

A SUFFICIENT CONDITION FOR HYPERBOLICITY OF PARTIAL DIFFERENTIAL OPERATORS WITH CONSTANT COEFFICIENT PRINCIPAL PART

BY

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ABSTRACT. Let P be a differential operator with principal part P_m , and suppose that P_m has constant coefficients and is hyperbolic. It is shown that the condition for hyperbolicity of P when P has constant coefficients, namely, that P is weaker than P_m is also a sufficient condition for hyperbolicity in the case where P does not have constant coefficients. Some generalizations are also made to the case where P is a square matrix of differential operators.

1. **Introduction.** Consider a partial differential operator $P(D) = \sum_{|\alpha| \leq m} a_\alpha(t, x) D^\alpha$ of order m in $n + 1$ independent variables where α is a multi-index $(\alpha_0, \alpha_1, \dots, \alpha_n)$ and

$$D^\alpha = \left(\frac{1}{i} \frac{\partial}{\partial t} \right)^{\alpha_0} \cdots \left(\frac{1}{i} \frac{\partial}{\partial x_n} \right)^{\alpha_n}.$$

There is as yet no general theory which allows one to make the assertion that such an operator is hyperbolic iff the a_α satisfy certain conditions. The traditional definition says that an operator is hyperbolic iff its principal part $P_m(\xi) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha$ is such that $P_m(\xi + \tau N) = 0$ for ξ real and $N = (1, 0)$ implies that τ is real. However, at best this should be regarded as a necessary condition. This can be best illustrated by considering the case of constant coefficients where there has been developed a good theory of hyperbolicity.

In this case one says that a differential operator P is hyperbolic with respect to a covector N if there exists a fundamental solution E , i.e., a solution of the equation $P(D)E = \delta$, such that $\text{supp } E$ is contained in a proper cone, i.e., a closed cone contained in the set $H \cup \{0\}$ where $H = \{x: \langle x, N \rangle > 0\}$.

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Gårding [2] was able to obtain a simple algebraic characterization of hyperbolicity, namely P is hyperbolic with respect to N iff $P_m(N) \neq 0$ and $P(\xi + \tau N) \neq 0$ for ξ real and $|\operatorname{Im} \tau| > c$, c a constant. To be more precise about the sufficiency of this condition one needs the following:

DEFINITION 1.1. If P is hyperbolic w.r.t. N , and P_m is the principal part of P , we denote by $\Gamma(P, N)$ the set of all real θ such that the polynomial $P_m(\theta + \tau N)$ has only negative roots τ .

We can now state the following.

THEOREM 1.2 (HÖRMANDER [4, p. 137]). *Let $P(D)$ satisfy Gårding's condition for hyperbolicity with respect to N . Then there exists one and only one fundamental solution E of the operator $P(D)$ with support in the half space $H = \{x: \langle x, N \rangle \geq 0\}$. For this fundamental solution we have $\operatorname{supp} E$ is contained in the convex cone*

$$\Gamma^*(P, N) = \{x: \langle x, \theta \rangle \geq 0, \theta \in \Gamma(P, N)\}$$

but in no smaller convex cone with vertex at 0.

One can show that if P is hyperbolic then so is its principal part P_m and that in order for P_m to be hyperbolic the equation $P_m(\xi + \tau N) = 0$ must have only real roots τ for ξ real. However, one does not have in general that P is hyperbolic if P_m is. In order to determine the precise relationship between the hyperbolicity of P and that of P_m one needs the following.

DEFINITION 1.3. A polynomial P is said to be *stronger* than a polynomial Q iff

$$\left(\sum_{\alpha} |D_{\xi}^{\alpha} Q(\xi)|^2 \right)^{1/2} < C \left(\sum_{\alpha} |D_{\xi}^{\alpha} P(\xi)|^2 \right)^{1/2}.$$

One can now state the following.

THEOREM 1.4 (HÖRMANDER [4] AND SVENSSON [8]). *If the principal part P_m of P is hyperbolic w.r.t. N , then P is hyperbolic w.r.t. N iff P is weaker than P_m . In this case P and P_m are equally strong.*

So much for the case of constant coefficients. The reason for its complete success is that when one Fourier transforms a differential operator with constant coefficients one obtains a polynomial. Obviously one is not so lucky in the case of variable coefficients. Certain other cases, namely the cases where P has simple characteristics or is a symmetric first order operator, can be handled by the method of energy integrals. Peyser [6] considers the extension of energy integral methods to the case where P has constant coefficients. He found that the method worked when the polynomial satisfied a condition which he called *proper hyperbolicity*.

Svensson [8] was then able to show that an operator is properly hyperbolic iff it is hyperbolic.

In order to explain these conditions we assume that there is an inner product on the space of independent variables and that $P(\tau, \xi) = P(\tau N + \xi)$ for ξ real and in N^\perp . Then the hyperbolicity of P w.r.t. N is equivalent to $P_m(1, 0) \neq 0$ and $P(\tau, \xi) \neq 0$ for ξ real and $|\text{Im } \tau| > c$. In particular the principal part of P, P_m , is hyperbolic w.r.t. N iff $P_m(1, 0) \neq 0$ and $P_m(\tau, \xi) = 0, \xi$ real implies τ is real.

Assume now that P_m is hyperbolic. Then we have

$$P_m(\tau, \xi) = \prod_{k=1}^m (\tau - \lambda_k^m(\xi))$$

where $\lambda_k^m(\xi)$ is real for ξ real and we have assumed without loss of generality that the coefficient of τ^m is 1.

Now define

$$P_m^{(i)}(\tau, \xi) = \frac{\partial^i}{\partial \tau^i} P_m(\tau, \xi) = m(m-1) \cdots (m-i+1) \prod_{k=1}^{m-i} (\tau - \lambda_k^{m-i}(\xi))$$

where again one has $\lambda_k^{m-i}(\xi)$ real for ξ real.

Define also

$$(P_m^{(i)})_j = \frac{P_m^{(i)}(\tau, \xi)}{\tau - \lambda_j^{m-i}} = m(m-1) \cdots (m-i+1) \prod_{k=1; k \neq j}^{m-i} (\tau - \lambda_k^{m-i}(\xi)).$$

One now has

THEOREM 1.5 (SVENSSON [8]). *Suppose $P(\tau, \xi) = \sum_{i=0}^m P_i(\tau, \xi)$ where $P_i(\tau, \xi)$ is a polynomial homogeneous of degree i , and suppose that $P_m(\tau, \xi)$ is hyperbolic and that the coefficient of τ^m is 1, then P is hyperbolic iff*

$$P_{m-i}(\tau, \xi) = \sum_{j=1}^{m-i+1} \gamma_j^i(\xi) (P_m^{(i-1)})_j(\tau, \xi), \quad i \geq 1$$

where the $\gamma_j^i(\xi)$ are uniformly bounded for ξ real.

REMARK. This theorem states that P is hyperbolic iff it is *properly hyperbolic*.

Flaschka and Strang [1] have conjectured that in the general case with constant hyperbolic principal part that the condition that the lower order terms are weaker than the principal part is necessary and sufficient for hyperbolicity. In this paper we will show that this condition is sufficient for hyperbolicity by

generalizing the method of energy integrals which Peysner devised. Finally we discuss the generalization of the technique to the case of systems of equations and obtain results similar to those of Kopáček and Suchá [5], Svensson [8] and Peysner [7].

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3. Statement of the problem. We wish to consider the case of a partial differential operator with constant principal part, i.e.,

$$P = P_m(D) + \sum_{|\alpha| < m} a_\alpha(t, x) D^\alpha, \text{ where } a_\alpha(t, x) \in \mathcal{B}(\mathbf{R}^{n+1}),$$

that is, the space of infinitely differentiable functions which are bounded along with all of their derivatives. We make the further assumption that the principal part $P_m(D)$ is hyperbolic in the time direction and that at each point the lower order part is weaker than $P_m(D)$.

An example of the type of problem which we wish to solve is given $f \in E'(\mathbf{R}^{n+1})$ find a $u \in \mathcal{D}'(\mathbf{R}^{n+1})$ such that $Pu = f$ and $\text{supp } u \subset K + L$ where K is a compact set and L is a proper cone. The obvious way to go about solving this problem is to write the equation as

$$P_m(D)u = - \sum_{|\alpha| < m} a_\alpha(t, x) D^\alpha u + f$$

and then perform the implied iteration. In order to show that this iteration converges we will have to use the energy integrals which are the subject of §4.

We will also be able to determine the regularity properties of u from those of f . In fact, if $f \in E'(\mathbf{R}^{n+1}) \cap H^{(s)}(\mathbf{R}^{n+1})$ where $H^{(s)}(\mathbf{R}^{n+1})$ is the Sobolev space as defined in Hörmander [4, p. 45] and s is an integer, then we have $u \in (H^{(s)})^{\text{loc}}(\mathbf{R}^{n+1})$.

To obtain these results, we will consider the Cauchy problem with homogeneous initial data in a slab $V_T = \{(t, x): 0 \leq t \leq T\}$. Define $\mathcal{D}_+ \subset C^\infty(\mathbf{R}^{n+1})$ to be the set of functions u such that $\text{supp } u \cap \{(t, x): t \leq 0\} = \emptyset$ and $\text{supp } u \subset K + L$ where K is compact and L is a proper cone. Similarly define $\mathcal{D}_- \subset C^\infty(\mathbf{R}^{n+1})$ to be the set of functions u such that $\text{supp } u \cap \{(t, x): t \geq T\} = \emptyset$ and $\text{supp } u \subset K - L$ again where K is compact and L is a proper cone. Figures 1 and 2 illustrate the support properties of functions in \mathcal{D}_+ and \mathcal{D}_- respectively.

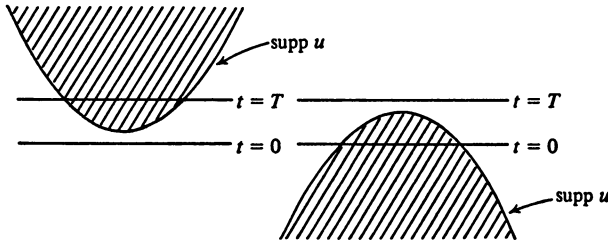


FIGURE 1. $u \in \mathcal{D}_+$

FIGURE 2. $u \in \mathcal{D}_-$

DEFINITION 3.1. $H_+^{(s)}(V_T)$, s a nonnegative integer, is the completion of \mathcal{D}_+ in the norm

$$\|v\|_s = \left(\int_{V_T} \sum_{|\alpha| \leq s} |D^\alpha v|^2 \right)^{1/2}.$$

Now for $u \in \mathcal{D}_+$, $s > 0$, define

$$\|u\|_{-s} = \sup_{v \in \mathcal{D}_-} \left| \int_{V_T} u \bar{v} \right| / \|v\|_s \leq \|u\|.$$

DEFINITION 3.2. $H_+^{(s)}(V_T)$, s a negative integer is the completion of \mathcal{D}_+ in the $\|\cdot\|_s$ norm.

DEFINITION 3.3. If $f \in H_+^{(s)}(V_T)$, s an integer, then we will say $u \in H_+^{(s)}(V_T)$ is a strong solution of $Pu = f$ with homogeneous initial data, if there exists a sequence $u_n \in \mathcal{D}_+$ such that $Pu_n \rightarrow f$ and $u_n \rightarrow u$ both in $H_+^{(s)}(V_T)$.

Now given our assumptions on P we will be able to show that for all $f \in H_+^{(s)}(V_T)$ there exists a unique strong solution $u \in H_+^{(s)}(V_T)$, and that for this solution $\text{supp } u \subset \text{supp } f + \Gamma^*(P_m, (1, 0))$. Using this result one can easily obtain the result for $f \in E'(\mathbb{R}^{n+1})$ and $u \in \mathcal{D}'(\mathbb{R}^{n+1})$. These existence theorems will be the subject of §5. In §6, we will make an easy generalization of these results to systems of differential operators.

4. Construction of energy integrals. Suppose that P is an operator which satisfies the assumptions of the previous section, in particular $P_m(D)$ is assumed hyperbolic. Since the set of differential operators of degree i which are weaker than P_m is a linear space, we can choose a basis $p_1^i, \dots, p_i^{n_i}$ for this set. Then our assumptions about P imply that

$$(4.1) \quad Pu = P_m(D)u + \sum_{i=0}^{m-1} \sum_{j=1}^{n_i} c_i^j(t, x) p_i^j(D)u$$

where the $c_i^j(t, x) \in \mathcal{B}(t, x)$. Furthermore it is obvious that any operator of form (4.1) satisfies our hypotheses.

We will now construct energy integrals for operators of form (4.1) following the method of Peysr [6]. Consider the quantity

$$- \operatorname{Im} (P_m^{(t-1)}(D)u \overline{P_m^{(t)}(D)u}).$$

One has by methods identical to those of Peysr that

$$(4.2) \quad - \operatorname{Im} (P_m^{(t-1)}(D)u \overline{P_m^{(t)}(D)u}) = \sum_{j=0}^m \partial A_j^{m-i}(u) / \partial x_j$$

where the A_j^{m-i} are Hermitian forms in the derivatives of order $m - i$ of u , and for $u \in \mathcal{D}_+$ or \mathcal{D}_- ,

$$(4.3) \quad \int_{S_{T'}} A_0^{m-i}(u) dx \geq 0$$

where $S_{T'}$ is the hyperplane $\{(t, x): t = T'\}$.

LEMMA 4.1. *If P satisfies our hypothesis and $u \in \mathcal{D}_+$ or \mathcal{D}_- then there is a constant c such that*

$$\int_{S_{T'}} \left| \sum_{j=1}^{n_i} c_j^i(t, x) p_j^i(D)u \right|^2 \leq c \int_{S_{T'}} A_0^i(u).$$

PROOF.

$$\begin{aligned} \int_{S_{T'}} \left| \sum_{j=1}^{n_i} c_j^i(t, x) p_j^i(D)u \right|^2 &\leq \int_{S_{T'}} \left(\sum_{j=1}^{n_i} |c_j^i(t, x)|^2 \right) \left(\sum_j |p_j^i(D)u|^2 \right) \\ &\leq c \int_{S_{T'}} \sum_j |p_j^i(D)u|^2. \end{aligned}$$

The result now follows from Lemma 3 of Peysr [6].

We are now ready to state the main energy inequality.

THEOREM 4.2. *If P is of form 4.1, then for each nonnegative integer s there exists a constant c which depends on P_m, T and the bounds of the c_j^i and their derivatives up to order s such that for T' in the interval $[0, T]$ and $u \in \mathcal{D}_+$ we have*

$$(4.4) \quad \sum_{|\gamma| \leq s} \sum_{i=1}^m \int_{S_{T'}} A_0^{m-i}(D^\gamma u) \leq c \sum_{|\gamma| \leq s} \iint_{V_{T'}} |D^\gamma Pu|^2.$$

PROOF. By integrating (4.2) over $V_{T'}$ and using the fact that $|\operatorname{Im}(ab)| \leq \frac{1}{2}(|a|^2 + |b|^2)$ we obtain

$$\int_{S_{T'}} A_0^{m-i}(D^\gamma u) \leq \frac{1}{2} \iint_{V_{T'}} |P_m^{(i-1)}(D)D^\gamma u|^2 + \frac{1}{2} \iint_{V_{T'}} |P_m^{(i)}(D)D^\gamma u|^2.$$

Summing over i and using Lemma 4.1 we obtain

$$\sum_{i=1}^m \int_{S_{T'}} A_0^{m-i}(D^\gamma u) \leq \frac{1}{2} \iint_{V_{T'}} |P_m(D)D^\gamma u|^2 + c \sum_{i=1}^m \iint_{V_{T'}} A_0^{m-i}(D^\gamma u).$$

Summing over γ for $|\gamma| \leq s$ gives

$$\begin{aligned} (4.5) \quad & \sum_{|\gamma| \leq s} \sum_{i=1}^m \int_{S_{T'}} A_0^{m-i}(D^\gamma u) \\ & \leq \frac{1}{2} \sum_{|\gamma| \leq s} \iint_{V_{T'}} |P_m(D)D^\gamma u|^2 + c \sum_{|\gamma| \leq s} \sum_{i=1}^m \iint_{V_{T'}} A_0^{m-i}(D^\gamma u). \end{aligned}$$

Now

$$P_m D^\gamma u = D^\gamma P_m u = D^\gamma P u - D^\gamma \left[\sum_i \sum_j c_i^j(t, x) p_i^j(D) u \right].$$

Applying Leibniz's rule to the quantity in brackets gives

$$\begin{aligned} P_m D^\gamma u &= D^\gamma P u - \gamma! \sum_{\alpha+\beta=\gamma} \sum_i \sum_j \left(\frac{D^\alpha c_i^j}{\alpha!} \right) \left(\frac{D^\beta p_i^j(D) u}{\beta!} \right) \\ &= D^\gamma P u - \gamma! \sum_{\alpha+\beta=\gamma} \sum_i \sum_j \left(\frac{D^\alpha c_i^j}{\alpha!} \right) \left(\frac{p_i^j(D) D^\beta u}{\beta!} \right). \end{aligned}$$

Thus

$$\begin{aligned} \iint_{V_{T'}} |P_m D^\gamma u|^2 &\leq \iint_{V_{T'}} |D^\gamma P u|^2 + c \sum_{\beta < \gamma} \sum_{i,j} \iint_{V_{T'}} |p_i^j(D) D^\beta u|^2 \\ &\leq \iint_{V_{T'}} |D^\gamma P u|^2 + c \sum_{\beta < \gamma} \sum_i \iint_{V_{T'}} A_0^{m-i}(D^\beta u) \\ &\leq \iint_{V_{T'}} |D^\gamma P u|^2 + c \sum_{|\beta| < s} \sum_i \iint_{V_{T'}} A_0^{m-i}(D^\beta u). \end{aligned}$$

Substituting this into (4.5) we obtain

$$\begin{aligned} & \sum_{|\gamma| \leq s} \sum_{i=1}^m \int_{S_{T'}} A_0^{m-i}(D^\gamma u) \\ & \leq \frac{1}{2} \sum_{|\gamma| \leq s} \iint_{V_{T'}} |D^\gamma P u|^2 + c \sum_{|\gamma| \leq s} \sum_{i=1}^m \iint_{V_{T'}} A_0^{m-i}(D^\gamma u). \end{aligned}$$

Now by a classical result about integral inequalities, this implies,

$$\sum_{|\gamma| \leq s} \sum_{i=1}^m \int_{S_{T'}} A_0^{m-i} (D^\gamma u) \leq c \sum_{|\gamma| \leq s} \iint_{V_{T'}} |D^\gamma Pu|^2.$$

COROLLARY 4.3. *If P is of form (4.1), $u \in \mathcal{D}_+$ and s a nonnegative integer, then $\|u\|_s \leq c \|Pu\|_s$ where $\|\cdot\|_s$ is the norm defined by*

$$\|u\|_s^2 = \sum_{|\gamma| \leq s} \iint_{V_T} |D^\gamma u|^2.$$

In order to obtain results about the existence of solutions of the equation $Pu = f$ for $f \in H_+^{(s)}(V_T)$ when s is negative, we will need energy inequalities for the adjoint of P, P^* :

$$\begin{aligned} P^*u &= \overline{P_m(-D)u} + \sum_i \sum_j \overline{p_i^j(-D)c_i^j(t, x)u} \\ &= P_m(D)u + \sum_\alpha \sum_i \sum_j [(-D)^\alpha \overline{c_i^j}] \overline{(p_i^j)^{(\alpha)}(-D)u}. \end{aligned}$$

Now $(p_i^j)^{(\alpha)}$ is weaker than P_m if p_i^j is and so the verification that P^* satisfies our hypothesis is reduced to the following: Claim if p is weaker than $P_m(D)$ then $\overline{p(-D)}$ is weaker than $P_m(D)$. For if $p(D) = \sum_\alpha a_\alpha D^\alpha$, then $p(-D) = \sum_\alpha a_\alpha (-D)^\alpha$ and $\overline{p(-D)} = \sum_\alpha \overline{a_\alpha} D^\alpha$. Thus if $f(\xi)$ is the polynomial corresponding to $\overline{p(-D)}$, then $f(\xi) = \overline{p(\xi)}$ for ξ real and thus $|\overline{f(\xi)}| = |p(\xi)|$. Now by Theorem 3.3.2(a) of Hörmander [4] this implies that $\overline{p(-D)}$ is weaker than P_m if $p(D)$ is. Thus P^* satisfies our hypothesis. One can now prove the analogs of Theorem 4.2 and Corollary 4.3 with $u \in \mathcal{D}_+$ replaced by $u \in \mathcal{D}_-$ and P by P^* .

5. Existence theory. We are now ready to prove existence theorems for the Cauchy problem for $Pu = f$ with homogeneous initial data.

THEOREM 5.1. *If P is of form (4.1) and f is an arbitrary element of $H_+^{(s)}(V_T)$, s an integer, then there exists a unique strong solution $u \in H_+^{(s)}(V_T)$ of the problem $Pu = f$ with homogeneous initial data. We have for this u that $\text{supp } u \subset \text{supp } f + \Gamma^*(P_m, (1, 0))$.*

The proof of this result will proceed in several steps. Step 1 is the proof of the result for nonnegative s . In the course of this proof we will show that for every $f \in \mathcal{D}_+$ there is a unique $u \in \mathcal{D}_+$ such that $Pu = f$. A similar result also holds for u and $f \in \mathcal{D}_-$ and $P^*u = f$. Step 2 uses the results of Step 1 to obtain estimates in $H_+^{(s)}$ for s negative which can then be used to prove the result for such s .

STEP 1. Assume that s is a nonnegative integer. Uniqueness is easy. If u and v are two strong solutions corresponding to f , then there exist two sequences u_n and v_n such that $Pu_n \rightarrow f$, $u_n \rightarrow u$, $Pv_n \rightarrow f$, and $v_n \rightarrow v$ all in $H_+^{(s)}(V_T)$. This implies that $P(u_n - v_n) \rightarrow 0$ in $H_+^{(s)}(V_T)$ and thus by Corollary 4.2 $u_n - v_n \rightarrow 0$. Therefore $u = v$.

For the existence of a strong solution u of the problem it suffices to find a sequence $u_n \in \mathcal{D}_+$ such that $Pu_n \rightarrow f$. Indeed, if u_n is such a sequence, Corollary 4.2 shows that u_n is Cauchy and hence must converge in $H_+^{(s)}(V_T)$. Suppose now that one has the following

LEMMA 5.2. *If P is of form (4.1) and $g \in \mathcal{D}_+$ then there exists a unique $u \in \mathcal{D}_+$ such that $Pu = g$. Furthermore we have for this u that $\text{supp } u \subset \text{supp } g + \Gamma^*(P_m, (1, 0))$.*

Now by choosing $f_n \in \mathcal{D}_+$ such that $\|f_n - f\|_s < \epsilon = 1/n$ and $\text{supp } f_n \subset \text{supp } f + B_\epsilon$ where B_ϵ is the ball of radius ϵ about 0, we have for the u_n given by the lemma that $Pu_n = f_n$ and that $\text{supp } u_n \subset \text{supp } f_n + \Gamma^* \subset \text{supp } f + \Gamma^* + B_\epsilon$ and thus the limit of the u_n , u has $\text{supp } u \subset \text{supp } f + \Gamma^*$, and is a strong solution of $Pu = f$.

PROOF OF LEMMA. We will perform the iteration

$$u_0 = 0,$$

$$P_m u_{i+1} = g - \sum_i \sum_j c_i^j(t, x) p_i^j(D) u_i.$$

That is,

$$u_{i+1} = E * \left(g - \sum_i \sum_j c_i^j(t, x) p_i^j(D) u_i \right)$$

where E is the fundamental solution of $P_m(D)$ with support in the cone $\Gamma^*(P_m, (1, 0))$.

Claim that all the u_i are in \mathcal{D}_+ and that in fact $\text{supp } u_i \subset \text{supp } g + \Gamma^*(P_m, (1, 0))$. Clearly this is true for u_0 . If it is true for u_i then $g - \sum_i \sum_j c_i^j p_i^j(D) u_i \in \mathcal{D}_+$ and thus by elementary facts about convolutions $u_{i+1} \in \mathcal{D}_+$. Furthermore

$$\begin{aligned} \text{supp } u_{i+1} &\subset \text{supp } E + (\text{supp } g \cup \text{supp } u_i) \\ &= (\text{supp } E + \text{supp } g) \cup (\text{supp } E + \text{supp } u_i) \\ &\subset (\Gamma^*(P_m, (1, 0)) + \text{supp } g) \cup (\Gamma^*(P_m, (1, 0)) + \Gamma^*(P_m, (1, 0)) + \text{supp } g) \\ &= (\Gamma^* + \text{supp } g) \cup (\Gamma^* + \text{supp } g) = \Gamma^* + \text{supp } g \end{aligned}$$

where we have used the fact that Γ^* is a convex cone. Thus if these u_l have a limit in \mathcal{D}_+ it is supported on $\Gamma^* + \text{supp } g$.

We will now use the energy inequalities of the previous section. One has

$$P_m(D)(u_1 - u_0) = g$$

$$P_m(D)(u_{l+1} - u_l) = -\sum c_l^j p_l^j(D)(u_l - u_{l-1}).$$

Now for T' fixed, t in the interval $[0, T']$ and

$$G_f(t) \equiv \sum_{|\gamma| \leq s} \sum_{i=1}^m \int_{S_t} A_0^{m-i} (D^\gamma(u_{i+1} - u_i))$$

we have that

$$G_0(t) \leq c \sum_{|\gamma| \leq s} \iint_{V_t} |D^\gamma g|^2 \leq c_1$$

and

$$G_f(t) \leq c \int_0^t dt' \sum_{|\gamma| \leq s} \sum_{i=1}^m \int_{S_{t'}} |D^\gamma \sum c_i^j p_i^j(D)(u_i - u_{i-1})|^2$$

$$\leq c_2 \int_0^t dt' G_{f-1}(t').$$

One obtains easily that $G_f(t) \leq c_1(c_2 t)^j/j!$ and therefore $G_f(t) \rightarrow 0$ uniformly for $0 \leq t \leq T'$. Furthermore

$$\sum_{|\gamma| \leq s} \iint_{V_{T'}} |D^\gamma(u_l - u_k)|^2 \leq c \int_0^{T'} dt' \sum_{i=k}^{l-1} G_f(t')$$

$$\leq c T' \sum_{i=k}^{l-1} \frac{(c_2 T')^i}{i!}$$

which implies that u_l is Cauchy in the $H_+^{(s)}(V_{T'})$ norm. Letting s and T' go to infinity one obtains that u_l converges in $C^\infty(\mathbb{R}^{n+1})$ to an element of \mathcal{D}_+ . This completes the proof of the lemma.

STEP 2. Suppose now that s is a negative integer and that $f \in H_+^{(s)}(V_T)$. The proof will go through much as before if we can establish the extension of Corollary 4.3 to the case of negative s .

For $f \in \mathcal{D}_-$ denote by Ef the unique element in \mathcal{D}_- such that $P^*Ef = f$. Then we have for $u \in \mathcal{D}_+$

$$\begin{aligned} \|Pu\|_s &= \sup_{v \in \mathcal{D}_-} |(v, Pu)|/\|v\|_{-s} = \sup_{v \in \mathcal{D}_-} |(P^*v, u)|/\|v\|_{-s} \\ &= \sup_{f \in \mathcal{D}_-} |(P^*Ef, u)|/\|Ef\|_{-s} = \sup_{f \in \mathcal{D}_-} |(f, u)|/\|Ef\|_{-s}. \end{aligned}$$

But by Corollary 4.2, $\|Ef\|_{-s} \leq c\|f\|_{-s}$ and thus

$$\|Pu\|_s \geq c \sup_{f \in \mathcal{D}_-} |(f, u)|/\|f\|_{-s} = c\|u\|_s.$$

This completes the proof of Theorem 5.1.

Let us now consider the problem $Pu = f$ where $f \in E'(\mathbb{R}^{n+1})$ and where $u \in \mathcal{D}'(\mathbb{R}^{n+1})$ with $\text{supp } u \subset K + L$, K compact and L a proper cone. The assumption that $f \in E'$ implies that $f \in H^{(s)}(\mathbb{R}^{n+1})$ for some s . We can by choosing the origin of time appropriately assure that $\text{supp } f \cap \{(t, x) : t \leq 0\} = \emptyset$, and we can choose T large enough so that $\text{supp } f \cap \{(t, x) : t \geq T\} = \emptyset$. Then we have by methods similar to those of Hörmander [4, §2.5] that $f \in H^{(s)}_+(V_T)$ and

$$\|f\|_{H^{(s)}_+(V_T)} = \|f\|_{H^{(s)}(\mathbb{R}^{n+1})}.$$

Thus we have a unique strong solution $u \in H^{(s)}_+(V_T)$. Furthermore, we also have a strong solution in $H^{(s)}_+(V_{T'})$ for $T' > T$ which can be restricted to $H^{(s)}_+(V_T)$ to yield the already given solution. Proceeding in this manner one can construct a $u \in H^{(s)\text{loc}}(\mathbb{R}^{n+1})$ with $u = 0$ for $t < 0$ such that $Pu = f$ and such that $\text{supp } u \subset \text{supp } f + \Gamma^*$. Furthermore, it is easy to show that among the elements of $\mathcal{D}'(\mathbb{R}^{n+1})$ such that $Pu = f$ and $\text{supp } u \subset K + L$, K compact and L a proper cone, the solution is unique. Summarizing we have

THEOREM 5.3. *If P satisfies our hypothesis and $f \in E'(\mathbb{R}^{n+1})$ then the equation $Pu = f$ has a solution $u \in \mathcal{D}'(\mathbb{R}^{n+1})$ with $\text{supp } u \subset \text{supp } f + \Gamma^*(P_m, (1, 0))$ and this solution is unique in the class of $u \in \mathcal{D}'(\mathbb{R}^{n+1})$ with $\text{supp } u \subset K + L$ where K is compact and L is a proper cone. One has also that $f \in H^{(s)}$ implies $u \in H^{(s)\text{loc}}$, where s is an arbitrary integer.*

6. Systems. We will now suppose that we have a square system of differential equations $Pu = \sum P_{ij}(D)u_j = f_i$. Certain special operators of this type will admit a treatment similar to the one we have already given for scalar operators. The most obvious such operators are of the form $Pu = P_m(D)I +$ lower order terms where $P_m(D)$ is a scalar operator with constant coefficients which is homogeneous of degree m and hyperbolic in the time direction. If one makes the further assumption that for each matrix position the lower order terms are

weaker than P_m , then one can push through the analogs of all the theorems of §§4 and 5. For instance,

THEOREM 6.1. *If $Pu = P_m(D)u + Qu$ where P_m is a scalar operator homogeneous of degree m with constant coefficients and hyperbolic in the time direction and Q is a matrix of differential operators with coefficients in \mathcal{B} such that at each point Q_{ij} is weaker than P_m , then for each $f \in H_+^{(s)}(V_T)$, s an integer, there exists a unique $u \in H_+^{(s)}(V_T)$ such that u is a strong solution of $Pu = f$ with homogeneous initial data. Furthermore $\text{supp } u \subset \text{supp } f + \Gamma^*(P_m, (1, 0))$.*

We also have for such P that if $f \in E'$ then there exists a unique $u \in \mathcal{D}'$ such that $Pu = f$ and $\text{supp } u \subset K + L$, K compact and L a proper cone. For this u we have $\text{supp } u \subset \text{supp } f + \Gamma^*(P_m, (1, 0))$.

The proof of this theorem and of all the other theorems from §§4 and 5 go through virtually without change.

One can obtain results for other systems of equations when it is possible to throw them into the above form. For instance if there exists an operator R such that RP is in the above form, then one has that for any $f \in E'$ there exists at most one $u \in \mathcal{D}'$ with $\text{supp } u \subset K + L$, K compact and L a proper cone such that $Pu = f$. Similarly if PR is in the above form one obtains an existence theorem.

Finally let us consider the problem when $P = A + B$ where A is a matrix of differential operators with constant coefficients. One makes the further assumption that there exists another operator C again with constant coefficients such that $CA = p(D)I$ where p is a hyperbolic polynomial and such that CB is weaker than the principal part of p . Clearly one has uniqueness for such an equation. One has existence also. In fact, if $CPu = Cf$ where $f \in E'$ and $u \in \mathcal{D}'$ with $\text{supp } u \subset K + L$, K compact and L a proper cone then $Pu = f$. For $C(Pu - f) = 0$ with $\text{supp } (Pu - f) \subset K + L$, therefore since our assumptions on C imply that it is a hyperbolic operator we have $Pu = f$. To see why C is hyperbolic one needs the result of Gårding and Svensson [8], that a matrix of operators with constant coefficients is hyperbolic iff its determinant is hyperbolic. However, that C has hyperbolic determinant follows easily from the fact that $CP = pI$ where p is a hyperbolic polynomial.

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