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DEFORMATIONS OF GROUP ACTIONS

BY

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ABSTRACT. Let G be a finite group and M be a compact piecewise linear (PL) manifold. Define a PL G-isotopy to be a level-preserving PL action of G on $M \times [0, 1]$. In this paper PL G-isotopies are studied and PL G-isotopic actions (which need not be equivalent) are characterized.

1. **Introduction.** Let G be a finite group. An *action* of G on a space X is a homomorphism of groups ϕ: $G \longrightarrow \mathrm{Homeo}(X)$, $g \longrightarrow \phi^g$. A *G-isotopy* is a level-preserving action θ on $X \times I$, $I = [0, 1]$. We then denote by θ_t the action on X given by $\theta_t^g(x) = \pi \circ \theta^g(x, t)$, where π: $X \times I \longrightarrow X$ is the projection on the first factor. Also we denote by $\theta_t \times 1$ the action on $X \times I$ which is θ_t at each level. Two actions ϕ and ψ on X are *equivalent* if there is an equivariant homeomorphism $h: (X, \psi) \longrightarrow (X, \phi)$, i.e., $h \circ \psi^g = \phi^g \circ h$ for all g in G.

The problem which we treat in this paper can now be stated as follows: If θ is a G-isotopy on $X \times I$, when and how nearly is θ_0 equivalent to θ_1? In particular, when is θ equivalent to $\theta_0 \times 1$?

In the smooth category there is the following result of Palais and Stewart [12] (see also Palais [11]):

THEOREM. *Let θ be a smooth G-isotopy on $M \times I$, where M is a compact smooth manifold. Then there is a smooth, level-preserving, equivariant diffeomorphism $(M \times I, \theta_0 \times 1) \longrightarrow (M \times I, \theta)$ which is the inclusion when restricted to $M \times 0$.*

This result is true, in fact, for smooth actions of arbitrary compact Lie groups, but is false if either the compactness or smoothness assumption is dropped. The focus of this paper, then, is on the case of not necessarily smooth G-isotopies on compact spaces. Most of the positive results refer to piecewise linear (PL) actions on a polyhedron or on a PL manifold.

In §2 we take care of preliminaries and give some background on PL actions.

In §3 we examine actions on disks and spheres under the relation of G-isotopy and present a collection of examples of nontrivial G-isotopies.

In §4 we investigate in detail how closely G-isotopic actions must resemble each other and derive conditions under which G-isotopic actions must be equivalent. First

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we observe (Theorem 4.1) that if G is a finite p-group then G-isotopic actions must have fixed point sets with isomorphic Z_p-homology groups. Second we show that a PL G-isotopy on a compact polyhedron contains only finitely many inequivalent levels (Corollary 4.3) and examples are constructed to show the necessity of the piecewise linearity and compactness assumptions (Examples 4.4, 4.5). Third we show (Theorem 4.6) that a certain local unknottedness property is sufficient to guarantee that a PL G-isotopy θ on $Y \times I$ is equivalent to the trivial G-isotopy $\theta_0 \times 1$, where Y is a compact polyhedron. In particular this shows that PL G-isotopic free actions on a compact polyhedron are PL equivalent. Finally we describe (Theorems 4.8, 4.9) precisely how PL G-isotopic actions can differ, showing that the examples of §3 detail essentially all the possible differences.

The results presented here constitute a part of the author's thesis, written at the University of Michigan under the direction of Professor Frank Raymond.

2. **Preliminaries.** We assume knowledge of the basic concepts of transformation groups and of elementary PL topology as given, for example, in Bredon [2] and Hudson [7], respectively.

It is convenient to let $A(G, X)$ denote the set of all actions of G on X. If h is a homeomorphism of X and ϕ is in $A(G, X)$, then $h \cdot \phi$ denotes the action given by $(h \cdot \phi)^g = h \circ \phi^g \circ h^{-1}$, for all g in G. Thus $h\colon (X, \phi) \longrightarrow (X, h \cdot \phi)$ is an equivariant homeomorphism. If ϕ is an action, then $\mathrm{Fix}(\phi)$ will denote the set of fixed points of ϕ.

If Y is a polyhedron (the underlying space of a locally finite, finite-dimensional simplicial complex), then $A_{\mathrm{PL}}(G, Y)$ denotes the set of PL actions, i.e., actions ϕ such that each homeomorphism ϕ^g is PL. We emphasize here that throughout this paper G will denote a finite group.

Since most of this paper deals with PL actions, we mention here for future reference two basic elementary lemmas about PL actions. The proofs are both reasonably straightforward exercises in subdivision technique, and hence are omitted.

LEMMA 2.1. *Let ϕ be in $A_{\mathrm{PL}}(G, Y)$ and K be a triangulation of Y. Then there is a subdivision of K on which ϕ defines a group of simplicial isomorphisms.*

We wish to make explicit the fact that a PL G-isotopy is a G-isotopy in $A_{\mathrm{PL}}(G, Y \times I)$. In particular it must be PL as an action on $Y \times I$ and not just PL at each level.

LEMMA 2.2. *Let Y be a compact polyhedron and θ be a G-isotopy in $A_{\mathrm{PL}}(G, Y \times I)$. Then any triangulation of $Y \times I$ with respect to which θ is simplicial has a subdivision on which θ is simplicial and in which the levels containing vertices are triangulated as subcomplexes.*

3. **Actions on disks and spheres.** Let ϕ be in $A(G, S^{n-1})$. Then there is an associated action $C(\phi)$ in $A(G, D^n)$ defined by

$$C(\phi)^g(y) = \begin{cases} |y|\phi^g(y/|y|) & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$$

Here $|y|$ denotes the usual euclidean norm. A trivial calculation shows that $C(\phi)$ is indeed a well-defined action, which we refer to as the action obtained from ϕ by "coning". Note that if ϕ is smooth, $C(\phi)$ will be smooth only if ϕ is a linear action; however, if ϕ is PL, then $C(\phi)$ is also PL. If ϕ is in $A(G, D^n)$, let $\phi|S^{n-1}$ denote the action which is the restriction of ϕ to S^{n-1}.

PROPOSITION 3.1. *Any action ϕ in $A(G, D^n)$ is canonically G-isotopic to the action $C(\phi|S^{n-1})$.*

PROOF. Define a G-isotopy as follows:

$$\phi_t^g(y) = \begin{cases} 0 & \text{if } y = 0 \text{ and } t = 1, \\ (1-t)\phi^g(y/(1-t)) & \text{if } 0 \leqslant |y| \leqslant 1-t \text{ and } t \neq 1, \\ |y|\phi^g(y/|y|) & \text{if } 1-t \leqslant |y| \leqslant 1 \text{ and } y \neq 0. \end{cases}$$

One easily checks that ϕ_t gives a well-defined, level-preserving action of G on $D^n \times I$. Clearly $\phi_0 = \phi$ and $\phi_1 = C(\phi|S^{n-1})$ as desired.

Note that the G-isotopy produced above is PL if ϕ is. It follows immediately from Proposition 3.1 that two actions in $A(G, D^n)$ are G-isotopic if and only if their restrictions to S^{n-1} are. Further, any action in $A(G, D^n)$ is G-isotopic to an action with contractible fixed point set and contractible orbit space, since the cone action has these properties. Proposition 3.1, together with the preceding remarks, is the basis for several examples of nontrivial G-isotopies.

EXAMPLE 3.2. Let M be a compact contractible n-manifold $(n \geqslant 4)$ with nonsimply connected boundary. (See Curtis [4].) Then $M \times D^1$ is homeomorphic to D^{n+1}, and there is an obvious action ϕ in $A(Z_2, D^{n+1})$ such that $\mathrm{Fix}(\phi) = M \times 0$. By the preceding propositions ϕ is G-isotopic to $C(\phi|S^n)$. However, ϕ is not equivalent to $C(\phi|S^n)$ since the fixed point set of the latter is homeomorphic to the cone over bd M, and hence not locally euclidean at the origin.

EXAMPLE 3.3. Let G denote the icosahedral group (of order 60). There is a well-known (see [2, pp. 55ff.]) action ϕ in $A(G, D^n)$, n large, such that ϕ has no fixed points. On the other hand ϕ is G-isotopic to $C(\phi|S^{n-1})$, which has a fixed point at the origin.

EXAMPLE 3.4. Let W be a PL h-cobordism between a lens space L of dimension greater than or equal to 5 and having fundamental group Z_p, with $p \geqslant 5$ and $p \neq 6$, and a manifold L' which is not PL homeomorphic to L. Milnor [10] showed that there are infinitely many (PL distinct) choices for L'. The universal covering space $\widetilde{W}$ of W is then an h-cobordism between $\widetilde{L} = S^n$ and $\widetilde{L}'$. Hence $\widetilde{W} \cong S^n \times I$ and covering transformations give an action of $G = Z_p$ on $S^n \times I$ which is linear on $S^n \times 0$. Coning over the nonlinear end, $S^n \times 1$, yields an action ϕ of G on D^{n+1} such that $\phi|S^n = \phi|\mathrm{bd}\ D^{n+1}$ is linear—although ϕ is not PL equivalent to a linear action. By Proposition 3.1, ϕ is G-isotopic to $C(\phi|S^n)$, a linear action. Thus ϕ and $C(\phi|S^n)$ yield G-isotopic but inequivalent PL actions with a single fixed point. Using standard facts about h-cobordisms (see, e.g., [9]) one sees that ϕ is topologically equivalent to $C(\phi|S^n)$, a linear action.

EXAMPLE 3.5. Let L be a k-dimensional lens space with fundamental group Z_p, $p \geqslant 3$ an odd integer, and M be L minus the interior of an open ball, so bd $M \cong S^{k-1}$. Smoothly embed M in D^n; n odd, $n \geqslant 2k + 3$, so that bd $M = M \cap \mathrm{bd}\ D^n$. Then according to Jones [8, 3.2] there is an action ϕ of $G = Z_2$ on D^n with $\mathrm{Fix}(\phi) = M$. Again by Proposition 3.1, ϕ is G-isotopic to $C(\phi|S^{n-1})$. But $\mathrm{Fix}(C(\phi|S^{n-1})) \cong D^k$. (For an easier way of finding examples of such actions see Bredon [2, pp. 49ff.].)

EXAMPLE 3.6. Bing [1] constructs an action of $G = Z_2$ on S^3 with fixed point set a wildly embedded 2-sphere, where the set of wild points is a Cantor set. Deleting the interior of a small invariant 3-disk around a nonwild point gives an action ϕ on the complementary 3-disk D with a wildly embedded 2-disk for fixed point set. However, $\phi|S^2 = \phi|\mathrm{bd}\ D$ is nice—in fact, by classical results, equivalent to a linear action. Hence, by Proposition 3.1, ϕ is G-isotopic to $C(\phi|S^2)$, which is equivalent to a linear action. Since ϕ cannot be equivalent to a linear action this gives another example of inequivalent but G-isotopic actions.

Note that in each of these examples the action ϕ is equivalent to $(\phi|S^{n-1}) \times 1$ on a small boundary collar $S^{n-1} \times I$, hence each G-isotopy produced by Proposition 3.1 is equivalent to ϕ at each level except the final $C(\phi|S^{n-1})$ level. The following observation is an amusing application of Proposition 3.1.

PROPOSITION 3.7. *Let* ϕ *be in* $A_{\mathrm{PL}}(G, S^n)$. *Then* ϕ *is* PL *G-isotopic to an action equivalent to the join of a fixed point free and a trivial action* $\psi * 1$, *in* $A_{\mathrm{PL}}(G, S^{n-r-1} * S^r)$.

The proof follows easily using Proposition 3.1 and induction on dimension, together with the fact that the closed complement of a PL n-disk in S^n is also a PL n-disk.

Note that if ϕ is a semifree action, then the G-isotopies produced by 3.7 are also semifree, so the resulting ψ is free. If G is a p-group (or if ϕ is semifree) the integer r is uniquely determined as the dimension of $\mathrm{Fix}(\phi)$.

An obvious modification of Example 3.3 shows that r is not uniquely determined in more general situations. It is interesting to ask how unique the action ψ is. Even in the case of Z_p actions the action ψ is not unique up to G-isotopy, as the following example shows.

EXAMPLE 3.8. In Example 3.4 a free action of Z_p on $S^n \times I$ was constructed which is linear on the end $S^n \times 0$ and not equivalent to a linear action on $S^n \times 1$. Let ϕ be the action on S^{n+1} obtained by coning over each end of $S^n \times I$. Then to "desuspend" ϕ as in Proposition 3.7 one can clearly choose ψ to be either the linear action or the nonlinear action. These are inequivalent actions, and are not even G-isotopic, by Theorem 4.7 of the next section. The actions are, however, clearly "G-concordant", where by a G-concordance we mean an action on $Y \times I$ which preserves $Y \times 0$ and $Y \times 1$, but which is not necessarily level-preserving.

This latter observation turns out to be the case in general: The fixed point free action ψ is unique up to G-concordance. This fact follows from the following easy observation. The proof, which used the techniques of Propositions 3.1 and 3.7, together with Theorem 4.1 of the next section, is left to the reader.

PROPOSITION 3.9. *Actions* ϕ *and* ψ *in* $A_{\mathrm{PL}}(G, S^n)$ *are* PL *G-concordant if and only if their suspensions* $\phi * 1$ *and* $\psi * 1$ *in* $A_{\mathrm{PL}}(G, S^n * S^0)$ *are* PL *G-isotopic.*

4. When are G-isotopic actions equivalent? Let Y be a polyhedron and let θ in $A(G, Y \times I)$ be a G-isotopy.

THEOREM 4.1. *If* G *is a p-group, p prime, then the inclusions* $\mathrm{Fix}(\theta_i) \subset \mathrm{Fix}(\theta)$, $i = 0, 1$, *induce isomorphisms* $H_*(\mathrm{Fix}(\theta_i); Z_p) \approx H_*(\mathrm{Fix}(\theta); Z_p)$.

PROOF. It suffices to consider the inclusion $\mathrm{Fix}(\theta_0) \subset \mathrm{Fix}(\theta)$. Since $Y \times 0$ is invariant under the action θ, a standard inequality in Smith theory (see Bredon [2, p. 144] for the result when $G = Z_p$; the general result for p-groups follows easily by induction on the order of the group) implies that

$$\sum_{i=0}^{n} \operatorname{rank} H_i(\mathrm{Fix}(\theta), \mathrm{Fix}(\theta_0); Z_p) \leqslant \sum_{i=0}^{n} \operatorname{rank} H_i(Y \times I, Y \times 0; Z_p),$$

for all n. The right-hand side, however, is 0. Thus $H_*(\mathrm{Fix}(\theta), \mathrm{Fix}(\theta_0); Z_p) = 0$, implying the desired isomorphism.

In particular, Theorem 4.1 says that the inclusions $\mathrm{Fix}(\theta_i) \subset \mathrm{Fix}(\theta)$ induce a one-to-one correspondence of path components. Also observe that this shows that a G-isotopy of a free action must be a free G-isotopy; for restricting to a p-subgroup which acts with fixed points would yield a contradiction to the theorem. Of course Theorem 4.1 is true much more generally. All that is really needed is that the inclusions $Y \times i \subset Y \times I$ induce Z_p-homology isomorphisms and that each $Y \times i$ is invariant under the action.

In light of the examples of §3 it is reasonable to ask how many inequivalent levels there may be in a nontrivial G-isotopy. Theorem 4.2, Corollary 4.3, and Examples 4.4 and 4.5 provide an answer to this question.

THEOREM 4.2. *Let* Y *be a polyhedron and* θ *in* $A_{\mathrm{PL}}(G, Y \times I)$ *be a* PL *G-isotopy. Suppose there is a triangulation of* $Y \times I$ *so that* θ *is simplicial and such that all vertices lie in* $Y \times \{0, 1\}$. *Then there is a level-preserving* PL *equivalence*

$$(Y \times (0, 1), \theta_{1/2} \times 1) \longrightarrow (Y \times (0, 1), \theta)$$

which is the identity on $Y \times 1/2$.

PROOF. Observe that each (y, s) in $Y \times (0, 1)$ lies in the interior of a unique simplex A of $Y \times I$ (its "carrier") and within this simplex the point (y, s) lies on a unique line segment whose end points lie in faces of A in $Y \times 0$ and in $Y \times 1$; that is, A can be written as the join $A_0 * A_1$ where $A_0 = A \cap Y \times 0$ and $A_1 = A \cap Y \times 1$.

Since θ is linear on simplices, θ linearly permutes the line segments described above; this fact allows one to set up the claimed equivalence. For each y in Y let $L(y)$ denote the unique line segment defined as above, passing through $(y, 1/2)$, lying in $A = A_0 * A_1$, starting at $(w, 0)$ and ending at $(z, 1)$.

Then, using the linear structure of simplex A, map the line segment $y \times I$ PL homeomorphically onto $L(y)$ by

$$(y, t) \mapsto (1 - t)(w, 0) + t(z, 1).$$

Note that this gives a level-preserving map.

Doing this for each y in Y yields a level-preserving equivariant PL map $(Y \times I, \theta_{1/2} \times 1) \longrightarrow (Y \times I, \theta)$ which one easily verifies to be a homeomorphism of $Y \times (0, 1)$ with itself. This completes the proof of the theorem.

COROLLARY 4.3. *Let* Y *be a compact polyhedron and let* θ *in* $A_{\mathrm{PL}}(G, Y \times I)$ *be a* PL *G-isotopy. Then* θ_t, $0 \leqslant t \leqslant 1$, *passes through at most finitely many mutually* PL *inequivalent levels.*

PROOF. By Lemmas 2.1 and 2.2 there is a triangulation of $Y \times I$ with respect to which θ is simplicial and such that the levels which contain vertices are triangulated as subcomplexes. By Theorem 4.2, θ_t changes equivalence type at most at these finitely many levels containing vertices. This completes the proof.

Now we present examples which show the necessity of the compactness and piecewise linearity in Corollary 4.3.

EXAMPLE 4.4. A Z_2-isotopy on D^5 which passes through infinitely many mutually inequivalent levels. The construction proceeds in several steps:

(1) Let p be an odd prime. According to Bredon [2, pp. 49ff.] there is a smooth involution f_p: $S^5 \longrightarrow S^5$ with fixed point set $L(p)$, a 3-dimensional lens space with fundamental group Z_p. Deleting the interior of a small invariant disk neighborhood of a fixed point on which f_p is equivalent to a linear involution yields

a smooth involution g_p: $D^5 \longrightarrow D^5$ which on S^4 is equivalent to the standard linear involution with S^2 as fixed point set. Thus we may assume that the involutions g_p all agree on S^4. Note that the fixed point set of g_p is $L(p)$ minus a disk, and hence has fundamental group Z_p.

(2) Let $p_1, p_2, \cdots$ be the sequence of odd primes. For each positive integer k let h_k: $D^5 \longrightarrow D^5$ be the involution which is g_{p_k} (shrunk down) on the subdisk $(1/k)D^5$ and is linear on the annular region $D^5 - (1/k)D^5$.

(3) Construct a level-preserving involution H on $D^5 \times I$ as follows: At levels $1/k$, $k = 1, 2, \cdots$, set $H_{1/k} = h_k$; let $H|S^4 \times I$ and H_0 be linear; fill in between levels $1/k$ and $1/(1+k)$ by coning from the point $(0, (1/k + 1/(1+k))/2)$ in a manner similar to that in Proposition 3.1.

One checks easily that this gives a well-defined Z_2-isotopy, which is PL except at one point. Further, the fixed point sets at levels $1, 1/2, 1/3, \cdots$ have pairwise distinct fundamental groups, hence the actions at these levels are all inequivalent.

EXAMPLE 4.5. A smooth Z_2-isotopy on $R^5 \times I$ passing through infinitely many inequivalent levels. We need only modify the preceding example slightly. Let J: $(D^5 - 0) \times I \longrightarrow (R^5 - \text{int } D^5) \times I$ be a level-preserving diffeomorphism. Then with H as above there is the level-preserving, smooth, involution $K = J \circ H \circ J^{-1}$, which clearly goes through infinitely many inequivalent levels. Further, $K|S^4 \times I$ is orthogonal at every level and hence we may extend K to all of $R^5 \times I$ smoothly, by coning at each level, to get the desired smooth Z_2-isotopy.

The preceding work indicates that in some loose sense the obstructions to proving that a given G-isotopy θ in $A(G, Y \times I)$ is equivalent to the "trivial" G-isotopy $\theta_0 \times 1$ are local in nature. We propose now to make this observation precise.

Let Y be a polyhedron and θ in $A_{\mathrm{PL}}(G, Y \times I)$ be a PL G-isotopy. The G-isotopy θ is *locally unknotted* if for each (y, t) in $Y \times I$, with $t < 1$, there is a θ_t-invariant (closed) PL neighborhood U of y in Y and an equivariant PL embedding h: $(U \times I, \theta_t \times 1) \longrightarrow (Y \times [t, 1], \theta)$ onto a neighborhood of (y, t) in $Y \times [t, 1]$, such that $h(z, 0) = (z, t)$ for all z in U. For $t = 1$ require that there be a θ_1-invariant PL neighborhood U and an equivariant embedding h: $(U \times I, \theta_1 \times 1) \longrightarrow (Y \times I, \theta)$ onto a neighborhood of $(y, 1)$ in $Y \times I$, such that $h(z, 1) = (z, 1)$ for all z in U. If there is a *level-preserving* equivariant PL homeomorphism $(Y \times I, \theta_0 \times 1) \longrightarrow (Y \times I, \theta)$ whose restriction to $Y \times 0$ is the inclusion then θ is said to be *unknotted.*

Observe that any free PL G-isotopy is automatically locally unknotted and that Examples 3.4, 3.5, 3.6 and 3.7 give knotted PL G-isotopies on $D^n \times I$. (We note that Example 3.6 is "topologically unknotted" although PL knotted.)

THEOREM 4.6. *Let Y be a compact polyhedron, and θ in $A_{\mathrm{PL}}(G, Y \times I)$ be a G-isotopy. Then θ is unknotted if and only if θ is locally unknotted.*

PROOF. An unknotted G-isotopy is seen to be trivially locally unknotted. Thus assume that θ is locally unknotted. The major portion of the proof is then devoted to showing the following statement:

LEMMA 4.6.1. *For sufficiently small* $\epsilon > 0$ *there is an equivariant level-preserving* PL *homeomorphism* $(Y \times [0, \epsilon], \theta_0 \times 1) \longrightarrow (Y \times [0, \epsilon], \theta)$, *which is the inclusion on* $Y \times 0$.

Assuming Lemma 4.6.1 for the moment, the proof is completed in the following tedious but elementary fashion.

First note that using Lemma 4.6.1 we can find for every t, $0 \leqslant t < 1$, a t', $t < t' \leqslant 1$, and an equivariant level-preserving PL homeomorphism $(Y \times [t, t'], \theta_t \times 1) \longrightarrow (Y \times [t, t'], \theta)$ which is the identity when restricted to $Y \times t$.

There is also $t < 1$ and an equivariant level-preserving PL homeomorphism $(Y \times [t, 1], \theta_t \times 1) \longrightarrow (Y \times [t, 1], \theta)$ which is the inclusion on $Y \times t$. To see this note that, turning the above argument upside down, there is $t < 1$ and an equivariant level-preserving PL homeomorphism h: $(Y \times [t, 1], \theta_1 \times 1) \longrightarrow (Y \times [t, 1], \theta)$ such that h restricted to $Y \times 1$ is the inclusion. Then define

$$h^*: (Y \times [t, 1], \theta_t \times 1) \longrightarrow (Y \times [t, 1], \theta) \text{ by } h^* = h \circ (h_t^{-1} \times 1).$$

One easily checks that h^* is the desired homeomorphism.

Now put all of this together using the compactness of the unit interval to obtain a partition $0 = t_0 < t_1 < \cdots < t_{k+1} = 1$ of I together with equivariant level-preserving PL homeomorphisms, for $i = 0, 1, \cdots, k$,

$$h^i: (Y \times [t_i, t_{i+1}], \theta_{t_i} \times 1) \longrightarrow (Y \times [t_i, t_{i+1}], \theta)$$

such that h^i restricted to $Y \times t_i$ is the inclusion. Finally patch together the h^i's to obtain the desired unknotting H: $(Y \times I, \theta_0 \times 1) \longrightarrow (Y \times I, \theta)$. Define H on $[t_i, t_{i+1}]$ by $H = h^i \circ (h_t^{i-1} \times 1) \circ \cdots \circ (h_{t_1}^0 \times 1)$. One easily checks that this gives a well-defined, level-preserving, equivariant PL homeomorphism. Modulo Lemma 4.6.1 this completes the proof of the theorem.

Lemma 4.6.1 is proved in two steps. First observe that there is an equivariant PL embedding c: $(Y \times I, \theta_0 \times 1) \longrightarrow (Y \times I, \theta)$, not necessarily level-preserving, onto a neighborhood of $Y \times 0$ in $Y \times I$, such that c restricted to $Y \times 0$ is the inclusion.

The existence of the equivariant embedding c follows immediately from a "local equivariant collaring implies equivariant collaring" theorem of Bredon [2, pp. 224–230] by making the minor modifications necessary to add piecewise linearity to both hypothesis and conclusion. Bredon's argument uses "local collaring respecting orbit type implies collaring respecting orbit type" in the orbit space; this can easily be done piecewise linearly. Then this collar is lifted to the total space using the covering homotopy theorem of Palais (see [2, pp. 92ff.]). This lift is of necessity PL. We take this opportunity, however, to point out that Connelly's short proof [3] of the collaring theorem, in which piecewise linearity is an optional variant, adapts quite easily to the equivariant case, working completely within the total space, thus eliminating the need to apply the covering homotopy theorem.

Second, we alter c on a neighborhood of $Y \times 0$ to make it level-preserving there. This part of the proof is similar to that of the "compatible collar" theorem of Hudson and Zeeman.

Let K and L be triangulations of $Y \times I$ so that $\theta_0 \times 1$ and θ, respectively, are simplicial actions, and so that the equivariant embedding c is a simplicial map $K \longrightarrow L$. Choose $0 < \epsilon < \delta < 1$ so that no vertices of K are in $Y \times (0, \delta)$ and so that $Y \times [0, \epsilon] \subset c(Y \times [0, \delta]) \subset Y \times I$. (This requires compactness.)

Subdivide K adding vertices only in $Y \times \delta$ to obtain K_1 such that $Y \times \delta$ is a subcomplex and $\theta_0 \times 1$ is still simplicial, as in Lemma 2.2. This induces via c a subdivision L_1 of L which adds vertices only in $c(Y \times \delta)$ so that c: $K_1 \longrightarrow L_1$ is simplicial. Subdivide L_1 adding vertices only in $Y \times \epsilon$ obtaining L_2 so that $Y \times \epsilon$ is a subcomplex and ϕ is still simplicial on L_2. We can do this since ϕ is level-preserving. This pulls back to a subdivision K_2 of K_1 so that c: $K_2 \longrightarrow L_2$ is simplicial and the respective actions are simplicial. Now $c^{-1}(Y \times \epsilon)$ is a subcomplex of K_2 and all vertices of K_2 lie in $Y \times 0$, $c^{-1}(Y \times \epsilon)$, and $Y \times [\delta, 1]$.

Now define an equivariant stretching PL homeomorphism f: $(Y \times I, \theta_0 \times 1) \longrightarrow (Y \times I, \theta_0 \times 1)$ such that f is the identity on $Y \times 0 \cup Y \times [\delta, 1]$ and such that $f(c^{-1}(Y \times \epsilon)) = Y \times \epsilon$.

We define f on vertices and extend it linearly on simplexes. Consider a vertex v of K_2 which lies in $c^{-1}(Y \times \epsilon)$. This vertex v lies in the interior of a unique simplex A of K_1 (its carrier). Simplex A can now be expressed as the join $A_0 * A_1$ where A_0 is the face of A lying in $Y \times 0$ and A_1 is the face of A lying in $Y \times \delta$. Then v lies on a unique line segment l whose end points lie in A_0 and in A_1. Now l meets $c^{-1}(Y \times \epsilon)$ only at v, since $c(l)$ is a straight line segment going from $Y \times 0$ to $c(Y \times \delta)$ and hence cuts $Y \times \epsilon$ at just one point, namely $c(v)$. On the other hand l meets $Y \times \epsilon$ in just one point, which we define to be $f(v)$. This defines f on the vertices of $c^{-1}(Y \times \epsilon)$. On the rest of the vertices of K_2 define f to be the identity. Now, thinking of f as a map from K_2 into K_1 extend f linearly on simplices of K_2. This clearly gives a PL homeomorphism $Y \times I \longrightarrow Y \times I$. We just have to check that f commutes with the action $\theta_0 \times 1$, and it suffices to check this on vertices. This follows immediately, however, since $\theta_0 \times 1$ is simplicial on K_1; for, $\theta_0 \times 1$ permutes the line segments l described above, and on these line segments f clearly commutes with $\theta_0 \times 1$. Now f induces a subdivision K_3 of K_1 so that f: $K_2 \longrightarrow K_3$ is simplicial and K_3 has vertices only in $Y \times 0$, $Y \times \epsilon$, and $Y \times [\delta, 1]$.

Define h: $(Y \times [0, \epsilon], \theta_0 \times 1) \longrightarrow (Y \times I, \theta)$ by $h = cf^{-1}$. Then h: $K_3 | Y \times [0, \epsilon] \longrightarrow L_2$ is a simplicial equivariant homeomorphism mapping $Y \times \epsilon$ to $Y \times \epsilon$. Since K_3 has no vertices in $Y \times (0, \epsilon)$, h must be level-preserving. This completes the proof of Lemma 4.6.1 and so the proof of the theorem.

We note that Example 4.5 of a smooth Z_2-isotopy on R^n shows the necessity of compactness in the theorem, since it is not hard to see that any smooth G-isotopy is locally unknotted.

Since any free PL action is automatically locally unknotted, we have the following corollary.

COROLLARY 4.7. *If Y is a compact polyhedron, then any free* PL *G-isotopy on $Y \times I$ is unknotted.*

In particular, since any G-isotopy between free actions is free, by Theorem 4.1, this means that if ϕ and ψ are free PL actions on Y which are PL G-isotopic, then there is a PL isotopy F: $Y \times I \longrightarrow Y \times I$ such that $F_0 = 1$ and $F_1 \cdot \phi = \psi$.

It is interesting to ask whether there is a notion of unknottedness which makes Theorem 4.6 true in the topological category—certainly the equivariant collaring theorem holds, and it would suffice to be able to find the level-preserving collar of Lemma 4.6.1. In particular, is any locally smooth action (in the sense of Bredon [2]) unknotted? The topological version of Corollary 4.7 does hold, by a completely different proof, see [5] and [6].

We conclude this section with two theorems which show precisely the relationship between G-isotopic actions in the PL category.

THEOREM 4.8. *Let M be a closed* PL *n-manifold and θ be a G-isotopy in $A_{\mathrm{PL}}(G, M \times I)$. Then there is a sequence of actions $\phi_0, \cdots, \phi_r$ in $A_{\mathrm{PL}}(G, M)$ such that $\phi_0 = \theta_0$, $\phi_r = h \cdot \theta_1$ where h is* PL *isotopic to 1, and, for $1 \leq i < r$, $\phi_{i+1} = \phi_i$ except on the invariant disjoint union of the interiors of* PL *n-balls.*

PROOF. The idea is to push $M \times 0$ up through invariant submanifolds of $M \times I$, one cell and its translates at a time, until one reaches $M \times 1$.

Call a sequence of actions as in the statement of the theorem a "sequence from θ_0 to θ_1." Let K be a triangulation of $M \times I$ so that θ is simplicial and the levels $0 = t_0 < t_1 < \cdots < t_k = 1$ containing vertices are triangulated as subcomplexes, by Lemmas 2.1 and 2.2. It suffices to consider just the case $t_1 = 1$. For if, for example, we have sequences from θ_0 to θ_{t_1} and from θ_{t_1} to θ_{t_2}, say $\theta_0 = \phi_0, \cdots, \phi_r = f \cdot \theta_{t_1}$, and $\theta_{t_1} = \psi_0, \cdots, \psi_s = h \cdot \theta_{t_2}$, then we obtain a new sequence from θ_0 to θ_{t_2}

$$\theta_0 = \phi_0, \ldots, \phi_r, \quad f \cdot \psi_0, \ldots, f \cdot \psi_s = (f \circ h) \cdot \theta_{t_2}.$$

Therefore suppose that $t_1 = 1$ and that all vertices of K lie in $M \times \{0, 1\}$.

Next it suffices to find a sequence of actions from θ_0 to $\theta_{1/2}, \theta_0 = \phi_0, \cdots, \phi_r = f \cdot \theta_{1/2}$. For by the same argument there would be a sequence $\theta_1 = \psi_0, \cdots, \psi_s = h \cdot \theta_{1/2}$. Then

$$\theta_0 = \phi_0, \ldots, \phi_r,$$
$$(f \circ h^{-1}) \circ \psi_s, \ldots, (f \circ h^{-1}) \cdot \psi_0 = (f \circ h^{-1}) \cdot \theta_1$$

would be the desired sequence from θ_0 to θ_1.

Let K' be the first derived subdivision of K described as follows: Star each simplex A of K at an interior point $\hat{A}$, where $\hat{A}$ is the barycenter of A if $A \subset M \times \{0, 1\}$, and $\hat{A}$ is chosen to lie in $A \cap M \times 1/2$ otherwise. Also be sure to choose $(\theta^g A)^\wedge = \theta^g \hat{A}$ for all g in G. Then θ is simplicial with respect to K' and $M \times 1/2$ is a subcomplex of K'. Let K_0 and K_0' denote the restrictions of K and K' to $M \times 0$.

Construct a sequence of subcomplexes $M_0, M_1, \cdots, M_m$ of K' such that

(1) $M_0 = M \times 0$, $M_m = M \times 1/2$, and each M_i is θ-invariant;

(2) $M_{i+1} \cap M_i = M_i - J_i' = M_{i+1} - J_{i+1}$, where J_i' and J_{i+1} are each θ-invariant disjoint unions of the interiors of PL n-balls;

(3) there is a sequence of PL homeomorphisms $f_i: M_i \longrightarrow M$ such that $f_0 = \pi | M \times 0, f_m = \pi \circ h | M \times 1/2$, where h is a PL homeomorphism of $M_m = M \times 1/2$ which is PL isotopic to 1, and $f_{i+1} = f_i$ on $M_{i+1} \cap M_i$, all i. Here as usual $\pi: M \times I \longrightarrow M$ is the projection map.

Given this construction the proof is completed by defining $\phi_i = f_i \cdot (\theta | M_i)$. Then $\phi_0 = \theta_0$, $\phi_m = f_m \cdot (\theta | M \times 1/2) = (\pi \circ h) \cdot (\theta | M \times 1/2) = k \cdot \theta_{1/2}$, where k is an appropriate homeomorphism which is PL isotopic to 1; and $\phi_{i+1} = \phi_i$ except on $M_{i+1} - (M_{i+1} \cap M_i)$, an invariant disjoint union of the interiors of n-balls.

Let $A_1, \cdots, A_m$ be the simplexes of K_0 arranged in order of decreasing dimension, just one representative for each "orbit" (if A is a simplex and g is in G, either $A = \theta^g A$ or A and $\theta^g A$ have disjoint interiors). Define M_i inductively by $M_0 = K_0'$ and $M_{i+1} = \mathrm{cl}(M_i - J_i') \cup J_{i+1}$ where

$$J_i' = \bigcup_g \theta^g \text{ int star}(\hat{A}_{i+1}, M_i), \quad g \text{ in } G;$$

and $J_{i+1} = \bigcup_g \theta^g$ int link (A_{i+1}, M_i^+). Here $M_0^+ = K'$ and

$$M_{i+1}^+ = \mathrm{cl}\left[M_{i+1}^+ - \bigcup_g \theta^g \text{ star}(\hat{A}_{i+1}, M_i^+)\right]$$
$$= \mathrm{cl}\left[M \times I - \bigcup_g \bigcup_{j \leq i+1} \theta^g \text{ star}(\hat{A}_j, K')\right],$$

the part of $M \times I$ lying above M_{i+1}. One checks that J_i' is a disjoint union of open n-balls which have a common boundary with those of J_{i+1}. Also M_i is PL homeomorphic to M. This all follows since one can verify inductively that

$$\text{star}(\hat{A}_{i+1}, M_i^+) \cap \theta^g \text{ star}(\hat{A}_{i+1}, M_i^+) \subset \text{bd star}(\hat{A}_{i+1}, M_i).$$

(The easiest approach to this seems to be via dual cells, as below.) It is now easy to verify properties (1) and (2) above. So it remains only to construct the PL homeomorphisms $f_i: M_i \longrightarrow M$.

The details of the construction are elementary but a little complicated, since we wish to be sure the final result is PL isotopic to the identity. The point is that if $\pi|M_i$ were one-to-one for each i, then we could define $f_i = \pi|M_i$ and be done. Since this is not the case in general, we work a little harder.

It is useful to give another description of the M_i in terms of dual cells. For a simplex A in K let $D(A, K') = \{\hat{B}_1\hat{B}_2 \cdots \hat{B}_j: A \leqslant B_1 < \cdots < B_j, B_i \text{ in } K\}$ be the dual cell to A in K'. (See Hudson, for example, [7, pp. 29ff.].) If A is a k-simplex, then $D(A, K')$ is an $(n + 1 - k)$-cell. The dual cells give a cell decomposition of $M \times I$ with respect to which θ is cellular. If A happens to lie in K_0, there are both $D(A, K_0')$ and $D(A, K')$.

With $A_1, \cdots, A_m$ as before then we may observe that

$$J_i' = \bigcup_g \theta^g \text{ int}(A_{i+1}' * \text{bd } D_{i+1}) \text{ and } J_{i+1} = \bigcup_g \theta^g \text{ int}[\text{bd } A_{i+1}' * D_{i+1}],$$

where $D_{i+1} = D(A_{i+1}, K') \cap (M \times \tfrac{1}{2})$ and "$*$" denotes join. We now use the dual cell description of M_i to define f_i.

Let L be the cell complex subdividing $M \times I$ whose cells are of the form $A \times 0, A \times 1$, or $A \times I$, where A is a simplex of K_0. Let L' be the first derived subdivision of L obtained by inductively starring cells $A \times 0, A \times 1$, and $A \times I$ at $(\hat{A}, 0)$, $(\hat{A}, 1)$, and $(\hat{A}, \tfrac{1}{2})$, where $\hat{A}$ is the barycenter of A, A in K_0. Then for simplexes A of $L_0 = K_0$, $D(A, L')$ makes sense and is clearly setwise equal to $D(A, L_0') \times [0, \tfrac{1}{2}]$. There is a level-preserving PL homeomorphism h: $M \times [0, \tfrac{1}{2}] \longrightarrow M \times [0, \tfrac{1}{2}]$ such that $h|M \times 0$ is the identity, $h(D(A, L')) = D(A, K')$, for all A in $L_0 = K_0$, and there is a subdivision L^* of L' adding vertices only in $M \times \tfrac{1}{2}$ so that h: $L^* \longrightarrow K'$ sends simplices linearly into simplices. One constructs h inductively over the dual cells $D(A, L')$. To define h, let $B_1, \cdots, B_q$ be (all) the simplices of L_0 in order of decreasing dimension. Suppose h is defined on $D(B_1, L') \cup \cdots \cup D(B_i, L')$, and then define h on $D(B_{i+1}, L') = D(B_{i+1}, L_0') \times [0, \tfrac{1}{2}]$. Observe that h is already defined on bd $D(B_{i+1}, L_0') \times [0, \tfrac{1}{2}]$; extend in any way, subdividing as necessary, to

$$D(B_{i+1}, L_0') \times \tfrac{1}{2} \to D(B_{i+1}, K') \cap (M \times \tfrac{1}{2});$$

then extend, by coning from B_{i+1}, to all of $D(B_{i+1}, L')$.

Now define $f_i: M_i \longrightarrow M$ to be $\pi \circ (h^{-1}|M_i)$. Since h^{-1} is by construction PL isotopic to the identity, we have only to verify inductively that π is one-to-one on $h^{-1}(M_i)$, for all i. This is clear for $i=0$; suppose it is true for i, and consider $h^{-1}(M_{i+1})$. By induction $\pi|h^{-1}(M_{i+1} \cap M_i)$ is one-to-one. It clearly suffices to show that $\pi|h^{-1}(\text{bd}\, A'_{i+1} * D_{i+1})$ is one-to-one. But by construction

$$h^{-1}(\text{bd}\, A'_{i+1} * D_{i+1}) = \text{bd}\, A'_{i+1} * h^{-1}(D_{i+1}) = \text{bd}\, A_{i+1} * [D(A_{i+1}, L'_0) \times ½];$$

also $\pi|\text{bd}\, A_{i+1}$ is the identity, $\pi[D(A_{i+1}, L'_0) \times ½] = D(A_{i+1}, L'_0)$ in a one-to-one way, and π preserves join coordinates here. Thus π is one-to-one on $h^{-1}(\text{bd}\, A_{i+1} * D_{i+1})$ as desired. This completes the proof of Theorem 4.8.

In our final result we improve the preceding theorem to obtain $\phi_r = \theta_1$, rather than $\phi_r = h \cdot \theta_1$ for some h PL isotopic to 1.

THEOREM 4.9. *Let M be a closed* PL *n-manifold and θ be a G-isotopy in $A_{\text{PL}}(G, M \times I)$. Then there is a sequence of actions $\phi_0, \cdots, \phi_k$ in $A_{\text{PL}}(G, M)$ such that $\phi_0 = \theta_0$, $\phi_k = \theta_1$, and each $\phi_{i+1} = \phi_i$ except on the invariant disjoint union of the interiors of* PL *n-balls.*

PROOF. By Theorem 4.8 there is a sequence $\theta_0 = \phi_0, \cdots, \phi_r = h \cdot \theta_1$, where h is PL isotopic to 1_M. Let H: $M \times I \longrightarrow M \times I$ be a PL isotopy such that $H_0 = h$ and $H_1 = 1_M$.

Now use an equivariant version of the proof (see Hudson [7, pp. 130ff.]) of "isotopy implies isotopy by moves" to find a sequence $h_1, \cdots, h_s$ of PL homeomorphisms of M such that $h_1 = h$, $h_s = 1_M$, and $h_{i+1} = h_i$ except on the θ_1-invariant disjoint union of the interiors of n-balls. One then defines $\psi_i = h_1 \cdot \theta_1$ to obtain a sequence from $\phi_r = h_1 \cdot \theta_1$ to $h_s \cdot \theta_1 = \theta_1$.

For completeness we sketch the construction of the h_i's. Let K triangulate M so that θ_1 is simplicial and so that θ_1^g leaves $A \cap \theta_1^g A$ pointwise fixed, for any A in K and g in G. One need only pass to a second derived subdivision to obtain this property—see Bredon [2, pp. 115ff.]. Suppose α: $K \longrightarrow I$ is linear on simplices. Hudson shows that if the diameter of $\alpha(K)$ is sufficiently small then the map $\alpha_* = \pi \circ H(i, \alpha)$ (i.e., $\alpha_*(x) = \pi \circ H(x, \alpha(x))$, where π: $M \times I \longrightarrow M$ is the projection) is a PL homeomorphism. Thus we may find a finite sequence of such maps α: $K \longrightarrow I$ such that α_1 is identically zero, α_s is identically one, $\alpha_{i+1} = \alpha_i$ except on the θ_1-translates of the open star of some vertex v_{i+1} in K, and the diameter of each $\alpha_i(K)$ is "sufficiently small." Then define $h_i = \alpha_{i*}$. This completes the proof of the theorem.

Finally, we remark that Theorems 4.8 and 4.9 clearly put a premium on understanding actions on disks. In particular, given ϕ in $A(G, D^n)$ one would like to know in some sense all ψ in $A(G, D^n)$ such that $\psi|S^{n-1} = \phi|S^{n-1}$.

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