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A LOCAL RESULT FOR SYSTEMS OF RIEMANN-HILBERT BARRIER PROBLEMS[1]

BY

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ABSTRACT. The Riemann-Hilbert barrier problem (for n pairs of functions)

$$G\Phi^+ = \Phi^- + g$$

is investigated for the square integrable functions on a union of analytic Jordan curves C bounding a domain in the complex plane. In the special case, where at each point t_0 of C the symbol G has at most two essential cluster values $G_1(t_0)$, $G_2(t_0)$, then the condition $\det[(1-\lambda)G_1(t_0) + \lambda G_2(t_0)] \neq 0$, for all t_0 in C and all λ $(0 \leqslant \lambda \leqslant 1)$, implies the Riemann-Hilbert operator is Fredholm. In the case, where for some t_0 in C and some λ_0 $(0 \leqslant \lambda_0 \leqslant 1)$, $\det[(1-\lambda_0)G_1(t_0) + \lambda_0 G_2(t_0)] = 0$, the Riemann-Hilbert operator is not Fredholm. An application is given to systems of singular integral equation on $L^2(E)$, where E is a measurable subset of C.

Let D^+ be an open connected region in the complex plane $\mathbf{C}$ with boundary C consisting of $m+1$ nonintersecting rectifiable analytic Jordan curves $C_0, C_1, \cdots, C_m$. The domain complementary to $D^+ \cup C$ is denoted by D^-. It is assumed that C_0 is the boundary of the unbounded component of D^- and that the boundary C is oriented positively with respect to D^+. The notation $L_n^p(C)$ $(1 \leqslant p < \infty)$ will be used for the Lebesgue spaces of $\mathbf{C}^n$-valued, p-integrable (with respect to arc length measure on C) functions on C. In the case $p = \infty$, $L_n^\infty(C)$ will denote the essentially bounded $\mathbf{C}^n$-valued measurable functions on C. If A is any nonempty set, then A_{M_n} will stand for the collection of $n \times n$ matrices with entries from A.

For f in $L_n^p(C)$ $(p \geqslant 1)$ we define the Cauchy transform of f by $Cf(z) = (2\pi i)^{-1}\int_C f(t)\,(t-z)^{-1}\,dt$, $z \notin C$. Obviously, Cf is a $\mathbf{C}^n$-valued function separately analytic in D^+ and D^-. Moreover, the nontangential limits of Cf from D^+ and D^- exist a.e. on C. These nontangential limits $f^\pm$ satisfy a.e. the Plemelj identities

$$(0.1)\qquad \begin{aligned} f^+(t) - f^-(t) &= f(t), \\ f^+(t) + f^-(t) &= (\pi i)^{-1}\int_C f(s)\,(s-t)^{-1}\,ds. \end{aligned}$$

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The singular integral $Qf(t) = (\pi i)^{-1} \int_C f(s)\,(s-t)^{-1}\,ds$ is interpreted as a Cauchy principal value. For the definition of the Cauchy principal value the reader is referred to Muskhelishvili [10, p. 25]. Further, it is an easy consequence of M. Riesz's theorem on the boundedness of the conjugate integral (see e.g. Zygmund [14, p. 253]) that the map $f \longrightarrow Qf$ is bounded on $L_n^p(C)$, for any $p > 1$. Of course, this implies that the maps $f \longrightarrow f^{\pm}$ are bounded on $L_n^p(C)$, for any $p > 1$.

Let G be in $L_{M_n}^{\infty}(C)$. We are interested in the Riemann-Hilbert operator $R_G\colon L_n^p(C) \longrightarrow L_n^p(C)$ $(p > 1)$ defined by

$$R_G f = Gf^+ - f^-. \tag{0.2}$$

If g is in $L_n^p(C)$, then the problem of solving the equation $R_G f = g$ in $L_n^p(C)$ is equivalent to finding a $\mathbf{C}^n$-valued function Φ separately analytic in D^+ and D^- vanishing at infinity whose nontangential limits $\Phi^{\pm}$ from $D^{\pm}$ are in $L_n^p(C)$ and satisfy the classical Riemann-Hilbert barrier problem $G\Phi^+ = \Phi^- + g$.

The problem under consideration in this paper is the question of when the operator R_G is a Fredholm operator. Recall, a bounded linear operator on a Banach space is called Fredholm in case it has closed range and the null spaces of the operator and its adjoint are finite dimensional (Russian authors call such operators Noetherian). We will deal exclusively with the operators R_G on $L_n^2(C)$ and will be preoccupied with local conditions on the symbol G which determine whether or not R_G is Fredholm.

There is a string of papers by I. B. Simonenko [13], see also the 6 references in [13], which establish that the question of deciding when R_G is Fredholm is a local question, i.e. depends only on the behavior of the symbol G in a neighborhood of each point on C. More recently, other authors Douglas and Widom [7], Douglas and Sarason [6] and Douglas ([4], [5]) have investigated similar local questions for the special case where the operator R_G acts on $L^2(\mathbf{T})$; here, $\mathbf{T}$ denotes the unit circle in $\mathbf{C}$. These authors, notably in [6], [5], use function algebra and C^*-algebra methods.

Our concern will be with sybmols G where the cluster behavior of G at points t_0 in C is particularly nice. We will assume that for some points t_0 in C there are at most two matrices X that satisfy

(0.3) For every $\epsilon > 0$ and neighborhood N of t_0 the set $\{t \in C\colon \|G(t) - X\| < \epsilon\} \cap N$ has positive measure.

In equation (0.3) and in the remainder of this paper the norm of an $n \times n$ matrix A, denoted by $\|A\|$, will refer to the norm of the matrix considered as an operator on $\mathbf{C}^n$ with the Euclidean metric.

In spite of the fact that the class of symbols under consideration appear restrictive our investigations have applications to systems of singular integral operators on $L^2(E)$, for measurable $E \subset C$, when the coefficients admit continuous extension to C. We briefly discuss this application in §4.

§1 of this paper is concerned with developing technical machinery aimed at reducing the problems to the generic case where $C = \mathbf{T}$. §2 gives sufficient conditions for the operators R_G to be Fredholm. In §3 sufficient conditions are given which guarantee that the operator R_G is not Fredholm. §4 contains applications and remarks.

The author would like to thank Edward Azoff for valuable discussions, in particular, for simplification in the proof of Lemma 2.1.

1. **Technical preliminaries.** It is easy to verify that the operator R_G: $L^p_n(C) \longrightarrow L^p_n(C)$ is Fredholm if and only if the operators R_G: $L^p_n(C_i) \longrightarrow L^p_n(C_i)$, $i = 0, 1, \cdots, m$, are Fredholm; here, we have set G_i equal to the restriction of G to C_i (see, e.g. Simonenko [13, Lemma 7]).

For $i = 0, 1, \cdots, m$, there is a conformal map β_i from the bounded domain D_i interior to C_i to the unit disc in the complex plane. The maps β_i extend to be one-to-one analytic maps of C_i to $\mathbf{T}$. (See Nehari [11, pp. 179 and 186].) The argument in Simonenko [13, Lemma 8] can be used to establish that R_{G_i}: $L^p_n(C_i) \longrightarrow L^p_n(C_i)$ is Fredholm if and only if $R_{G_i \circ \beta_i^{-1}}$: $L^p_n(\mathbf{T}) \to L^p_n(\mathbf{T})$ is Fredholm.

The notations H^p $(1 \leqslant p \leqslant \infty)$ will be used for the usual Hardy spaces on the unit circle and H^p_n the $\mathbf{C}^n$-valued analogues. The map $f \longrightarrow f^+$ on $L^2_n(\mathbf{T})$ is precisely the orthogonal projection of $L^2_n(\mathbf{T})$ onto H^2_n. If G is in $L^\infty_{M_n}(\mathbf{T})$, then the Toeplitz operator with symbol G is the operator T_G defined on H^2_n by $T_G f = (Gf)^+$. It is easy to establish that T_G is Fredholm on H^2_n if and only if R_G is Fredholm on $L^2_n(\mathbf{T})$.

Simonenko [13, Theorem 1] has established that the Riemann-Hilbert operator R_G: $L^p_n(C) \longrightarrow L^p_n(C)$ is Fredholm if and only if for every t_0 in C there is a neighborhood $N(t_0)$ and G_{t_0} in $L^\infty_{M_n}(C)$ such that $R_{G_{t_0}}$ is Fredholm and $G \mid N(t_0) = G_{t_0}$. Douglas [5, Corollary 4.7] has given a C^*-algebra proof of the corresponding result on H^2_n. In view of the remarks in the first part of this section these localization results for Riemann-Hilbert operators on $L^2_n(C)$ and Toeplitz operators on H^2_n are equivalent.

The following lemma appears in the scalar case in Brown and Halmos [1]. The simple proof, which we include for completeness, is due to Widom.

LEMMA 1.1. *Suppose G is in $L^\infty_{M_n}(\mathbf{T})$ and for some $\epsilon > 0$,* Re $G(t) \geqslant \epsilon I$ *a.e. Then T_G is invertible.*

PROOF. For $\delta > 0$ small, we have almost everywhere

$$\|I - \delta G(t)\| \leqslant \|I - \delta \operatorname{Re} G(t)\| + \delta \|G(t)\| < 1.$$

It follows that $T_{\delta G} = \delta T_G$ is invertible. This completes the proof.

2. R_G **Fredholm.** Throughout this section it will be assumed that at every point $t_0 \in C$ the symbol G has at most two cluster values $G_1(t_0)$, $G_2(t_0)$ satisfying (0.3).

LEMMA 2.1. *Suppose that for some* t_0 *in* C

$$\det[\lambda G_1(t_0) + (1 - \lambda)G_2(t_0)] \neq 0, \quad \textit{for } 0 \leqslant \lambda \leqslant 1.$$

Then there is a neighborhood N_{t_0} *of* t_0, $\epsilon > 0$, *and invertible matrices* P, Q *such that for almost every* t *in* N_{t_0}, $\operatorname{Re}(PG(t)Q) \geqslant \epsilon I$.

PROOF. Set $M = G_1(t_0)G_2^{-1}(t_0)$. Obviously, $\det[\lambda I + (1 - \lambda)M] \neq 0$ $(0 \leqslant \lambda \leqslant 1)$. There exists an invertible matrix W such that $WMW^{-1} = S + N$ is in Jordan canonical form, where S is diagonal and N is a nilpotent. Since $\det[\lambda I + (1 - \lambda)S] \neq 0$ $(0 \leqslant \lambda \leqslant 1)$ it is clear that there is a unitary matrix U (in fact, diagonal) such that $\operatorname{Re} U \geqslant \alpha I$ and $\operatorname{Re} US \geqslant \alpha I$, for some $\alpha > 0$. The nilpotent N is similar to βN, for any $\beta \neq 0$. This follows because N and βN have the same Jordan form. Indeed, by considering the block form of S and N, for any $\beta > 0$ there exists an invertible matrix V_β such that $V_\beta N V_\beta^{-1} = \beta N$ and $V_\beta S V_\beta^{-1} = S$. Clearly, by taking β small enough, we have $\operatorname{Re} U \geqslant \gamma I$ and $\operatorname{Re}(UV_\beta WMW^{-1}V_\beta^{-1}) \geqslant \gamma I$, for some $\gamma > 0$. In other words, if $P = UV_\beta W$ and $Q = G_1^{-1}(t_0)W^{-1}V_\beta^{-1}$, then $\operatorname{Re}(PG_i(t_0)Q) \geqslant \gamma I$, for $i = 1, 2$.

The lemma follows since almost every value of $G(t)$ in a neighborhood of t_0 can be made arbitrarily close to $G_1(t_0)$ or $G_2(t_0)$ by taking the neighborhood small enough. This completes the proof.

Douglas and Widom [7] have asked the following question: Given a compact convex set $\mathcal{K}$ of invertible matrices do there exist invertible matrices P, Q such that $\operatorname{Re} PKQ \geqslant \epsilon I > 0$, for all K in $\mathcal{K}$? Lemma 2.1 answers this question affirmatively when $\mathcal{K}$ is a line segment.

The main result of this section is

THEOREM 2.1. *Let* G *be in* $L^\infty_{M_n}(C)$ *and suppose that for every* t_0 *in* C *there are at most two cluster values* $G_1(t_0)$, $G_2(t_0)$ *satisfying* (0.3). *Assume further, that* $\det[\lambda G_1(t_0) + (1 - \lambda)G_2(t_0)] \neq 0$, *for all* t_0 *in* C *and* $0 \leqslant \lambda \leqslant 1$. *Then the operator* R_G *defined by* (0.2) *on* $L^2_n(C)$ *is Fredholm.*

PROOF. In view of the remarks in §1 we can assume $C = \mathbf{T}$ and establish that T_G is Fredholm. It follows from Lemma 2.1 that for every point t_0 in

$\mathbf{T}$ there exist invertible matrices P_{t_0} and Q_{t_0} and a neighborhood N_{t_0} such that $H(t) = P_{t_0}G(t)Q_{t_0}$ satisfies $\operatorname{Re} H(t) \geqslant \epsilon I > 0$, for almost all t in N_{t_0}. Define,

$$G_{t_0}(t) = \begin{cases} G(t), & t \in N_{t_0}, \\ G_1(t_0), & t \notin N_{t_0} \end{cases}$$

and

$$H_{t_0}(t) = \begin{cases} H(t), & t \in N_{t_0}, \\ P_{t_0}G_1(t_0)Q_{t_0}, & t \notin N_{t_0}. \end{cases}$$

Then

$$T_{G_{t_0}} = T_{P_{t_0}} T_{H_{t_0}} T_{Q_{t_0}}$$

and since $T_{P_{t_0}}, T_{Q_{t_0}}$ are obviously invertible and $T_{H_{t_0}}$ is invertible by Lemma 1.1, it follows that $T_{G_{t_0}}$ is invertible. Consequently, T_G is locally Fredholm and, therefore, by Simonenko [13, Theorem 1] or Douglas [5, Corollary 4.7] the operator T_G is Fredholm. This completes the proof.

3. R_G **not-Fredholm.** In this section we will obtain sufficient conditions on G which guarantee that the operator R_G on $L_n^2(C)$ is not-Fredholm. The investigation involves a localization to Hardy spaces of representing measures for H^∞. This technique was first used by D. Sarason (private communication) in the study of Toeplitz operators on H^2.

The maximal ideal space of the algebra $L^\infty(\mathbf{T})$ is denoted by X. The space X is normally fibered over the unit circle, so that, $X = \bigcup_{|\lambda|=1} X_\lambda$; where, the fibre X_λ consists of the homomorphisms of $L^\infty(\mathbf{T})$ that assign the value λ to the function $\chi(e^{it}) = e^{it}$. The notation Y will be used for the maximal ideal space of H^∞. For λ in $\mathbf{T}$, the fibre Y_λ consists of the homomorphisms of H^∞ which assign the value λ to χ. The "algebra on the fibre" A_λ is the restriction of $\hat{H}^\infty$ (the space of Gelfand transforms) to the fibre X_λ.

We will be using many of the properties of X, Y, X_λ, Y_λ and A_λ. Hoffman's book [9, Chapter 10] is the best reference for these properties. Indeed, we will need the following facts:

(i) X is the Shilov boundary of the algebra H^∞ and if γ is a homomorphism of H^∞ in the fibre Y_λ, then γ has a unique representing measure supported on X_λ.

(ii) A_λ is a closed subalgebra of $C(X_\lambda)$ whose maximal ideal space is Y_λ and whose Shilov boundary is X_λ.

(iii) Y_λ is connected.

(iv) If W, V are disjoint closed sets in X_λ, then there is an F in A_λ such that $F|_W = 0$ and $|F| = 1$ on V.

Hardy spaces of representing measures have been studied by many authors (see, Gamelin [8, Chapter 4] and the references there). Toeplitz operators on Hardy spaces of representing measures were studied by Devinatz [3]. For our purposes we will need the $\mathbf{C}^n$-valued Hardy spaces of a representing measure μ of H^∞ supported on a fibre X_λ. For convenience we assume the measure μ is supported on the fibre X_1. Let $L_n^p(\mu)$ $(1 \leqslant p \leqslant \infty)$ denote the $\mathbf{C}^n$-valued Lebesgue spaces. The notation $H_n^p(\mu)$ $(1 \leqslant p < \infty)$ will be used for the closure of $A_1 \otimes \mathbf{C}^n$ in $L_n^p(\mu)$ and $H_n^\infty(\mu)$ the weak*-closure of $A_1 \otimes \mathbf{C}^n$ in $L_n^\infty(\mu)$. If P denotes the orthogonal projection of $L_n^2(\mu)$ onto $H_n^2(\mu)$ and $\Phi \in L_{M_n}^\infty(\mu)$, then the Toeplitz operator T_Φ on $H_n^2(\mu)$ is defined by $T_\Phi f = P\Phi f$.

Assume now that the fibre X_1 is the disjoint union of a fixed pair of nonempty closed sets X_1^+ and X_1^-. It follows easily from properties (i) and (iii) above that there is a homomorphism of H^∞ with measure μ_0 supported on X_1 such that $\mu_0(X_1^\pm) > 0$.

Let $\mathfrak{A}$ be the subalgebra of $L^\infty(\mu_0)$ consisting of functions which are constant a.e. $[\mu_0]$ on $X_1^\pm$. We will be concerned with the collection $\mathcal{B}_{M_n}$ consisting of symbols G in $L_{M_n}^\infty(\mu_0)$ which can be factored in the form $G = G_1G_2G_3$, where G_1^*, G_3 are invertible elements in $(A_1)_{M_n}$ and G_2 is in $\mathfrak{A}_{M_n}$.

Before stating a theorem on Toeplitz operators with symbols in $\mathcal{B}_{M_n}$ we prove

LEMMA 3.1. *Let Φ be in $\mathfrak{A}_{M_n}$. If* $\det \Phi = 0$ *on X_1^+ or X_1^-, then T_Φ is not left-invertible on $H_n^2(\mu_0)$.*

PROOF. Assume without loss of generality that $\det \Phi = 0$ on X_1^+. It is easy to see that there is an invertible matrix S such that the first column of $\Psi = S\Phi S^{-1}$ is zero on X_1^+. Since, $T_\Psi = T_S T_\Phi T_{S^{-1}}$ and T_S, $T_{S^{-1}}$ are invertible it suffices to show that T_Ψ is not left-invertible. By property (iv) there exists an f in $H_n^2(\mu_0), f \neq 0$ and $f(X_1^-) = 0$. Let F be the $\mathbf{C}^n$-valued $H^2(\mu_0)$ function whose first entry is f and remaining entries are zero. Clearly, $T_\Psi F = 0$ and this completes the proof.

The following theorem is essentially a result on vector valued "self-adjoint" Toeplitz operators on $H_n^2(\mu_0)$.

THEOREM 3.1. *Let G be in $\mathcal{B}_{M_n}$, $G = G_1G_2G_3$, where G_1^*, G_3 are invertible in $(A_1)_{M_n}$ and G_2 is constant with values $G_2^\pm$ on $X_1^\pm$. If*

$\det[\lambda_0 G_2^+ + (1-\lambda_0)G_2^-] = 0$, *for some* λ_0, $0 \leq \lambda_0 \leq 1$, *then the operator* T_G *on* $H_n^2(\mu_0)$ *is not left-invertible.*

PROOF. Obviously, it suffices to show T_{G_2} is not left-invertible. We can assume by the preceding lemma that $G_2^\pm$ are invertible. Set $M = G_2(G_2^+)^{-1}$. Clearly, M is in $\mathfrak{A}_{M_n}$ and T_{G_2} is left-invertible if and only if T_M is left invertible. On X_1^- the symbol M equals $M^- = G_2^-(G_2^+)^{-1}$. Choose an invertible S so that SM^-S^{-1} is in the Jordan canonical form. It follows that $N = SMS^{-1}$ is in $\mathfrak{A}_{M_n}$ and is in Jordan canonical form at each point of X_1. Moreover, T_{G_2} is left-invertible if and only if T_N is left-invertible. If $N^\pm$ denote the values of N on $X_1^\pm$ (N^+ = identity) and if the diagonal entries of N^- are $\eta_{11}^-, \eta_{22}^-, \cdots, \eta_{nn}^-$, then by rechoosing S if necessary we can assume $\lambda_0 + (1-\lambda_0)\eta_{11}^- = 0$. Consider the self-adjoint Toeplitz operator T_φ on $H^2(\mu_0)$ where $\varphi = 1$ on X_1^+ and $\varphi = \eta_{11}^-$ on X_2^-. The operator T_φ is not left-invertible. (Devinatz, [3, Theorem 1]. Actually, Devinatz's theorem is only proven for the case of representing measures of Dirichlet algebra, however, the proof lifts to the case under consideration here.) The triangular form of N makes it obvious that T_N is not left-invertible. This completes the proof.

The next step is to obtain a theorem for the operators R_G on $L_n^2(C)$ from Theorem 3.1. We work first with the case where $C = \mathbf{T}$.

Again it is assumed that the fibre X_1 has been partitioned into a fixed pair of nonempty closed sets $X_1^\pm$ and μ_0 is a representing measure of a homomorphism of H^∞ supported in X_1 such that $\mu_0(X_1^\pm) > 0$. The notation $\tilde{\mathfrak{A}}$ will denote the subalgebra of $L^\infty(\mathbf{T})$ consisting of functions whose Gelfand transforms are constant on $X_1^\pm$. Clearly, $(\tilde{\mathfrak{A}})^\wedge \subset \mathfrak{A}$. If G is in $\tilde{\mathfrak{A}}_{M_n}$, then $G^\pm$ will denote the values of $\hat{G}|X_1^\pm$. The main theorem concerning Toeplitz operators with symbols in $\tilde{\mathfrak{A}}_{M_n}$ is the following:

THEOREM 3.2. *Assume that* G *is in* $\tilde{\mathfrak{A}}_{M_n}$ *and that for some* λ_0, $0 \leq \lambda_0 \leq 1$, $\det[\lambda_0 G^+ + (1-\lambda_0)G^-] = 0$. *Then* T_G *is not-Fredholm on* H_n^2.

The proof will involve some results of Rabindranathan [12] which we now state.

LEMMA 3.2 (*Rabindranathan* [12, *Lemma* 4.1]). *Let* Φ *be an invertible element in* $L_{M_n}^\infty(\mathbf{T})$. *Then* $\Phi = UA$, *where* A *is an invertible element in* $H_{M_n}^\infty$ *and* U *is unitary valued a.e.*

The following result is a matrix analogue of a result of Douglas and Sarason [6]. It can easily be derived from Lemma 4.3 of Rabindranathan [12].

LEMMA 3.3. *Let* U *be a unitary valued element of* $L_{M_n}^\infty(\mathbf{T})$. *Then* T_U *on* $H_n^2(\mathbf{T})$ *is left-Fredholm if and only if there is an* F *in* $(H^\infty + C)_{M_n}$ *(here,*

C denotes the continuous functions on $\mathbf{T}$) *such that*

$$\|U - F\|_\infty \equiv \operatorname*{ess\,sup}_{e^{it} \in \mathbf{T}} \|U(e^{it}) - F(e^{it})\| < 1.$$

The final result needed to prove Theorem 3.2 is the following elementary lemma.

LEMMA 3.4. *Suppose μ is a representing measure of a homomorphism of H^∞ supported on the fibre X_1. Let U be in $L^\infty_{M_n}(\mu)$ and unitary valued. If there exists an F in $H^\infty_{M_n}(\mu)$ such that*

$$\|U - F\|_\infty \equiv \operatorname*{ess\,sup}_{\gamma \in X_1} \|U(\gamma) - F(\gamma)\| < 1,$$

then T_U is left-invertible.

PROOF OF THEOREM 3.2. Suppose the operator T_G is Fredholm. Then G is an invertible element in $L^\infty_{M_n}(\mathbf{T})$ and, therefore, by Lemma 3.2, $G = UA$ where U is unitary and A is an invertible element in $H^\infty_{M_n}$. The operator T_U is certainly left-Fredholm and, hence, there exists by Lemma 3.3, an F in $(H^\infty + C)_{M_n}$ such that $\|U - F\|_\infty < 1$. From the equality

$$\|U - F\|_\infty = \sup_{\gamma \in X} \|\hat{U}(\gamma) - \hat{F}(\gamma)\|$$

and the fact that $\hat{C}|_{X_1} \subset A_1$, we obtain an H in $(A_1)_{M_n}$ such that

$$\sup_{\gamma \in X_1} \|\hat{U}(\gamma) - H(\gamma)\| < 1.$$

This says that the operator $T_{\hat{U}}$ is left-invertible on $H^2_n(\mu)$ for any representing measure μ for a homomorphism of H^∞ supported on X_1. In particular, $T_{\hat{U}}$ is left-invertible on $H^2_n(\mu_0)$. However, $\hat{U} = \hat{G}\hat{A}^{-1}$, where $\hat{A}^{-1}$ is invertible in $(A_1)_{M_n}$ and $\hat{G}$ is in $\mathfrak{A}_{M_n}$. (In other words $\hat{U}$ is in $\mathcal{B}_{M_n}$.) From the hypothesis $\det[\lambda_0 G^+ + (1 - \lambda_0)G^-] = 0$ and Theorem 3.1 we conclude that $T_{\hat{G}}$ is not left-invertible on $H^2_n(\mu_0)$. This contradiction completes the proof.

In case C is the union of the contours $C_0, C_1, \cdots, C_m$, then the map $\beta\colon C \longrightarrow \mathbf{T}$ $(\beta|C_i \equiv \beta_i)$ induces a natural "fibration" of $L^\infty(C)$ at points of C. Indeed, if t_0 is in C, every partition of $X_{\beta(t_0)}$ into a pair of closed sets $X^\pm_{\beta(t_0)}$ and corresponding algebra $\tilde{\mathfrak{A}}^{\beta(t_0)} \subset L^\infty(\mathbf{T})$ consisting of functions whose Gelfand transforms are constant on $X^\pm_{\beta(t_0)}$, gives rise to the algebra $\tilde{\mathfrak{A}}^{t_0}$ consisting of functions f in $L^\infty(C)$ such that $f \circ \beta^{-1}$ is in $\tilde{\mathfrak{A}}^{\beta(t_0)}$. The following corollary is immediate:

COROLLARY 3.1. *Let t_0 be in C. Assume G is in $\tilde{\mathfrak{A}}^{t_0}_{M_n}$ (for some decomposition of $X_{\beta(t_0)}$). Let $G^\pm$ be the values of $G \circ \beta^{-1}$ on $X^\pm_{\beta(t_0)}$. If*

$\det[\lambda_0 G^+ + (1-\lambda_0)G^-] = 0$, *for some* λ_0, $0 \leqslant \lambda_0 \leqslant 1$, *then* R_G *is not-Fredholm on* $L_n^2(C)$.

4. **Systems of singular integral equations.** Let E be a bounded measurable subset of C and let A, B be in $L_{M_n}^\infty(E)$. It will always be assumed that $A \pm B$ are invertible in $L_{M_n}^\infty(E)$. Consider the singular integral operator S on $L_n^2(E)$ defined by

$$Sf(s) = A(s)f(s) + B(s)Qf(s), \qquad s \in E. \tag{4.1}$$

It follows from the Plemelj identities (0.1) that

$$Sf = (A+B)f^+ + (A-B)f^-, \qquad f \text{ in } L_n^2(E).$$

Consider the symbol G in $L_{M_n}^\infty(C)$ defined by

$$G(s) = \begin{cases} (B-A)^{-1}(A+B)(s), & s \in E, \\ I, & s \notin E. \end{cases} \tag{4.2}$$

The following lemma is easily established.

LEMMA 4.1. *Let* S *denote the singular integral operator defined by* (4.1) *on* $L_n^2(E)$ *and* G *the symbol in* $L_{M_n}^\infty(C)$ *defined in* (4.2). *Then* S *is Fredholm on* $L_n^2(E)$ *if and only if* R_G *is Fredholm on* $L_n^2(C)$.

The technique of associating the barrier operator R_G, where the symbol G, is defined by (3.2), with the operator S on $L_n^2(E)$ was exploited for the case $n = 1$ in [2]. In fact, the results in [2] are motivation for this paper.

Suppose the matrix symbols A, B appearing in $L_{M_n}^\infty(E)$ are restrictions of continuous matrix functions $\hat{A}, \hat{B}$ on C to E. In this case the cluster behavior of the symbol G appearing in (4.2) is easy to describe. Let t_0 be in C. The matrix $\hat{G}(t_0) = (\hat{B} - \hat{A})^{-1}(\hat{A} + \hat{B})(t_0)$ is a cluster value of G at t_0 if and only if every neighborhood of t_0 intersects E in a set of positive measure. The identity matrix is a cluster value of G at t_0 if and only if every neighborhood of t_0 intersects the complement of E in a set of positive measure. The matrices $\hat{G}(t_0)$ and I are the only possible cluster values. Further, if β: $C \longrightarrow \mathbf{T}$ is the map introduced in §1 and $\chi_{\beta_i(E)}$ denotes the characteristic function of $\beta_i(E)$, then, for t_0 in C_i, $X_{\beta_i(t_0)}$ has a natural partition into the pair of disjoint closed sets

$$X^+_{\beta_i(t_0)} = \{\gamma \in X_{\beta_i(t_0)}: \hat{\chi}_{\beta_i(E)} = 1\}$$

and

$$X^-_{\beta_i(t_0)} = \{\gamma \in X_{\beta_i(t_0)}: \hat{\chi}_{\beta_i(E)} = 0\}.$$

The symbol G belongs to the algebra $\tilde{\mathfrak{A}}^{t_0}_{M_n}$ relative to this decomposition.

Our final result is a direct consequence of Theorem 2.1 and Corollary 3.1.

THEOREM 4.1. *Let S be the singular integral operator defined in* (4.1) *on $L_n^2(E)$. Assume that A, B are the restrictions of continuous matrix functions $\hat{A}$, $\hat{B}$ defined on C to E. The operator S is Fredholm if and only if for every point t_0 in C such that every neighborhood of t_0 intersects both E and its complement in a set of positive measure*

$$\det[\lambda\hat{G}(t_0) + (1-\lambda)I] \neq 0, \qquad 0 \leqslant \lambda \leqslant 1.$$

REMARKS. Nothing has been said about the index of the operator R_G. Except in the routine case where the symbol G is upper triangular we have been unable to give a natural computation of the index comparable to the result in [7]. One obstacle is the impossibility of putting a continuous symbol into upper triangular form in a continuous manner.

Even in the scalar case, when the symbol G is constant on each element of a partition of X_1 into more than two disjoint closed sets, no result corresponding to Corollary 3.1 is known.

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