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RIGHT-BOUNDED FACTORS IN AN LCM DOMAIN[1]

BY

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ABSTRACT. A right-bounded factor is an element in a ring that generates a right ideal which contains a nonzero two-sided ideal. Right-bounded factors in an LCM domain are considered as a generalization of the theory of two-sided bounded factors in an atomic 2-fir, that is, a weak Bezout domain satisfying the acc and dcc for left factors. Although some elementary properties are valid in a more general context most of the main results are obtained for an LCM domain satisfying (M) and the dcc for left factors; the condition (M) is imposed to insure that prime factorizations are unique in an appropriate sense. The right bound b^* of a right bounded element b is considered in general, then in case b is a prime, and finally in case b is indecomposable. The effect of assuming that right bounds are two-sided is also considered.

0. Introduction. The theory of bounded factors in a principal ideal domain is well established [11]. More recently, this was generalized to 2-firs (i.e. weak Bezout domains) satisfying the acc and dcc for left factors [6]. Our purpose here is twofold: (i) to study *right*-bounded factors, and (ii) to carry this out in the more general context of right LCM domains (intersection of any two principal right ideals is principal), a class of rings which was described in [2] and [3].

It was shown in [2] that for right LCM domains satisfying an additional mild hypothesis (M) factorization into primes is unique up to order of factors and projective factors. In §1 we collect this and other related facts that will be needed. Right-bounded factors in a ring R are considered in §2; their right bounds exist if R is a complete right LCM domain (intersection of any collection of principal right ideals is principal). In §3 we consider right-bounded primes. The right bound p^* of a prime p is described in some detail. For example, it is shown that if R is an LCM domain satisfying (M) and the dcc for left factors then p^* can be factored into primes that are projective (in fact transposed) to p. The possibility of factoring p^* which is right invariant into further right invariant factors is also discussed. In §4 we consider right bounded elements

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that are indecomposable; their right bounds are shown to be indecomposable in an appropriate sense if R is an LCM domain satisfying (M) and having the acc and dcc for left factors. In §5 we show how the results in the preceding sections can be improved if right bounds are assumed to be two-sided.

1. Preliminaries. All rings considered here have no proper divisors of zero and have a unity. A ring R in which the intersection of any two principal right ideals is again principal is a *right LCM domain.* In contrast, a 2-fir is a right LCM domain in which the sum of two principal right ideals having nonzero intersection is also principal. The most immediate example of a right LCM domain that is not a 2-fir is the ring of polynomials in (more than one) commuting indeterminate over a field. Additional examples of right LCM domains that need not be 2-firs can be found by considering rings of formal power series over a principal right ideal domain (cf. [7, Theorem 9]). In this section we gather the prerequisite facts related to right LCM domains some of which may be found in [2]. In particular, any proofs that are omitted below are given in [2].

If R is a right LCM domain and if $aR \cap bR \neq 0$ then the least common right multiple, $[a, b]_r$, of a and b exists and in fact generates $aR \cap bR$. The highest common left factor, $(a, b)_l$, need not exist. However, if $ab' = ba' \neq 0$ then $(a', b')_r$ does exist and satisfies

(1) $$ab' = ba' = [a, b]_r(a', b')_r.$$

In a left LCM domain the left-right analogue of (1) is

(2) $$ab' = ba' = (a, b)_l[a', b']_l.$$

In particular, if R is an *LCM domain* (i.e. both right and left) then $(a, b)_l$ exists whenever $aR \cap bR \neq 0$. In this case the set $[xR, R] = \{yR | xR \subseteq yR \subseteq R\}$ which is partially ordered by inclusion will be a lattice whenever $x \neq 0$; for this reason we occasionally write $aR \vee bR$ for $(a, b)_l R$.

If $[a, b]_r$ exists then $x[a, b]_r = [xa, xb]_r$ for any $x \neq 0$; the corresponding result for the greatest common left divisor is obtained by applying (2) to the equation $xab' = xba'$. We summarize in the following.

PROPOSITION 1.1. *Let R be an LCM domain. If $aR \cap bR \neq 0$ and x is any nonzero element of R then*

(i) $x[a, b]_r = [xa, xb]_r$,

(ii) $x(a, b)_l = (xa, xb)_l$.

A ring in which the intersection of any family of principal right ideals is again principal will be called a *complete* right LCM domain. In this case the poset $[xR, R]$ $(x \neq 0)$ is a complete semilattice and hence a lattice. In addition, the least common right multiple and highest common left factor of any family of elements a_i exist provided $\bigcap a_iR \neq 0$.

For two elements a, a' in a ring R we define $a \operatorname{tr} a'$ if there is a relation $ab' = ba'$ in which $(a, b)_l = 1$ and $[a, b]_r = ba'$; in this situation a and a' are said to be *transposed* since the posets $[aR, R]$ and $[ba'R, bR]$ $(\cong [a'R, R])$ are transposed intervals. If R is commutative then $a \operatorname{tr} a'$ is equivalent to $aR = a'R$, and if R is a 2-fir then $a \operatorname{tr} a'$ is equivalent to $R/aR \cong R/a'R$, i.e. a and a' are similar (cf. [8] where a 2-fir is called a weak Bezout domain).

PROPOSITION 1.2. *In an LCM domain the relation* tr *is transitive.*

PROOF. Let $a \operatorname{tr} a'$ and $a' \operatorname{tr} a''$. Thus $ab' = ba' = [a, b]_r$ and $a'c' = ca'' = [a', c]_r$ with $(a, b)_l = (a', c)_l = 1$. Using Proposition 1.1 we have $(ab', bc)_l = b$ and putting this together with $(a, b)_l = 1$ we obtain $(a, bc)_l = 1$. Considering the relation $a(b'c') = (bc)a''$, we shall have shown $a \operatorname{tr} a''$ once we show that $[a, bc]_r = (bc)a''$. Now $a'R \cap cR = ca''R$ and so $ba'R \cap bcR = bca''R$; replacing $ba'R$ by $aR \cap bR$ in the last equation we obtain $aR \cap bcR = bca''R$ as desired.

In an LCM domain the relation tr is left-right symmetric because of (1) and (2); however it is not a symmetric relation (see Example 2.9 below). We therefore define a and a' to be *projective*, and write $a \operatorname{pr} a'$, if there is a sequence $a = a_1, a_2, \cdots, a_n = a'$ in R in which either $a_i \operatorname{tr} a_{i+1}$ or $a_{i+1} \operatorname{tr} a_i$. Projectivity is an equivalence relation in any ring and is left-right symmetric in an LCM domain.

By a *prime* (= atom) we understand a nonunit $p \neq 0$ in a ring R that has no proper factors; this is equivalent to $[pR, R] = \{pR, R\}$. Maximal finite chains in $[aR, R]$ correspond to complete factorizations of a into primes. We shall say that a ring R has the *acc* (*dcc*) *for left factors* if the poset $[aR, R]$ of "left factors" of a has the acc (dcc) for each $a \neq 0$ in R; one also says that R has the *acc* (*restricted dcc*) *for principal right ideals.* Since the posets $[aR, R]$ and $[Ra, R]$ are dually isomorphic [4] the acc (dcc) for left factors is equivalent to the dcc (acc) for right factors. If R has the acc and dcc for left factors then each nonunit $a \neq 0$ in R has a complete factorization into primes.

Uniqueness of prime factorizations may be established if R is a right LCM domain satisfying the following condition which is left-right symmetric in an LCM domain and which automatically holds in a 2-fir:

(M) $\quad [x, y]_r = [x, yz]_r, (x, y)_l = (x, yz)_l$ implies z is a unit.

We note in passing that if R is either an LCM domain or a complete right LCM domain then the lattices $[aR, R]$ are modular precisely when R satisfies (M). The following uniqueness theorem is proved in [2].

THEOREM 1.3. *Let* R *be a right LCM domain satisfying* (M). *If* $p_1 p_2 \cdots p_n = q_1 q_2 \cdots q_m$ *where* p_i *and* q_i *are primes then* $n = m$ *and* $p_i \operatorname{pr} q_{\pi(i)}$ *for some permutation* π *of the subscripts.*

The proof of Theorem 1.3 uses the fact that the relation tr preserves primes as well as units. Although these properties follow from the fact that transposed intervals are isomorphic if R is an LCM or a complete right LCM domain satisfying (M) they can be obtained more generally. Since the proofs are not given in [2] we include them here.

PROPOSITION 1.4. *Let* R *be a ring in which* $a \operatorname{tr} a'$.

(i) a *is a unit if and only if* a' *is a unit.*

(ii) *Let* R *satisfy* (M); *if* a *is a prime then* a' *is prime, and the converse holds if, in addition, R is a right LCM domain.*

PROOF. The proof of (i) is easy and shall be omitted. To prove (ii) let $ab' = ba' = [a, b]_r$ with $(a, b)_l = 1$. Assume that a is prime and let $a' = zy$. Then $aR \cap bR = bzyR = aR \cap bzR$. We consider two cases. First, if $aR \subseteq dR$ and $bzR \subseteq dR$ for some nonunit d; then $aR = dR$ since a is prime; thus $bzR \subseteq aR \cap bR = bzyR$ which implies y is a unit. On the other hand if a and bz do not have a common left factor other than units then $(a, bz)_l = 1$; thus $(a, b)_l = 1$ also, and we may apply (M) and conclude that z is a unit. This shows that a' has no proper factors and is therefore prime.

To prove the converse let $a = zy$. Then $(z, b)_l = 1$ because $(zy, b)_l = 1$. Now $zyR \cap bR \subseteq zR \cap bR$ and we again consider two cases. First if the containment is equality then y is a unit by (M). In the other case we may put $zR \cap bR = bz'R$ (since R is a right LCM domain) so that $ba'R \subsetneq bz'R$. Since a' is prime we must have z' a unit, and since $z \operatorname{tr} z'$, z must also be a unit. Thus a has no proper factors and is therefore prime.

We shall also need the following result (cf. [2, Theorem 5]).

THEOREM 1.5. *Let* R *be a right LCM domain satisfying* (M) *and let* $a \operatorname{tr} a'$. *If* $a = a_1 a_2$ *then* $a' = a_1' a_2'$ *where* $a_i \operatorname{tr} a_i'$.

If R is a right LCM domain satisfying (M) then because of Theorem 1.3 we may define the *dimension* of a nonzero element a in R, $\dim(a)$, to be n if a is the product of n primes and ∞ otherwise. Thus $\dim(a) = 0$ if a is a unit and $\dim(a) = 1$ if a is prime. Using Theorem 1.5 we see that if $a \operatorname{tr} a'$ then $\dim(a) = \dim(a')$.

We conclude this section with the following proposition which will be needed later.

PROPOSITION 1.6. (i) *Let* R *be a right LCM domain. If* q *is a prime* *left factor of* ab *then either* q *is a left factor of* a *or* $q \operatorname{tr} q'$ *and* q' *is a left factor of* b.

(ii) *Let* R *be an LCM domain satisfying* (M). *If* q *is a prime left factor of a product of primes* $p_1 p_2 \cdots p_n$ *then* $q \operatorname{tr} p_j$ *for some* j.

PROOF. Let q be a prime left factor of ab and suppose that q is not a left factor of a. Then $(a, q)_l = 1$; putting $[a, q]_r = aq'$ we find $q \operatorname{tr} q'$ and q' is a left factor of b.

To prove (ii) write $a = p_1, b = p_2 \cdots p_n$; letting q be a left factor of ab we find by (i) that either q is a left factor of p_1, in which case $qR = p_1 R$ (and so $q \operatorname{tr} p_1$), or $q \operatorname{tr} q'$ where q' is a left factor of $p_2 \cdots p_n$ and is also prime. The result now follows by Proposition 1.2 and induction on $\dim(p_1 \cdots p_n)$.

2. **Right-bounded factors.** A right ideal I in a ring R is said to be *bounded* if R/I is a bounded R-module, that is, if $(R/I)^r = \{x \in R | Rx \subseteq I\}$ is nonzero. A nonzero element $b \in R$ is said to be *right bounded* if bR is a bounded right ideal (cf. [10, p. 226]). We shall use the notation

$$I_b = (R/bR)^r = \{x \in R | Rx \subseteq bR\}.$$

Thus b is right bounded if and only if bR contains a nonzero two-sided ideal of R, the largest such being I_b.

If $b, c \in R$, let $I_b(c) = \{x \in R | cx \in bR\}$. Thus $I_b(c)$ is a right ideal of R and is related to I_b by

$$I_b = \bigcap_{c \in R} I_b(c). \tag{3}$$

Now $I_b(c)$ has the form $I_b(c) = b'R$ if and only if $bR \cap cR = cb'R$. In particular R is a right LCM domain if and only if each $I_b(c)$ is a principal right ideal.

Let R be a right LCM domain and let $b \in R$ be right large (i.e. bR an essential right ideal). Thus for any $c \in R$ there exists $b' \in R$ such that $I_b(c) = b'R \neq 0$. If $d = (b, c)_l$ exists then writing $b = db_1, c = dc_1$ we have $(b_1, c_1)_l = 1$ and $[b_1, c_1]_r = c_1 b'R$; thus $b_1 \operatorname{tr} b'$ where b_1 is a right factor of b. This shows that $I_b = \bigcap \{b'R | b_1 \operatorname{tr} b'$ for some right factor b_1 of $b\}$.

More generally, if $b \in R$ where R is any ring and $b_1 \operatorname{tr} b'$ for a right factor b_1 of b, say $b = ab_1$, then choosing $c_1 \in R$ such that $b_1 R \cap c_1 R = c_1 b'R$ (by the definition of $b_1 \operatorname{tr} b'$) we have $bR \cap ac_1 R = ac_1 b'R$, i.e. $b'R = I_b(ac_1) \supseteq I_b$. We have established the following.

THEOREM 2.1. *Let* b *be an element in a ring* R. *Then*

$$I_b \subseteq \bigcap\{b'R \mid b_1 \text{ tr } b' \textit{ for some right factor } b_1 \textit{ of } b\}.$$

The containment becomes equality if b *is right large and if* R *is a right LCM domain in which* $(x, y)_l$ *exists whenever* $xR \cap yR \neq 0$, *and in particular if* R *is either an LCM domain or a complete right LCM domain.*

Let R be a complete right LCM domain and let b be a right bounded element of R. According to (3), I_b has the form $I_b = b^*R$; in general, b^* is called *the right bound* of b. Rephrasing Theorem 2.1 we may describe the right bound of an element as follows.

THEOREM 2.2. *Let* R *be a complete right LCM domain and let* b *be a right-bounded element of* R. *Then* b *has right bound* b^* *given by*

$$b^*R = \bigcap\{b'R \mid b_1 \text{ tr } b' \textit{ for some right factor } b_1 \textit{ of } b\}. \tag{4}$$

From another point of view we have $b^*R = \bigcap_{c \in R} I_b(c)$ where $I_b(c) = b'R$ if and only if $bR \cap cR = cb'R$. The last equation defines the monomorphism ϕ of R-modules given by

$$\phi\colon R/b'R \longrightarrow R/bR, \quad 1 + b'R \longmapsto c + bR$$

and vice versa. Thus if R is a complete right LCM domain and $b \in R$ is right bounded then the right bound b^* of b may be described by

$$b^*R = \bigcap\{b'R \mid \exists \text{ monomorphism } \phi\colon R/b'R \longrightarrow R/bR\}.$$

However this last equation turns out to be less useful than (4).

We recall that a nonzero element $b \in R$ is *right invariant* if bR is a two-sided ideal of R (i.e. $Rb \subseteq bR$). Thus b is right invariant if and only if $b \in I_b$. If b is right invariant then *any factor of* b *is actually a left factor*, for if $b = xay$ then choosing x' such that $xb = bx'$ we find that $b = ayx'$.

Let $b \in R$ have right bound b^* as in (4). Then b is a factor of b^*, and b^* is right invariant because I_b is a two-sided ideal. Conversely, if a is right invariant and if b is a factor of a then, since b is actually a left factor of a, b will be bounded by a. We have established the following characterization of right boundedness.

THEOREM 2.3. *An element in a complete right LCM domain is right bounded if and only if it is a factor of a right invariant element.*

We note several other consequences of Theorem 2.2.

PROPOSITION 2.4. *Let* R *be a complete right LCM domain. If* a *and* b *are right invariant then so are* $[a, b]_r$ *and* $(a, b)_l$. *In particular the set of right invariant elements of* R *forms a lattice.*

PROOF. Let $m = [a, b]_r$ and $d = (a, b)_l$. Clearly m is right invariant since $mR = aR \cap bR$. Now since $aR \subseteq dR$, $bR \subseteq dR$, and a and b are right invariant we must have $aR \subseteq d^*R$, $bR \subseteq d^*R$ and consequently $dR \subseteq d^*R$. This shows that $dR = d^*R$ and so d is right invariant.

PROPOSITION 2.5. *Let R be a complete right LCM domain satisfying* (M). *If a tr a' and if a is right invariant and finite dimensional then $aR = a'R$.*

PROOF. By Theorem 2.2, $aR = a^*R \subseteq a'R$; also $\dim(a) = \dim(a')$ and since this number is finite we must have $aR = a'R$.

COROLLARY 2.6. *Let R be a complete right LCM domain satisfying* (M). *If a is right invariant and finite dimensional and if $(a, b)_l = 1$ then $aR \cap bR = baR$.*

PROOF. Let $ab' = ba'$ be a generator of $aR \cap bR$. Then a tr a' whence $a'R = aR$ and so $ba'R = baR$.

We recall that the quotient ring $RS^{-1} = \{rs^{-1} | r \in R, s \in S\}$ of R with respect to S is defined provided that S is a right Ore system in R, i.e., a submonoid of R^* (the monoid of nonzero elements of R) satisfying $bR \cap cS \neq \varnothing$ for each $b \in S$, $c \in R^*$. It is not difficult to prove that the set of all right invariant elements together with all of their factors is a right Ore system in any integral domain (cf. [5]). Hence the set of all right bounded elements in a complete right LCM domain is a right Ore system by Theorem 2.3. We can prove this more generally for any right LCM domain as follows.

THEOREM 2.7. *Let R be a right LCM domain. The set B of all right-bounded elements of R is a right Ore system in R.*

PROOF. Let $b, d \in B$ so that $0 \neq I_b \subseteq bR$ and $0 \neq I_d \subseteq dR$. We then have $0 \neq I_b I_d \subseteq bRI_d \subseteq bI_d \subseteq bdR$. It follows that $I_b I_d \subseteq I_{bd}$ and in particular $bd \in B$. Therefore B is a submonoid of R^* (and contains all of the units of R). Let $b \in B$ and $c \in R^*$. Then $0 \neq I_b \subseteq I_b(c) = b'R$ for some b'. Hence $b' \in B$, and since $cb'R = bR \cap cR$ we may choose $c' \in R$ such that $bc' = cb'$. This shows that $bR \cap cB \neq \varnothing$.

EXAMPLE 2.8. Let $R = F[x, \sigma]$ be the skew polynomial ring, where F is a commutative field, σ is a monomorphism of F into a proper subfield of itself, and multiplication in R is defined by the formula $ax = x\sigma(a)$. Clearly x and therefore each nonzero monomial is right invariant. In fact, it is not difficult to show that the set of right invariant elements of R is precisely the set of nonzero monomials (cf. [11, p. 38]). On the other hand the only left invariant elements of R are the units. For, if $f = x^n b_n + \cdots + xb_1 + b_0$ is left invariant then choosing

$a \in F \backslash \sigma[F]$ we have $fa = a'f$ for some $a' \in F$ which leads to $b_n a = \sigma^n(a')b_n$ forcing $n = 0$ (by the choice of a) and therefore $f \in F$.

Since R is a principal right ideal domain (PRI domain) it is a complete right LCM domain. It follows from Theorem 2.3 that R has no nonunit left bounded elements while the set of right-bounded elements is precisely the set of nonzero monomials.

EXAMPLE 2.9. Let F and σ be as in Example 2.8 and let $H = F[[x, \sigma]]$ be the ring of (skew) formal power series in an indeterminate x over F in which coefficients are written on the right of x and multiplication is determined by $ax = x\sigma(a)$. We extend σ to H by defining $\sigma(x) = x$. Let $R = H[[y, \sigma]]$ be the ring of formal power series in y over H in which coefficients are written on the left of y and multiplication is defined by the formula $yh = \sigma(h)y$. Thus each element of R has the form $f = \Sigma x^i a_{ij} y^j$ where $a_{ij} \in F$. Since H is a PRI domain [12] it follows that R is a right LCM domain [7, Theorem 9]; on the other hand if we view R as the ring of formal power series in x over the PLI domain $F[[y, \sigma]]$ it follows that R is a left LCM domain.

It can be shown that each lattice $[fR, R]$ $(f \neq 0)$ is of finite length. Thus R is an LCM domain having both the acc and dcc for left factors.(2) Note however that R is not a 2-fir.

We observe that $xy = yx$ is central in R since for any $a \in F$, $axy = x\sigma(a)y = xya$. Also, x is right but not left invariant and y is left but not right invariant (σ is not an epimorphism). The right bound of y is $y^* = xy$ while $x^* = x$; thus the right bound of a prime need not be the right bound of each of its prime factors. We consider the right bound of primes more closely in the next section.

Finally we show that y tr x. For, if $a \in F\backslash\sigma[F]$ then $yR \cap ayR = ayxR$ and $yR \vee ayR = R$; in fact the equation $y(xa) = (ay)x$ is left and right coprime, i.e., $(y, ay)_l = (x, xa)_r = 1$. However, it is not true that x tr y; otherwise Theorem 2.1 yields $xR \subseteq I_x \subseteq yR$ which is not possible.

3. **Right-bounded prime factors.** Let R be a right LCM domain having the dcc for left factors. Thus R is a complete right LCM domain. If $b \in R$ is right bounded then the intersection in (4) may be taken to be finite and irredundant, that is,

$$b^*R = b'_1R \cap \cdots \cap b'_nR \tag{5}$$

where b_i tr b'_i for right factors b_i of b and where no b'_iR can be omitted. Let us assume that R is an LCM domain satisfying (M) and that b is finite dimensional; thus each b'_i and therefore b^* is finite dimensional. If

(2) ADDED IN PROOF. It can be shown that R does not satisfy (M).

$mR = b'_1R \cap \cdots \cap b'_{n-1}R$ then $b^* = [m, b'_n]_r$, say $b^* = ma'_n = b'_nc$. If $d = (m, b'_n)_l$ with $m = dm'$, $b'_n = db''_n$ then b''_n tr a'_n. Since b''_n is a right factor of b'_n and b_n tr b'_n we may apply the left-right analogue of Theorem 1.5 to find a right factor a_n of b_n such that a_n tr b''_n. Thus $b^* = ma'_n$ where a_n tr a'_n (Proposition 1.2) for some right factor a_n of b. By induction on $\dim(b^*)$ we obtain

(6) $$b^* = a'_1 \cdots a'_n,\ a_i \text{ tr } a'_i \text{ for right factors } a_i \text{ of } b.$$

This result has a number of consequences.

THEOREM 3.1. *Let R be an LCM domain satisfying* (M) *and having the dcc for left factors. If b is right bounded and a product of primes then its right bound b^* is a product of primes that are transposed to prime factors of b. All prime factors of b^* are projective to prime factors of b, and if q is a prime right factor of b^* then p* tr *q for some prime factor p of b.*

PROOF. All assertions except the last one follow directly from (6), Theorem 1.3, and the left-right analogue of Theorem 1.5. Turning to the last statement let q be a prime right factor of b^*. By the left-right analogue of Proposition 1.6(ii) we have p' tr q where p' is a prime factor of some a'_j, and since a_j tr a'_j there is a prime factor p of a_j such that p tr p' (again by Theorem 1.5). Hence p tr q by Proposition 1.2 and p is a factor of b.

If p is a prime with right bound p^* then equation (4) has the form

$$p^*R = \bigcap \{p'R \mid p \text{ tr } p'\}.$$

Applying Theorem 3.1 in this case we obtain the following explicit description of p^*.

THEOREM 3.2. *Let R be an LCM domain satisfying* (M) *and the dcc for left factors. If p is a right-bounded prime in R then its right bound p^* can be factored*

(7) $$p^* = p'_1 \cdots p'_n \quad \textit{with } p \text{ tr } p'_i.$$

All prime factors of p^* are projective to p, and if q is a prime right factor of p^* then p tr *q.*

Under the hypotheses of Theorem 3.2 the right bound of a prime has finite dimension. At the end of this section we give examples of primes whose right bounds have infinite dimension. We note the following criterion for distinguishing the right bounds of primes in a 2-fir (cf. [6] for the case of a (two-sided) bounded prime).

COROLLARY 3.3. *Let R be a 2-fir having the dcc for left factors. If p and q are primes with right bounds p^* and q^* respectively then $p^*R = q^*R$ if and only if p and q are similar.*

Notwithstanding the last result we have seen in Example 2.9 that for primes p and q in a ring R, $p \operatorname{tr} q$ need not imply that $p^*R = q^*R$; we must also assume $q \operatorname{tr} p$. We now turn to the question of factoring p^* into right-invariant factors.

PROPOSITION 3.4. *Let R be an LCM domain. If q is a prime right factor of bc where b is right invariant then q is either a left factor of b or a right factor of c.*

PROOF. Applying the left-right analogue of Proposition 1.6(i) we find that either q is a right factor of c or $p \operatorname{tr} q$ for some right factor p of b. In the latter case we have $bR \subseteq I_b \subseteq qR$ by Theorem 2.1, i.e. q is a left factor of b.

THEOREM 3.5. *Let R be an LCM domain and let $p \in R$ be a prime with right bound p^*. If p^* is the right bound of some prime right factor then p^* has no proper factorization into right invariant elements.*

PROOF. Let q be a prime right factor of p^* such that $p^* = q^*$. If $p^* = bc$ where b and c are right invariant then by Proposition 3.4, q and therefore q^* is either a left factor of b or of c. Since b and c are left factors of q^* the proof is complete.

The hypothesis on p^* in Theorem 3.5 cannot be dropped as an example at the end of this section shows. Under the hypotheses of Theorem 3.2 we can show that p^* cannot be factored into the product of two right-invariant elements that are relatively prime; this is due to the fact that p is *indecomposable*, a topic which we take up in the next section. We can sharpen Theorem 3.5 a bit but first we note the following whose proof is obvious.

PROPOSITION 3.6. *Let R be a complete right LCM domain and let $a \in R$ be right invariant. Then a has no proper right-invariant factor if and only if a is the right bound of each of its factors.*

THEOREM 3.7. *Let R be a complete right LCM domain having either the acc or the dcc for left factors and let p be a prime with right bound p^*. Then p^* has no proper right-invariant factor if and only if p^* is the right bound of each of its prime factors.*

PROOF. Let p^* be the right bound of each of its prime factors and suppose that b is a factor of p^* which is right invariant; thus $p^* = bc$ for some c. Using either the acc or the dcc we may select a prime factor q of b which will be a left factor. Then $p^*R \subseteq bR \subseteq q^*R = p^*R$. The converse follows from Proposition 3.6.

COROLLARY 3.8. *Let* R *be a 2-fir having the dcc for left factors and let* p *be a right-bounded prime in* R. *Then* p^* *is the right bound of each of its prime factors; thus* p^* *has no proper right-invariant factor.*

PROOF. If q is any prime factor of p^* then q is similar to p by unique factorization and hence $p^* = q^*$ by Corollary 3.3. Thus we may apply Theorem 3.7 to complete the proof.

EXAMPLE 3.9. The following example is taken from [9]. Let $H = F(t_i)[y, \sigma]$ where $F(t_i)$ is the commutative field generated by an infinite number of indeterminates t_i over a field F, σ is the monomorphism of $F(t_i)$ defined by $\sigma(t_i) = t_{i+1}$, and multiplication in H is defined by $ay = y\sigma(a)$. Then H is a PRI domain to which σ may be extended by defining $\sigma(y) = t_1$. Thus σ maps H into $F(t_i)$. Let $R = H[x, \sigma]$ where multiplication is determined again by $hx = x\sigma(h)$. Then R is also a PRI domain [12] and consequently an LCM domain satisfying (M) which is right complete.

The ring R does not have the dcc for left factors as the following sequence of equations shows:

(8) $x = hx\sigma(h)^{-1} = h^2x\sigma(h)^{-2} = h^3x\sigma(h)^{-3} = \cdots$ (h a nonunit in H).

It can be shown just as in Example 2.8 that the right invariant elements of R are just the nonzero monomials $x^iy^ja_{ij}$ while the only left invariant elements of R are the units. The first equation in (8) shows that each nonzero element of H is bounded by x; consequently the nonzero polynomials in H that are not monomials all have right bound x. For example x is the right bound of the prime $1 + y$ but not of the prime y (which is its own right bound). Thus x is not the right bound of each of its prime factors. Since $(1 + y)^* = x$, equations (8) with $h = y$ show that the right bound of a prime can have proper factorizations into right invariant elements. In contrast to the hypothesis in Theorem 3.5 it is easy to see that x has no prime right factor; in the language of [1], x is an $\text{inf}^{(1)}$-prime (an infinite dimensional "prime"). This example also shows that the dcc is essential in Corollary 3.8.

Finally we note that if $K = HB^{-1}$ is the right quotient ring of H with respect to the set B of right bounded elements of H then K is a simple PRI domain [5]. Extending σ to K in the natural way we can then define $R = K[x, \sigma]$ rather than $H[x, \sigma]$ as above. The result is a PRI domain in which x is the right bound of every nonzero constant polynomial in x (i.e. every nonzero member of K); thus x is the right bound of each of its factors.

4. **Right-bounded indecomposable elements.** A nonzero element $b \in R$ is said to be *decomposable* if $bR = aR \cap cR$ with $(a, c)_l = 1$. This definition

is left-right symmetric in an LCM domain by equations (1) and (2). In a 2-fir this definition is equivalent to the condition that the R-module R/bR be decomposable as the direct sum of the factors R/aR and R/cR. If a, b, and c are all right invariant in the definition above then b is said to be *RI-decomposable.* As usual *indecomposable* means not decomposable.

Using Corollary 2.6 we obtain the following characterization of RI-decomposability.

PROPOSITION 4.1. *Let R be a complete right LCM domain satisfying* (M) *and let $b \in R$ be finite dimensional and right invariant. Then b is RI-decomposable if and only if $b = ac$ where a and c are right invariant with $(a, c)_l = 1$.*

In proving the main result of this section we use the following.

LEMMA 4.2. *Let R be an LCM domain which is right complete. If a is right invariant and a product of primes and if $(a, c)_l = 1$ then $(a, b)_l = (a, bc)_l$ for any $b \in R$.*

PROOF. First we assume that $(a, b)_l = 1$. If $(a, bc)_l R \subseteq pR$ where p is prime then p cannot be a left factor of b. Consequently $p \operatorname{tr} q$ for some left factor q of c by Proposition 1.6(i). Thus $aR \subseteq p^*R \subseteq qR$ and this contradicts $(a, c)_l = 1$. In general let $d = (a, b)_1$ with $a = da_1, b = db_1$ so that $(a_1, b_1)_l = 1$. Clearly $(a_1, c)_l = 1$ since a_1 must be a left factor of the right-invariant element a. Applying the first case we have $(a_1, b_1c)_l = 1$; multiplying this on the left by d (Proposition 1.1) we obtain the desired result.

THEOREM 4.3. *Let R be an LCM domain satisfying* (M) *and the dcc for left factors. Let b be finite dimensional and right bounded. If b is indecomposable then its right bound b^* is RI-indecomposable.*

PROOF. The dcc assures that $\dim(b^*)$ is finite (cf. Theorem 3.1). Suppose $b^*R = aR \cap cR$ where a and c are right invariant and $(a, c)_l = 1$; thus $b^*R = acR = caR$. We claim that

$$bR = (a, b)_l R \cap (c, b)_l R. \tag{9}$$

First we show that $(a, b)_l RcR \subseteq bR$. For,

$$\begin{aligned}(a, b)_l RcR &\subseteq (a, b)_l cR = (a, b)_l R \cap cR \qquad \text{(by Corollary 2.6)}\\ &= (a, bc)_l R \cap cR \qquad \text{(by Lemma 4.2)}\\ &= (aR \vee bcR) \cap cR\\ &= (aR \cap cR) \vee bcR \quad \text{(by (M), i.e. modularity)}\\ &= (acR \vee bcR) \subseteq bR.\end{aligned}$$

Similarly $(c, b)_l RaR \subseteq bR$. Thus if $x \in (a, b)_l R \cap (c, b)_l R$ then $xR = x(aR \vee cR) = xaR \vee xcR \subseteq bR$. This establishes (9). Clearly $(a, b)_l R \vee (c, b)_l R = R$ so by indecomposability of b we have $(a, b)_l R = bR$ or $(c, b)_l R = bR$; this yields respectively $aR \subseteq b^*R$ or $cR \subseteq b^*R$ contradicting the hypothesis.

Theorem 4.3 may also be stated in the following form (cf. [6] for the case of a bounded element in an atomic 2-fir):

COROLLARY 4.4. *Let R be an LCM domain satisfying* (M) *and having the acc and dcc for left factors. If b is right bounded and indecomposable then its right bound b^* is RI-indecomposable.*

5. **Two-sided bounds.** An element $b \in R$ which is a factor of a (right and left)-invariant element is said to be *bounded.* In this case both annihilators $(R/Rb)^l$ and $(R/bR)^r$ are nonzero. In an atomic 2-fir these two annihilators are equal [6, p. 5] so that

$$Rb^{\#} = (R/Rb)^l = (R/bR)^r = b^*R \tag{10}$$

where $b^{\#}$ and b^* are the left and right bounds of b respectively. In general however the left and right bounds of a bounded element need not be equal (i.e. up to unit factors): referring to Example 2.9 we have $y^* = xy$ (which is central) but $y^{\#} = y$. In general it can be shown (cf. [6, Proposition 3.2]) that *if $aR = Ra'$ then $aR = a'R$.* Thus if b is bounded and if (10) holds then $b^*R = b^{\#}R$; this common generator is called the *(two-sided) bound* of b.

Since the two-sided bound b^* of an element b is invariant we see that under the hypotheses of Theorem 3.1, q is a prime factor of b^* if and only if $p \operatorname{tr} q$ for some prime factor p of b. We shall elaborate for the case of a bounded prime. First we observe the following improvement of Theorem 3.5 in this case.

THEOREM 5.1. *Let R be an LCM domain and let $p \in R$ be a prime with two-sided bound p^*. Then p^* has no proper right-invariant or left-invariant factor.*

PROOF. If $p^* = ac$ then using the invariance of p^* one may check that a is right (left) invariant if and only if c is left (right) invariant. Let us assume that a is right invariant and apply Proposition 3.4; we find that p is either a left factor of a in which case $p^*R = aR$ or p is a right factor of c in which case $Rp^*(= Rp^{\#}) = Rc$.

THEOREM 5.2. *Let R be an LCM domain satisfying* (M) *and the dcc for left factors. Let $p \in R$ be a prime with two-sided bound p^*. The following conditions are equivalent for any prime $q \in R$:*

(i) q *has (two-sided) bound* p^*,
(ii) q *is a factor of* p^*,
(iii) p tr q.

PROOF. Obviously (i) implies (ii). Conversely if q is a factor of p^* then q is bounded by p^* so that both q^* and $q^\#$ divide p^* whence $q^*R = p^*R = Rp^* = Rq^\#$ by Theorem 5.1. The equivalence of (ii) and (iii) follows by Theorem 3.2 and the invariance of p^*.

Before considering the question of decomposability of two-sided bounds we need a bit more information on invariant elements. Since there is no distinction between "left" and "right" for multiples and divisors of invariant elements we omit the subscripts that appear in the notation of equations (1) and (2).

PROPOSITION 5.3. *Let* R *be an LCM domain. If* a *and* b *are invariant elements of* R *then so are* (a, b) *and* $[a, b]$. *In particular the set of invariant elements of* R *is a lattice.*

PROOF. If a and b are invariant in R then we may write

$$0 \neq mR = aR \cap bR = Ra \cap Rb = Rm'$$

from which it follows that $mR = m'R$ so that $m = [a, b]$ is invariant. Therefore if $m = ab' = ba'$ then a' and b' are also invariant and hence so is $[a', b']$. Referring to equation (2) we conclude that (a, b) is invariant.

An invariant element which has no proper invariant factors will be called an *I-prime.* In an LCM domain every two-sided bound of a prime is an I-prime by Theorem 5.1. In general the converse need not hold: the I-prime xy in Example 2.9 is the right bound of y, the left bound of x, but the two-sided bound of neither. Of course this situation cannot arise in an atomic 2-fir because every bounded element has a two-sided bound in this case.

We need the following analogue of Theorem 1.3 for invariant elements (cf. [10, p. 115]).

THEOREM 5.4. *Let* R *be an LCM domain having the* dcc *(or equivalently the* acc*) for invariant factors. Each invariant element of* R *that is not a unit is the product of I-primes; this factorization is unique up to order of factors and unit factors.*

Using the last two results we deduce that if R is an LCM domain having the dcc for left factors and if an invariant element in R is RI-indecomposable then it must be a power of an I-prime. Applying this with Theorem 4.3 we obtain the following theorem which describes, as a special case, an indecomposable element having two-sided bound.

THEOREM 5.5. *Let R be an LCM domain satisfying* (M) *and the dcc for left factors, and let $b \in R$ be indecomposable with invariant right bound b^*. Then b^* is a power of an I-prime; if this I-prime is the right bound of a prime p then all prime factors of b are transposed to p.*

PROOF. As we have remarked b^* is the power of an *I*-prime. Let us assume that $b^* = (p^*)^n$ where p is a prime. If q is a prime factor of b then q divides $(p^*)^n$ and so q divides p^* (e.g. by Proposition 3.4); thus $p \operatorname{tr} q$ by Theorem 3.2 and the fact that p^* must also be invariant. This concludes the proof of the theorem.

Referring to Example 2.9; we find that y^n is indecomposable and has invariant (in fact central) right bound given by $(y^n)^* = (xy)^n = (y^*)^n$.

Let $b \in R$ be indecomposable with $b^* = (p^*)^n$ as in Theorem 5.5. If R is an atomic 2-fir then it can be shown that $\dim(b) = n$ [6, Theorem 5.2]. Although we have not been able to prove this in the more general context of Theorem 5.5 the converse can be established. First we shall need the following generalization of Theorem 5.1 in this case.

LEMMA 5.6. *Let R be an LCM domain satisfying* (M) *and the dcc for left factors. If p^* is the two-sided bound of a prime p then the only right-invariant* (*or left-invariant*) *factors of $(p^*)^k$ are the powers of p^*.*

PROOF. First we observe that every prime factor q of $(p^*)^k$ has bound p^*; this follows by Theorem 5.2 and the fact that q must divide p^*. Let a be a right-invariant factor of $(p^*)^k$. In view of Theorem 5.1 we assume $k > 1$ and proceed by induction. If $(p^*)^k = ax$ then x must be left invariant. Assuming that a is not a unit we may write $a = p^*y$; putting this into the previous equation and cancelling we obtain

$$(p^*)^{k-1} = yx \tag{11}$$

which shows that y must be right invariant. Applying the induction hypothesis to (11) we find that y is a power of p^* (possibly a unit). Therefore a is a power of p^*.

THEOREM 5.7. *Let R be an LCM domain satisfying* (M) *and the dcc for left factors. Let $p \in R$ be a prime with two-sided bound p^*. Then*

(i) *there exist primes p_i with $p \operatorname{tr} p_i$ $(i = 1, \cdots, k)$ such that $(p_1 \cdots p_k)^* = (p^*)^k$,*

(ii) *any such product of primes $p_1 \cdots p_k$ is indecomposable.*

PROOF. First we note that if $p \operatorname{tr} p_i$ $(i = 1, \cdots, j)$ then $(p^*)^j R \subseteq p_1 \cdots p_j R$ since $p_i^* = p^*$ by Theorem 5.2; in particular

$$(p_1 \cdots p_j)^* = (p^*)^h, \quad h \leqslant j, \tag{12}$$

by Lemma 5.6.

To prove (i) we assume that $(p_1 \cdots p_n)^* = (p^*)^n$ and proceed inductively. If there is no prime q satisfying $(p_1 \cdots p_n q)^* = (p^*)^{n+1}$ where $p \operatorname{tr} q$ then $(p^*)^n R \subseteq (p_1 \cdots p_n q)^* R$ for each q for which $p \operatorname{tr} q$ (by (12)). Therefore

$$(p^*)^n R \subseteq \bigcap_{p \operatorname{tr} q} (p_1 \cdots p_n q)R = p_1 \cdots p_n q^* R = p_1 \cdots p_n p^* R$$

which leads to $(p^*)^{n-1} R \subseteq p_1 \cdots p_n R$ contradicting the inductive hypothesis.

To prove (ii) let $b = p_1 \cdots p_k$ where $p \operatorname{tr} p_i$ and $b^* = (p^*)^k$. If b is decomposable then $bR = aR \cap cR$ where a and c are each products of less than k prime factors all of which are transposed to p (Theorem 5.2); equation (12) shows that $(p^*)^{k-1} R \subseteq a^* R \cap c^* R \subseteq b^* R$ which is a contradiction; thus b is indecomposable.

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