\omega$, then $S \times p(S)^\omega \to p(S)^\omega$ is a separately continuous and weakly right uniformly continuous affine action of S which induces $p(S)^\omega \times p(S)^\omega \to p(S)^\omega$.*

PROOF. By Theorem 2.1, $p(S)^\omega$ with WO topology is a compact convex topological semigroup. It is clear that map $p(S)^\omega \times p(S)^\omega \to p(S)^\omega$ where $(U, V) \to UV$ is a separately continuous convex affine action of $p(S)^\omega$ on itself. For each $h \in AF(K) \subset C(p(S)^\omega) = W(p(S)^\omega)$, $T_V^\omega h(U) = h(UV) = r_V h(U)$ or $T_V^\omega h = r_V h$. By [3, Theorem 2.7], the map $V \to T_V^\omega h$ of $p(S)^\omega$ into $W(p(S)^\omega)$ is weakly continuous. It follows that this action of $p(S)^\omega$ is weakly right uniformly continuous. By the preceding theorem, the map $S \times p(S)^\omega \to p(S)^\omega$ where $(s, V) \to r_s V$ is a separately continuous and right uniformly continuous affine action of S which induces the action of $p(S)^\omega$.

REMARK 6.8. Corollary 6.7 has an analogue for right uniformly continuous actions. Let $p(S)^\alpha$ be the strong operator closure of $\{T_\theta: \theta \in p(S)\}$ in $B(A(S))$ where $T_\theta f = \Sigma_s \theta(s) r_s f$, $f \in A(S)$, the space of all (strongly) almost periodic functions on S. $p(S)^\alpha$ is a compact convex topological semigroup (with jointly continuous multiplication) with the strong operator topology of $B(A(S))$. The actions $S \times p(S)^\alpha \to p(S)^\alpha$ where $(s, V) \to r_s V$ and $p(S)^\alpha \times p(S)^\alpha \to p(S)^\alpha$ where $(U, V) \to UV$ are jointly continuous and right uniformly continuous affine actions on S and $p(S)^\alpha$ respectively and the first action induces the second. We omit the details.

THEOREM 6.9. *Let S be a topological semigroup with identity, then $W(S)$ has a left invariant mean iff S has the following fixed point property:*

(WF) *For any separately continuous and weakly right uniformly continuous affine action* $S \times K \to K$ *of* S *on a compact convex subset* K *of a separated locally convex space,* K *contains a common fixed point of* S.

$A(S)$ *has a left invariant mean iff* S *has the fixed point property*:

(SF) *For any separately continuous and right uniformly continuous affine action* $S \times K \to K$ *of* S *on a compact convex subset* K *of a separated locally convex space,* K *contains a common fixed point of* S.

PROOF. Assume that $W(S)$ $(A(S))$ has a LIM. Let $S \times K \to K$ be a separately continuous and weakly right uniformly continuous (right uniformly continuous) affine action of S on a convex compacta K. For each $h \in AF(K)$, $x \in K$, $T_x h \in W(S)$ $(A(S))$ by Lemma 6.2. Therefore the action $S \times K \to K$ is an A-representation of the pair S, $W(S)$ $(S, A(S))$ in the sense of [1, §2]. By [1, Theorem 1] the action $S \times K \to K$ must have a common fixed point in K.

Conversely, let S have fixed point property (WF) ((SF)). Consider the affine action $S \times p(S)^\omega \to p(S)^\omega$ where $(s, V) \to r_s V$ $(S \times p(S)^\alpha \to p(S)^\alpha$ where $(s, V) \to r_s V)$. By Corollary 6.7 (Remark 6.8) this action is separately continuous and weakly right uniformly continuous (right uniformly continuous). Hence it must have a common fixed point which is also a common fixed point of the induced action $p(S)^\omega \times p(S)^\omega \to p(S)^\omega$ where $(U, V) \to UV$ $(p(S)^\alpha \times p(S)^\alpha \to p(S)^\alpha$ where $(U, V) \to UV)$. Such a fixed point is a right zero of $p(S)^\omega$ $(p(S)^\alpha)$ which gives rise to a left invariant mean on $W(S)$ $(A(S))$ by Theorem 2.1 (its analogue for $A(S)$).

In Lau [12, Theorem 3.4], it was proved that $A(S)$ has a LIM iff S has the following fixed point property

(F) For any separately continuous equicontinuous affine action of S on a compact convex subset K of a separated locally convex space, K has a common fixed point for S.

Since every equicontinuous (and separately continuous) affine action is always right uniformly continuous (Lemma 6.3), it follows from Theorem 6.9 that if $A(S)$ has a LIM, then S has fixed point property (F). Hence Theorem 6.9 extends [12, Theorem 3.4, necessity condition]. This is only a partial extension since seimgroups in [12] need not have identity. However the two fixed point properties (F) and (SF) are in fact equivalent.

For groups, Theorem 6.9 sharpens the famous Kakutani's fixed point theorem [4, p. 457].

Now let LUC(S) (WLUC(S)) be the space of all left uniformly continuous (weakly left uniformly continuous) functions on S. That is $f \in$ LUC(S) (WLUC(S)) iff $f \in C(S)$ and the map $s \to l_s f$ of S into $C(S)$ is norm (weakly) continuous (see [16, §§3 and 4]). Mitchell proved in [16] that LUC(S) (WLUC(S)) has a LIM iff S has fixed point property (P_2) (respectively (P_4)) defined below.

(P_2) For any *jointly* continuous affine action of S on a compact convex subset K of a separated locally convex space, K has a common fixed point for S.

(P_4) For any *separately* continuous affine action of S on a compact convex subset K of a separated locally convex space, K has a common fixed point for S.

By [3, Theorem 2.7, left-handed version], $A(S) \subset$ LUC(S) and $W(S) \subset$ WLUC(S). Also each right uniformly continuous separately continuous affine action is jointly continuous (Lemma 6.3). As a result (P_2) $\Rightarrow$ (SF) and clearly (P_4) $\Rightarrow$ (WF). Therefore Theorem 6.9 fits in nicely into the pattern of results in [16, Theorem 3.2 and Theorem 4.4]. (Mitchell also introduced fixed point properties (P_1) and (P_3) in [16] arising from the study of MLIM.)

It should be remarked that we can also consider multiplicative left invariant means on $W(S)$ ($A(S)$) and characterise their existence in terms of fixed point properties of similar types of continuous actions (no longer affine) on a *compact Hausdorff* space. Naturally we have to employ the weakly almost periodic compactification S^ω (respectively, the almost periodic compactification S^α). The whole business is no more than a routine carry over. We omit the details.

7. **Conclusion and comments.** As a summary, we gather a list of characterisations of the existence of a left invariant mean on $W(S)$ in terms of the semigroups $p(S)$, $S_1 = \{T_\theta\colon \theta \in p(S)\}$, S^ω, $p(S)^\omega$ and fixed point properties. Those concerning S^ω are actually due to deLeeuw and Glicksberg [3].

THEOREM 7.1. *Let S be a topological semigroup with identity, the following conditions are all equivalent.*

(1) *$W(S)$ has a LIM.*

(2) *$C_{W(S)}(S_1)$ has a MLIM or LIM.*

(3) *W has a MLIM or LIM.*

(4) *$C(p(S)^\omega)$ has a MLIM or LIM.*

(5) *$p(S)^\omega$ has a unique minimal right ideal.*

(6) $E_1E_2 = E_2$ *for any* $E_1, E_2 \in K(p(S)^\omega)$.
(7) $p(S)^\omega$ *has a right zero.*
(8) $C(S^\omega)$ *has a LIM.*
(9) S^ω *has a unique minimal right ideal.*
(10) $E_1E_2 = E_2$ *for any projections* $E_1, E_2 \in K(S^\omega)$.
(11) S *has fixed point property* (WF).
(12) $p(S)^\omega$ *has fixed point property* (WF).

PROOF. (2), (3) and (4) are equivalent by Corollary 4.3. (1), (4) and (7) are equivalent by Theorem 2.1. (2), (5) and (6) are equivalent by [3, Theorem 4.10] applied to the weakly almost periodic semigroup of operators $S_1 = \{T_\theta: \theta \in p(S)\}$ in $B = W(S)$ and the fact that $K(p(S)^\omega)$ consists entirely of projections [3, Theorem 7.2]. Also (8) and (9) are equivalent by [3, Lemma 2.8] while (9) and (10) are equivalent by [3, Corollary 2.4]. By [3, Theorem 5.3], (1) $\Longleftrightarrow$ (8). Finally (1), (11) and (12) are equivalent by Theorem 6.9 applied to both S and $p(S)^\omega$ (recall that $C(p(S)^\omega) = W(p(S)^\omega)$).

Let S be a topological semigroup (with identity) such that $m(S)$ has a LIM, then the space $m(p(S))$ need not have a MLIM [15, p. 126]. In fact it is not clear whether $C(p(S))$ can have a MLIM ($p(S)$ regarded as having the weak topology of $l_1(S)$). However, in view of Theorem 2.1, $C(p(S)^\omega)$ (which is isomorphic to the subalgebra W of $C(p(S))$) does have a MLIM and it is enough to assume that $W(S)$ (instead of $m(S)$) has a LIM. In this sense, the compact semigroup $p(S)^\omega$ is a more suitable tool than $p(S)$, in studying the algebra $W(S)$ as far as invariant means are concerned.

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