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INTERPOLATION POLYNOMIALS WHICH GIVE BEST ORDER OF APPROXIMATION AMONG CONTINUOUSLY DIFFERENTIABLE FUNCTIONS OF ARBITRARY FIXED ORDER ON $[-1, +1]$

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Dedicated to Professor P. Turán

ABSTRACT. The object of this paper is to show that there exists a polynomial $P_n(x)$ of degree $\leq 2n-1$ which interpolates a given function exactly at the zeros of nth Tchebycheff polynomial and for which $\|f - P_n\| \leq C_k w_k(1/n, f)$ where $w_k(1/n, f)$ is the modulus of continuity of f of kth order.

1. **Introduction.** The classical theorems of D. Jackson extend the Weierstrass approximation theorem by giving quantitative information on the degree of approximation to a continuous function in terms of its smoothness. Specifically, Jackson proved

THEOREM 1. *Let $f(x)$ be continuous for $|x| \leq 1$ and have modulus of continuity $w(t)$. Then there exists a polynomial $P(x)$ of degree n at most such that $|f(x) - P(x)| \leq Aw(1/n)$ for $|x| \leq 1$, where A is a positive numerical constant.*

In 1951 S. B. Stečkin [3] made an important generalization of the Jackson theorem.

THEOREM 2 (S. B. STEČKIN). *Let k be a positive integer; then there exists a positive constant C_k such that for every $f \in C[-1, +1]$ we can find an algebraic polynomial $P_n(x)$ of degree n so that*

$$\|f - P_n\| \leq C_k w_k(1/n, f), \tag{1.1}$$

where $w_k(1/n, f)$ is the modulus of continuity of $f(x)$ of kth order.

During personal conversation in Poznań, S. B. Stečkin raised the following

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question concerning Theorem 2: Does there exist an algebraic polynomial $P_n(x)$ of degree Cn $(C > 1)$ interpolating at n points and which satisfies (1.1)? The case $k = 1$ was previously raised by P. L. Butzer [1] and solved consequently by G. Freud [2]. The object of this paper is to prove that an algebraic interpolatory polynomial satisfying (1.1) does exist. The approach we have adapted is to modify the classical Hermite-Fejér interpolation polynomials on the Tchebycheff nodes. Moreover the degree of the new interpolation process is still $2n - 1$. It may be interesting to point out that recently the author [5], [6] has solved the problem of obtaining a trigonometric polynomial which interpolates a given 2π periodic continuous function at $x_k = 2k\pi/n$, $k = 0, 1, \cdots, n-1$, and for which (1.1) is also true.

2. It is well known that the Hermite-Fejér interpolation polynomial of degree $\leqslant 2n - 1$ is defined by

$$H_n[f, x] = \sum_{k=1}^{n} f(x_{kn})(1 - xx_{kn})\left(\frac{T_n(x)}{n(x - x_{kn})}\right)^2 \tag{2.1}$$

where

$$x_{kn} = \cos((2k-1)\pi/2n), \qquad k = 1, 2, \cdots, n, \tag{2.2}$$

are the zeros of Tchebycheff polynomial $T_n(x) = \cos(n \text{ arc} \cos x)$. Let us express $H_n[f, x]$ as a linear combination of $T_0(x), T_1(x), \cdots, T_{2n-1}(x)$. For this purpose we define

$$C_0(f) = \frac{1}{n}\sum_{k=1}^{n} f(x_{kn}), \qquad C_j(f) = \frac{2}{n}\sum_{k=1}^{n} f(x_{kn})T_j(x_{kn}) \tag{2.3}$$

for $j = 1, 2, \cdots, 2n - 1$. A simple computation shows that

$$H_n[f, x] = \sum_{j=0}^{2n-1} C_j(f)\left(\frac{2n-j}{2n}\right)T_j(x).$$

This representation of $H_n[f, x]$ suggests the definition

$$R_n(f) = R_n[f, x] = \sum_{j=0}^{2n-1} C_j(f)\alpha_{j,M}T_j(x), \tag{2.4}$$

where M is an arbitrary fixed positive integer and

$$\alpha_{0,M} = 1, \quad \alpha_{j,M} + \alpha_{2n-j,M} = 1, \quad j = 1, 2, \cdots, n, \quad \alpha_{j,M} = 0, \quad j > 2n. \tag{2.5}$$

A simple example of $\alpha_{j,M}$ satisfying (2.5) is given by

$$\begin{aligned} \alpha_{j,M} &= (2n-j)^M/((2n-j)^M + j^M), & j &= 0, 1, \cdots, 2n-1, \\ &= 0, & j &\geqslant 2n. \end{aligned} \tag{2.6}$$

For our purpose we make further restrictions on $\alpha_{j,M}$. We denote

$$\text{(2.7)}\qquad \begin{aligned} \mu_{j,M} &= (1-\alpha_{j,M})/j^M, \qquad j = 1, 2, \cdots, 2n,\\ &= 0, \qquad\qquad\qquad\quad j = 0. \end{aligned}$$

Let us suppose that

$$\text{(2.8)}\qquad |\mu_{j+1,M} - \mu_{j,M}| = O(1/n^{M+1}), \qquad j = 1, \cdots, 2n-1,$$

and

$$\text{(2.9)}\qquad |\mu_{j+1,M} - 2\mu_{j,M} + \mu_{j-1,M}| = O(1/n^{M+2}), \qquad j = 1, \cdots, 2n-1.$$

We also require that

$$\text{(2.10)}\qquad 1 - \alpha_{1,M} = O(1/n^M),$$

$$\text{(2.11)}\qquad |\alpha_{j+1,M} - 2\alpha_{j,M} + \alpha_{j-1,M}| = O(1/n^2), \qquad j = 1, \cdots, 2n-1.$$

We now state our main theorem.

THEOREM 3. *Let $f(x)$ be continuous for $|x| \leqslant 1$. Then $R_n(f)$ as defined by* (2.4) *and* (2.5) *satisfy*

$$\text{(2.12)}\qquad R_n[f, x_{in}] = f(x_{in}), \qquad i = 1, 2, \cdots, n,$$

and

$$\text{(2.13)}\qquad R_n[1, x] = 1.$$

Moreover under the assumptions (2.7)–(2.11) *we have* ($f \neq$ *polynomial of degree* $\leqslant m-1$)

$$\text{(2.14)}\qquad \|R_n(f) - f\| \leqslant C_M w_{M-1}(1/n, f).$$

It is easy to verify that the choice of $\alpha_{j,M}$ given by (2.6) satisfies all the requirements needed in Theorem 3.

PROOF OF THEOREM 3. First we will prove that $R_n(f)$ as defined by (2.4) and (2.5) is an interpolation polynomial in x of degree $\leqslant 2n-1$ satisfying (2.12). For this purpose we express

$$\text{(3.1)}\qquad R_n[f, x] = \sum_{j=0}^{n-1} C_j(f)\alpha_{j,M}T_j(x) + \sum_{j=n+1}^{2n-1} C_j(f)\alpha_{j,M}T_j(x).$$

In view of the fact that

$$\text{(3.2)}\qquad T_{2n-j}(x_{in}) = -T_j(x_{in}), \qquad C_{2n-j}(f) = -C_j(f), \qquad j = 1, \cdots, n,$$

we obtain, on using (2.5) and (3.1),

$$R_n[f, x_{in}] = C_0(f) + \sum_{j=1}^{n-1} C_j(f)T_j(x_{in}).$$

From the definition of $C_j(f)$ as given in (2.3) it follows that

$$C_0(f) + \sum_{j=1}^{n-1} C_j(f)T_j(x_{in}) = f(x_{in}), \qquad i = 1, \cdots, n.$$

Therefore we obtain $R_n[f, x_{in}] = f(x_{in})$, $i = 1, \cdots, n$. This proves (2.12).

(2.13) is an immediate consequence of (2.3). For if $f(x) \equiv 1$ then $C_0(f) = 1$, $C_j(f) = 0$, $j = 1, \cdots, 2n-1$.

Next we hope to prove the existence of a positive constant L independent of n and x such that

$$\|R_n[f]\| \leqslant L\|f\|. \tag{3.3}$$

To prove this we need some preliminary notation and estimates. We denote the Fejér kernel by

$$\begin{aligned} t_j(\theta) &= 1 + \frac{2}{j}\sum_{i=1}^{j-1} (j-i)\cos i\theta, \qquad j = 2, 3, \cdots, \\ t_1(\theta) &\equiv 1. \end{aligned} \tag{3.4a}$$

Associated with this kernel we introduce

$$\tau_{j,k}(\theta) = \tfrac{1}{2}(t_j(\theta + \theta_{kn}) + t_j(\theta - \theta_{kn})). \tag{3.4}$$

It is easy to verify that

$$(j+1)\tau_{j+1,k}(\theta) - 2j\tau_{j,k}(\theta) + (j-1)\tau_{j-1,k}(\theta) = 2\cos j\theta \cos j\theta_{kn}, \tag{3.5}$$

and

$$\sum_{k=1}^{n} |\tau_{j,k}(\theta)| = n, \qquad j = 1, 2, \cdots. \tag{3.6}$$

From (2.3) and (2.4) it follows that

$$R_n[f, x] = \sum_{k=1}^{n} f(x_{kn})P_{kn}(x), \tag{3.7}$$

where

$$P_{kn}(x) = \frac{1}{n}\left[1 + 2\sum_{j=1}^{2n-1} \alpha_{j,M}T_j(x_{kn})T_j(x)\right]. \tag{3.8}$$

Now we prove that

$$\sum_{k=1}^{n} |P_{kn}(x)| \leqslant L, \tag{3.9}$$

from which (3.3) follows on using (3.7). For this purpose we express (3.8) in terms of $\tau_{j,k}(\theta)$ as defined in (3.9). On using (3.5) we obtain

$$P_{kn}(x) = \frac{1}{n}\sum_{l=1}^{2n-1} (\alpha_{l+1,M} - 2\alpha_{l,M} + \alpha_{l-1,M})l\tau_{l,k}(\theta) + \tau_{2n,k}(\theta)\alpha_{2n-1,M}.$$

On using (2.10), (2.11) and (3.6), (3.9) follows immediately. This proves (3.9) and in turn (3.3). By proving (3.3) it follows easily that $R_n[f, x]$ converges uniformly to $f(x)$ on $[-1, +1]$ for every continuous function on $[-1, +1]$. Here our aim is to obtain error estimates (2.14). The proof of (2.14) is based on following [4]

THEOREM 4 [S. B. STEČKIN]. *Let P be a natural number and U_n ($n = 1, 2, \cdots$) be a linear method of approximation of functions having the following properties*:

(i) *for any function* $\phi(\theta) \in C_{2\pi}$, $\|U_n(\phi)\| \leqslant L_0\|\phi\|$;

(ii) *for any function* $\phi(\theta) \in C_{2\pi}$ *for which* $\phi^{(P)}(\theta) \in C_{2\pi}$, $\|\phi - U_n(\phi)\| \leqslant L_P\|\phi^{(P)}\|/n^P$, $n = 1, 2, \cdots$. *Then for any function* $\phi(\theta) \in C_{2\pi}$ *we have* $\|\phi - U_n(\phi)\| \leqslant B_P(L_0 + L_P)w_P(1/n, \phi)$.

First let us choose U_n to be typical means of fourier series given by

$$(3.10)\quad X_{2n,M}(\phi, \theta) = \frac{1}{2}a_0 + \sum_{j=1}^{2n-1} (a_j \cos j\theta + b_j \sin j\theta)(1 - j^M/(2n)^M),$$

where a_j's, b_j's are fourier coefficients of $\phi(\theta)$. From a theorem of A. Zygmund [7] it follows that conditions (i) and (ii) are satisfied for $P = M - 1$ $(M > 1)$ and, therefore, we conclude from Stečkin's theorem that

$$(3.11)\quad |X_{2n,M}(\phi, \theta) - \phi(\theta)| \leqslant C_M w_{M-1}(1/n, \phi).$$

Theorem 4 and (3.11) described above lead to a simple proof of (2.14). The representation of $R_n[f, x]$ as given by (3.7) suggests that we consider a trigonometric polynomial

$$(3.12)\quad A_n[\phi, \theta] = \sum_{k=1}^{n} \phi(\theta_{kn})\left\{\frac{1}{n} + \frac{2}{n}\sum_{j=1}^{2n-1} \alpha_{j,M} \cos j(\theta - \theta_{kn})\right\}$$

where

$$(3.13)\quad \phi(\theta) = f(\cos\theta) \equiv f(x).$$

From (3.12) it follows that

$$(3.14)\quad A_n[1, \theta] \equiv 1,$$

$$(3.15)\quad A_n[\cos iu, \theta] - \cos i\theta = -\alpha_{2n-i}(\cos i\theta + \cos(2n - i)\theta],$$

for $i = 1, 2, \cdots, 2n - 1$. From (3.10); (3.12)–(3.15) it follows that

$$\begin{aligned} A_n[X_{2n,M}(t), \theta] - X_{2n,M}(\theta) \\ &= -(1 + \cos 2n\theta) \sum_{i=1}^{2n-1} a_i \cos i\theta\alpha_{2n-i}(1 - i^M/(2n)^M) \\ &\quad + \sin 2n\theta \sum_{i=1}^{2n-1} a_i \sin i\theta\alpha_{2n-i}(1 - i^M/(2n)^M). \end{aligned}$$

Since $\phi(\theta)$ is an even function of θ, its fourier coefficients b_i are all zero. Therefore we obtain

$$A_n[X_{2n,M}(t),\theta] - X_{2n,M}(\theta)$$
$$= -(1+\cos 2n\theta)\sum_{i=1}^{2n-1}(a_i\cos i\theta - b_i\sin i\theta)\alpha_{2n-i}(1-i^M/(2n)^M)$$
$$+\sin 2n\theta\sum_{i=1}^{2n-1}(b_i\cos i\theta - a_i\sin i\theta)\alpha_{2n-i}(1-i^M/(2n)^M).$$

On using an integral representation of fourier coefficients we obtain

$$X_{2n,M}(\theta) - A_n[X_{2n,M}(t),\theta]$$

$$(3.16)\qquad = \frac{(1+\cos 2n\theta)}{\pi}\sum_{i=1}^{2n-1} i\delta_{i,M}\int_0^{2\pi}\phi(u)\cos i(u-\theta)\,du$$
$$+\frac{\sin 2n\theta}{\pi}\sum_{i=1}^{2n-1} i\delta_{i,M}\int_0^{2\pi}\phi(u)\sin i(u-\theta)\,du,$$

where

$$(3.17)\qquad i\delta_{i,M} = \alpha_{2n-i,M}(1-i^M/(2n)^M);$$

we put

$$(3.18)\qquad F(\theta) = \frac{1}{\pi}\sum_{i=1}^{2n-1}\delta_{i,M}\int_0^{2\pi}\phi(u+\theta)\sin iu\,du,$$

and rewrite

$$(3.19)\quad X_{2n,M}(\theta) - A_n[X_{2n,M}(f),\theta] = (1+\cos 2n\theta)F'(\theta) - \sin 2n\theta\widetilde{F}'(\theta).$$

Now, we need to obtain estimates of $F'(\theta)$ and $\widetilde{F}'(\theta)$. For this purpose we assume that $\phi(\theta)$ is $(M-1)$ times continuously differentiable of function of θ. Integrating by parts $(M-1)$ times we obtain after elementary calculation that

$$(3.20)\qquad F(\theta) = \frac{(-1)^{M/2+1}}{\pi}\sum_{i=1}^{2n-1}\lambda_{i,M}\int_0^{2\pi}\phi^{(M-1)}(u+\theta)\cos iu\,du$$

for M even integer, where

$$(3.21)\qquad i^{M-1}\lambda_{i,M} = \delta_{i,M}.$$

From (3.4a) we obtain

$$2\sum_{i=1}^{2n-1}\lambda_{i,M}\cos iu$$
$$= \sum_{i=1}^{2n-1}(\lambda_{i-1,M} - 2\lambda_{i,M} + \lambda_{i+1,M})it_i(u) + 2n\lambda_{2n-1,M}t_{2n}(u).$$

Therefore on using (3.20) and (3.21) we obtain

$$\begin{aligned} F(\theta) = {} & \frac{(-1)^{M/2+1}}{2\pi}\int_0^{2\pi} \phi^{(M-1)}(u+\theta) \\ (3.22) \qquad & \times \left\{\sum_{i=1}^{2n-1} (\lambda_{i,M} - 2\lambda_{i,M} + \lambda_{i+1,M}) it_i(u) + 2n\lambda_{2n-1}t_{2n}(u)\right\} du. \end{aligned}$$

From (2.7), (3.17) and (3.21) it follows that

$$|\lambda_{i-1,M} - 2\lambda_{i,M} + \lambda_{i+1,M}| = O(1/n^{M+2}), \qquad i = 1, \cdots, 2n-1,$$
$$\lambda_{2n-1} = O(1/n^{M+1}).$$

From (3.4a) we obtain $\int_0^{2\pi} |t_i(u)|\, du = 2\pi$. On putting these estimates in (3.22) we obtain $|F(\theta)| \leqslant c_M \|\phi^{(M-1)}\|/n^M$. From (3.18) it follows that $F(\theta)$ is a trigonometric polynomial of order $\leqslant 2n$. On using a well-known theorem of S. N. Bernstein (see Zygmund [8, volume 1, p. 118]) we obtain

$$|F'(\theta)| \leqslant 2c_M \|\phi^{(M-1)}\|/n^{M-1}, \qquad |\widetilde{F}'(\theta)| \leqslant 2c_\phi \|\phi^{(M-1)}\|/n^{M-1}.$$

Therefore, under the assumption that $\phi^{(M-1)}(\theta) \in c_{2\pi}$ we obtain

$$(3.23) \qquad |X_{2n,M}(\theta) - A_n[X_{2n}(t), \theta]| \leqslant 6c_M \|\phi^{M-1}\|/n^{M-1}.$$

Following a proof similar to that given for (3.3) it follows that for every $\phi \in c_{2\pi}$ we have

$$(3.24) \qquad |A_n[\phi, \theta]| \leqslant B_M \|\phi\|.$$

Now we claim that for every $\phi^{(M-1)} \in c_{2\pi}$ we have

$$(3.25) \qquad |\phi(\theta) - A_n[\phi, \theta]| \leqslant B_M \|\phi^{(M-1)}\|/n^{M-1}.$$

This follows from

$$\begin{aligned} \phi(\theta) - A_n[\phi, \theta] = {} & \phi(\theta) - X_{2n,M}(\theta) + X_{2n,M}(\theta) - A_n[X_{2n,M}(t), \theta] \\ & + A_n[X_{2n,M}(t), \theta] - A_n[\phi, \theta], \end{aligned}$$

(3.23), (3.24) and (3.11). This proves (3.25).

(3.24) and (3.25) enable us to apply Stečkin Theorem 4 and we conclude that $|\phi(\theta) - A_n[\phi, \theta]| \leqslant B_M w_{M-1}(1/n, \phi)$.

But this inequality implies

$$|\phi(\theta) - \tfrac{1}{2}(A_n[\phi, \theta] + A_n[\phi, -\theta])| \leqslant B_M w_{M-1}(1/n, \phi).$$

Since $\phi(\theta) = f(\cos\theta) \equiv f(x)$ and $R_n[f, x] = \tfrac{1}{2}A_n[\phi, \theta] + A_n[\phi, -\theta]$, we obtain

$$(3.26) \qquad |f(x) - R_n[f, x]| \leqslant B_M w_{M-1}(1/n, \phi).$$

It is well known that $w_{M-1}(1/n, \phi) \leqslant c_M w_{M-1}(1/n, f)$. We obtain from (3.26) that

$$|f(x) - R_n[f, x]| \leqslant e_M w_{M-1}(1/n, f).$$

This proves (2.14) and thus completes the proof of Theorem 3 for M even integer. For M-odd positive integer > 1 a similar proof can be given.

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