FIXED POINTS OF POINTWISE ALMOST PERIODIC HOMEOMORPHISMS ON THE TWO-SPHERE

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ABSTRACT. A homeomorphism f of the two-sphere S^2 onto itself is defined to be pointwise almost periodic (p.a.p.) if the collection of orbit closures forms a decomposition of S^2 . It is shown that if $f: S^2 \rightarrow S^2$ is p.a.p. and orientation-reversing then the set of fixed points of f is either empty or a simple closed curve; if $f: S^2 \rightarrow S^2$ is p.a.p. orientation-preserving and has a finite number of fixed points, then f is shown to have exactly two fixed points.

1. Introduction. Every periodic mapping f of the two-sphere S^2 to itself is topologically equivalent either to the identity, to a rotation, a reflection, or to a rotation followed by a reflection ([5] and [9]). Thus, the set of fixed points of f is either empty or an i -sphere, $0 \leq i \leq 2$. If f satisfies the weaker condition of being almost periodic (equivalent to having equicontinuous iterates) or the still weaker condition of being weakly almost periodic (the collection of orbit closures forms a continuous decomposition of S^2), the fixed point set is again either empty or an i -sphere, $0 \leq i \leq 2$ ([11] and [12]).

In this paper we study the fixed point sets of pointwise almost periodic (p.a.p.) homeomorphisms on S^2 (the collection of orbit closures forms a decomposition of S^2). In the orientation-reversing case the set of fixed points must still be either empty or a 1-sphere (Theorem 6). In the orientation-preserving case, on the other hand, there may be a continuum of fixed points together with any finite or countable number of isolated fixed points (see §6). However, if there are only a finite number of fixed points in the orientation-preserving case, then there must be exactly two (Theorem 7).

The main theorems of this paper are contained in §§5 and 6. §3 gives a summary of the results in the theory of prime ends which we use to prove the main lemma in §4.

2. Definitions and notation. If A is a subset of a space X , $\text{Cl}(A)$ and $\text{Bd}(A)$ denote, respectively, the closure and boundary of A . A is *nondegenerate* if A is not a single point. If $f: X \rightarrow X$ is a map, then $f|A$ denotes the restriction

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of f to A . Homeomorphisms will always be onto.

A *domain* is a connected open set. If U is a domain in S^2 and M is a subset of a component R of $S^2 - \text{Cl}(U)$, then $\text{Bd}(R)$ is the *outer boundary* of U with respect to M .

If $f: X \rightarrow X$ is a homeomorphism and $A \subset X$, then $\text{orbit}(A)$, the *orbit* of A , is the union of the set of iterates $f^n(A)$, $n = 0, \pm 1, \pm 2, \dots$, ($f^0 = \text{Id}$). The set of *period two points* of f is the set $\{x \in X: f^2(x) = x\}$. Thus, the set of period two points includes the fixed points of f .

$f: X \rightarrow X$ is a *recurrent homeomorphism* if, given any $x \in X$ and any neighborhood U of x , there is a positive integer n and a negative integer m such that $\{f^n(x), f^m(x)\} \subset U$ [8, 10.18, p. 83].

$f: X \rightarrow X$ is a *pointwise almost periodic* (p.a.p.) homeomorphism if given any $x \in X$ and any neighborhood U of x , there is a finite set K of integers such that the orbit of x is contained in the union of the sets $f^n(U)$, $n \in K$ [8, 4.02, p. 31]. If X is a locally compact T^2 space, an equivalent definition is: $f: X \rightarrow X$ is p.a.p. if the collection $\{\text{Cl}(\text{orbit}(x)): x \in X\}$ forms a decomposition of X [8, 4.10, p. 32]. Note that if a homeomorphism is p.a.p. then it is recurrent.

If U is a domain in S^2 with a nondegenerate boundary, then a *crosscut* of U is an open arc in U whose closure is an arc which intersects $\text{Bd}(U)$ in two points. An *endcut* of U is a half-open arc in U whose closure is an arc which intersects $\text{Bd}(U)$ in one point. If A is a crosscut or an endcut of U , then a *subendcut* of A is an endcut of U which is contained in A .

3. Prime ends. This section contains the results from the theory of prime ends which we use in the next two sections. See also [12, §3] or [15].

Let U be a simply connected domain in S^2 with a nondegenerate boundary. A *C-transformation* of U onto the open unit disk D is a homeomorphism $T: U \rightarrow D$ such that the image of any crosscut of U is a crosscut of D , and the endpoints of such images of crosscuts of U are dense in the unit circle. C-transformations always exist [15, Appendix 2].

A collection of crosscuts $Q_1, Q_2, \dots$, of the simply-connected domain U is a *chain* if (a) the arcs $\text{Cl}(Q_1), \text{Cl}(Q_2), \dots$, are pairwise disjoint; (b) Q_n separates Q_{n-1} from Q_{n+1} in U ; (c) there is a point on $\text{Bd}(U)$ whose greatest distance from Q_n approaches 0 as $n \rightarrow \infty$. Corresponding to each Q_n there is a domain U_n of $U - Q_n$ containing Q_{n+1} . Note $U_1 \supset U_2 \supset \dots$.

If $\{Q_i\}$, $\{R_i\}$ are chains of crosscuts and $\{U_i\}$, $\{H_i\}$ are their respective corresponding domains, then $\{Q_i\}$, $\{R_i\}$ are *equivalent chains* if for every n there is an m such that $H_m \subset U_n$ and $U_m \subset H_n$. Equivalent chains are said to define the same *prime end*. Thus, a prime end of U is an equivalence class of chains of U .

If $Q_1, Q_2, \dots$ is a chain of crosscuts in U , then their images $T(Q_1), T(Q_2), \dots$ under the C -transformation $T: U \rightarrow D$ is a chain in D . If $\{Q_i\}, \{R_i\}$ are equivalent chains in U , then $\{T(Q_i)\}, \{T(R_i)\}$ are equivalent chains in D and converge to the same point on the boundary of D . Thus, T sets up a 1-1 correspondence between prime ends of U and points on the unit circle.

If $Q_1, Q_2, \dots$ is a chain defining the prime end E and $U_1, U_2, \dots$ are the corresponding domains of the chain, then the *impression* of E is the set $\bigcap_{i=1}^{\infty} \text{Cl}(U_i)$. The impression of E is easily seen to be independent of which defining chain is used. Note that $\text{Impression}(E) \subset \text{Bd}(U)$ and that distinct prime ends may have identical impressions.

Suppose U is a simply connected domain, $T: U \rightarrow D$ is a C -transformation, E is a prime end of U , and e is the point on the unit circle corresponding to E under T . A half-open arc B in U defines the prime end E if $T(B)$ is an endcut in D with e as a limit point. Among the half-open arcs defining the prime end E there is one A such that $\text{Cl}(A) - A$ is minimal (contained in $\text{Cl}(B) - B$ for every half-open arc B defining E). This minimal set is the set of *principal points* of E (for an alternate definition of principal point see [12, §3]). Note that a prime end defined by an endcut has just one principal point.

Given a homeomorphism $f: \text{Cl}(U) \rightarrow \text{Cl}(U)$, with $f(U) = U$, and a C -transformation $T: U \rightarrow D$, it follows that $TfT^{-1}: D \rightarrow D$ is a C -transformation which may be extended to a homeomorphism h of the closed unit disk onto itself [15, 4.10, p. 6; A1.7, p. 27]. If E is a prime end of U , e is the point on the unit circle corresponding to E under T , and $h(e) = e$, then E is a *fixed prime end* of f . If G is another prime end of U , p is the point on the unit circle corresponding to G under T , and p converges to e under positive iterates of h , then we say *the prime end G converges to the prime end E* under positive iterates of f . These last two definitions are independent of the choice of the C -transformation T .

The reader unfamiliar with prime ends might attempt to show as an exercise that if K is a pseudo-arc [1] and E is a prime end of $S^2 - K$ then $\text{Impression}(E) = K$; also, there exists a prime end E of $S^2 - K$ such that every point of K is a principal point of E . Completing this exercise, however, is not necessary for understanding the rest of the paper.

4. The Main lemma. Preliminary remark: In the proof of Lemma 1 below, various crosscuts and endcuts are constructed in a domain U . Our diagrams, however, will always show the images in the open unit disk of these crosscuts and endcuts under a C -transformation. To avoid clumsy notation the crosscuts and endcuts will be denoted with the same letters as their C -images in the diagrams.

LEMMA 1. Suppose $f: S^2 \rightarrow S^2$ is an orientation-preserving homeomorphism and U is an invariant, simply-connected domain with nondegenerate boundary. Suppose A and B are endcuts of U such that (1) the prime end E of U which is defined by A is fixed under f , and (2) the prime end F of U which is defined by B is distinct from E but converges to E under positive iterates of f . Then f is not recurrent on $\text{Bd}(U)$.

PROOF. Let b be the point $\text{Cl}(B) \cap \text{Bd}(U)$.

Case 1. b is not in the impression of the prime end E . Then let V be a neighborhood (in S^2) of b whose closure misses $\text{Impression}(E)$. Let $U_1 \supset U_2 \supset \dots$ be a sequence of subdomains of U such that $\text{Impression}(E) = \bigcap_{i=1}^{\infty} \text{Cl}(U_i)$. Then for some n , $U_n \cap V = \emptyset$. If f were recurrent at b , then subendcuts of infinitely many positive iterates of B would be contained in V , and thus would miss U_n . Then the prime end F would not converge to the prime end E .

Case 2. f is periodic at b , with least period n . If $f(b) = b$, $n = 1$, let Y_1 be an open arc in U such that $\text{Cl}(Y_1)$ is a simple closed curve and Y_1 contains a subendcut of B and a subendcut of $f(B)$. If $f^n(b) = b$, $n > 1$, let $Y_1, \dots, Y_n$ be a set of pairwise disjoint crosscuts of U such that each Y_i contains a subendcut defining the same prime end as $f^{i-1}(B)$ and a subendcut defining the same prime end as $f^i(B)$, $1 \leq i \leq n$. Note that $\text{Cl}(Y_n) \cap \text{Cl}(Y_1) = \{b\} = \{f^n(b)\}$. In both cases, $\text{Cl}(Y_1) \cup \dots \cup \text{Cl}(Y_n)$ forms a simple closed curve J such that J separates a subendcut of A from the closure of some endcut N of U (see Figure 1 for a sketch of the C -images of $Y_1, \dots, Y_n, A, N$). Note that $\text{Bd}(U) \cap \text{Cl}(N)$ cannot lie in $\text{Impression}(E)$.

Now, if J' is the arc in the unit circle bounded by the endpoints of the C -images of A and B and containing the endpoint of the C -image of $f(B)$, and if h is the (orientation-preserving) homeomorphism of the closed unit disk associated with f (see definition of convergent prime end, §3), then $h(J') \subset J'$. The endpoint of the C -image of N is in J' and thus converges to the fixed endpoint of J' under positive iterates of h .

But then the prime end determined by N converges to the prime end E , and $\text{Bd}(U) \cap \text{Cl}(N)$ cannot lie in $\text{Impression}(E)$. Hence, by Case 1, $\text{Bd}(U) \cap \text{Cl}(N)$ is not a recurrent point of f .

Case 3. b is in the impression of E , but is not a periodic point. We suppose f recurrent on $\text{Bd}(U)$ and derive a contradiction.

Our plan is to construct a simple closed curve J , made up of crosscuts of U plus an arc Y , such that J separates the endcut A and some point of $\text{Impression}(E)$, (to construct J we may have to modify f on some subdisks of U), then to obtain a certain subcontinuum L of $\text{Impression}(E)$ such that $L \cap Y \neq \emptyset$ but L misses one component of $S^2 - J$, and finally to obtain

FIGURE 1

the contradiction that $f(L) = L$ but $f(L)$ must intersect both components of $S^2 - J$. We now proceed with this plan.

Since E is a fixed prime end we may assume $f = \text{Id}$ on A . For, there is an open disk $Z \subset U$ and a subendcut A' of A such that $\text{Cl}(Z) \cap \text{Bd}(U) = \text{Cl}(A) \cap \text{Bd}(U)$, and $A' \cup f(A') \subset Z$. Then replace A by A' and f by f followed by a homeomorphism which is the identity outside Z and which is equal to f^{-1} on $f(A')$.

Choose a crosscut Q of $U - A$ such that (a) Q has one endpoint on A and the other on $\text{Bd}(U)$, (b) there is a positive integer n such that Q separates $f^{-1}(B) \cup B \cup f(B)$ from some subendcut of $f^n(B)$ in $U - A$, (c) $\text{Cl}(Q)$ is disjoint from $\text{Cl}(f^{-1}(B) \cup B \cup f(B))$ and from $f^n(b)$. The existence of Q follows from the fact that the prime end defined by B converges to the prime end defined by A .

Next, choose a crosscut X in U such that (d) the endpoints of X are b and $f(b)$, (e) X contains a subendcut defining the prime end, F , (f) the crosscuts $f^{-1}(X), X, f(X), \dots, f^n(X)$ form a pairwise disjoint collection (here, n is the integer mentioned in (b) of the preceding paragraph), and (g) Q separates $f^{-1}(X)$ from $f^n(X)$ in $U - A$. See Figure 2 for a sketch of C -images.

The existence of X follows from the facts that $f(b) \neq b$ and that F converges to E .

Next, choose an open (in S^2) neighborhood O of b such that

$\text{Cl}(O)$ is a 2-cell;

$\text{Cl}(O) \cap f(\text{Cl}(O)) = \emptyset$;

$[A \cup \text{Cl}([Q])] \cap [\text{Cl}(O) \cup f(\text{Cl}(O))] = \emptyset$;

$\text{Cl}(O) \cap [\text{Cl}(X) \cup f^{-1}(\text{Cl}(X))]$ is an arc; and

$\text{Cl}(O) \cap f^i(X) = \emptyset$ for $i = 1, 2, \dots, n$.

FIGURE 2

Claim. There is a homeomorphism $k: S^2 \rightarrow S^2$ such that:

- (1) $k = f$ on some neighborhood of $\text{Bd}(U) \cup \text{Cl}(O) \cup A$,
- (2) $k^i(X) = f^i(X)$ for $i = -1, 0, 1, \dots, n$, (3) for some integer $m > n$, $k^{-1}(X), X, k(X), \dots, k^m(X)$ is a pairwise disjoint collection, $k^i(X) \cap \text{Cl}(O) = \emptyset$, $n \leq i < m$, $k^m(X) \cap \text{Cl}(O) \neq \emptyset$, and Q separates $k^{-1}(X)$ and $k^i(X)$ in $U - A$, $n \leq i \leq m$.

Proof of claim. We construct k by modifying f on various subdisks of U .

Suppose $f^{n+1}(X) \cap Q \neq \emptyset$. Let W be the component of $U - (f^n(X) \cup f^{-1}(Q))$ such that $\text{Bd}(W)$ contains two disjoint subendcuts of $f^n(X)$. Let N be a crosscut of W such that $\text{Cl}(N) \cap \text{Bd}(U) = \emptyset$, the endpoints of N separate $f^n(b)$ and $f^{n+1}(b)$ from $f^{-1}(Q) \cap f^n(X)$ in $\text{Cl}(f^n(X))$, and N is contained in a small neighborhood of $f^n(X) \cup f^{-1}(Q)$. Since $f^n(X) \cap \text{Cl}(O) = \emptyset$ and $\text{Cl}(Q) \cap f(\text{Cl}(O)) = \emptyset$, we may choose N so that $\text{Cl}(N) \cap \text{Cl}(O) = \emptyset$, (see Figure 3 for C -images).

FIGURE 3

Let $Z \subset U$ be the disk bounded by N and the subarc of $f^n(X)$ cut off by the endpoints of N . Then Z misses $\text{Cl}(O)$ since $\text{Bd}(Z)$ misses $\text{Cl}(O)$. Let Z_1 be a small neighborhood of Z such that $\text{Cl}(Z_1)$ is a subdisk of U , $\text{Cl}(Z_1) \cap \text{Cl}(O) = \emptyset$, $\text{Cl}(Z_1) \cap A = \emptyset$, and $Z_1 \cap f^i(X) = \emptyset$, $i = -1, 0, 1, \dots, n-1$. Let $g: S^2 \rightarrow S^2$ be a homeomorphism such that $g = \text{Id}$ outside Z_1 , $g = \text{Id}$ on $f^n(X) - Z$, and $g(\text{Bd}(Z) - N) = \text{Cl}(N)$. Finally, let $h = fg$.

Note that (1) $h = f$ on some neighborhood of $\text{Bd}(U) \cup \text{Cl}(O) \cup A$, (2) $h^{n+1}(X) \cap Q = \emptyset$, and (3) $\{h^{-1}(X), X, h(X), \dots, h^{n+1}(X)\} = \{f^{-1}(X), X, f(X), \dots, f^n(X)\} \cup \{h^{n+1}(X)\}$ is a pairwise disjoint collection. (For, $h^{n+1}(X)$ can intersect the preceding images of X only in $f^{-1}(X)$; otherwise, the preceding images would not be disjoint. But Q separates $h^{n+1}(X)$ and $f^{-1}(X)$.)

If $\text{Cl}(h^{n+1}(X)) \cap \text{Cl}(O) \neq \emptyset$, then h is the homeomorphism we seek. If $\text{Cl}(h^{n+1}(X)) \cap \text{Cl}(O) = \emptyset$, we repeat the above process, modifying h to add another image of X to our collection. This process must finally terminate, however, because we are assuming that f is recurrent at b , and $b \in O$, so eventually some image of X will intersect $\text{Cl}(O)$. This completes the proof of our claim.

To simplify notation let us assume that f requires no modification, that $f^{n+1}(X) \cap Q = \emptyset$ and $\text{Cl}(f^{n+1}(X)) \cap \text{Cl}(O) \neq \emptyset$.

Let P be the arc $[\text{Cl}(X) \cup f^{-1}(\text{Cl}(X))] \cap \text{Cl}(O)$. We may assume that O was chosen small enough so that one of the components V of $\text{Cl}(O) - P$ is contained in $U - (A \cup Q)$. Then we must have $\text{Cl}(f^{n+1}(X)) \cap \text{Cl}(O)$ contained in $\text{Cl}(O) - V$. Let Y be an arc in $\text{Cl}(O) - V$ from $b (= \text{Cl}(X) \cap f^{-1}(\text{Cl}(X)))$ to a point of $\text{Cl}(f^{n+1}(X))$ such that, except for its endpoints, Y misses $\text{Cl}(f^{-1}(X) \cup X \cup f(X) \cup \dots \cup f^{n+1}(X))$. Then from the set $Y \cup \text{Cl}(X \cup f(X) \cup \dots \cup f^{n+1}(X))$ we may form a simple closed curve J .

Note that $f(Y - \{b\})$ does not intersect J , because $Y \subset \text{Cl}(O)$ and $f(\text{Cl}(O)) \cap \text{Cl}(O) = \emptyset$ (this is the reason for never modifying f on $\text{Cl}(O)$).

Also, note that $\text{Cl}(A)$ and $f(Y - \{b\})$ are in different components of $S^2 - J$. For, we may obtain an arc C from A to $f(b)$, which does not intersect $J - \{f(b)\}$, by starting at A and traveling along close to Q and then close to $f^{n-1}(X) \cup f^{n-2}(X) \cup \dots \cup f(X)$ until we hit $f(V)$, and then traveling through $f(V)$ up to $f(b)$. Then $C \cup f(Y)$ is an arc which intersects J in the piercing point $f(b)$, and thus its endpoints must lie in different components of $S^2 - J$.

Let H be the component of $S^2 - J$ containing the endcut A . We want to construct a subcontinuum L of $\text{Bd}(U)$ such that $L \subset \text{Cl}(H)$, $L \cap Y \neq \emptyset$, and $f(L) = L$. This will yield a contradiction.

Choose a chain $R_1, R_2, \dots$ of crosscuts of U defining E , with corresponding domains $U_1 \supset U_2 \supset \dots$, such that

$$U_1 \cap (f^{-1}(X) \cup X \cup \dots \cup f^{n+1}(X)) = \emptyset \text{ and } U_1 \cap f(V) = \emptyset.$$

Note that $f(b) \in \text{Cl}(U_i)$, $i = 1, 2, \dots$, since $b \in \text{Impression}(E)$, and E is a fixed prime end. Let A_1 be an arc in U_1 from A to a point very close to $f(b)$. Since $U_1 \cap f(V) = \emptyset$, we must have $A_1 \cap J \neq \emptyset$, hence $A_1 \cap Y \neq \emptyset$. Let B_1 be the subarc of A_1 from $A \cap A_1$ to the first point at which A_1 intersects Y . Then $B_1 \subset \text{Cl}(H)$.

Repeating this procedure, we may construct a sequence $B_1, B_2, \dots$ of disjoint arcs of U such that:

- (1) B_i intersects A and Y ,
- (2) $B_i \subset \text{Cl}(H)$,
- (3) $B_i \subset U_{m(i)}$, for some $m(i) \geq i$.

We may assume all B_i 's lie on the "same side" of A ($A \cup Q$ does not separate any $B_i - A$ from any $B_j - A$ in U) and that the B_i 's converge to a subcontinuum L of $\text{Bd}(U)$. (See Figure 4 for C -images.)

But then we may choose a half-open arc $T \subset U$ in a small neighborhood of $A \cup B_1 \cup B_2 \cup \dots$ such that $T \cap A = \emptyset$, T also defines the prime end E , and $\text{Cl}(T) - T = L$. Since f is orientation-preserving, both T and $f(T)$ lie on the same side of A . Thus, by [15, 2.2, p. 2] or [14, 3.38, p. 321], either $L \subset f(L)$ or $f(L) \subset L$. But we are assuming that f is recurrent on $\text{Bd}(U)$ so we must have $f(L) = L$ [16, 4.12, p. 247].

Note also that $L \subset \text{Cl}(H)$ and $L \cap Y \neq \emptyset$.

This easily yields a contradiction, for if $L \cap (Y - \{b\}) \neq \emptyset$, then, since $f(Y - \{b\}) \subset S^2 - \text{Cl}(H)$, we cannot have $f(L) = L$; and, if $b \in L$, then $f(L) = L$ is a continuum containing $f(b)$, but every nondegenerate subcontinuum of $\text{Bd}(U)$ containing $f(b)$ must intersect $S^2 - \text{Cl}(H)$, since $f(V) \cap \text{Bd}(U) = \emptyset$.

FIGURE 4

This final contradiction establishes Lemma 1.

5. Fixed points of orientation-reversing p.a.p. homeomorphisms.

LEMMA 2. Let $f: X \rightarrow X$ be a p.a.p. homeomorphism of the complete metric space X , K a compact invariant subset of X , and U an open subset of X such that $U - K \neq \emptyset$. Then there is an open subset V of U such that the orbit of V misses some neighborhood of K .

PROOF. For each positive integer n , let $R(n) = \{x \in U - K: \text{for some integer } m, \text{dist}(f^m(x), K) < 1/n\}$. $U - K$ is complete and each $R(n)$ is open in $U - K$. Therefore, some $R(n)$ is not dense in $U - K$; otherwise, $\bigcap_{n=1}^{\infty} R(n)$ is not empty by the Baire category theorem, but f cannot be p.a.p. at points of $\bigcap_{n=1}^{\infty} R(n)$. Hence, for some positive integer m , there is an open set V in $U - K$ which misses $R(m)$. V is the required open set, and the proof is complete.

LEMMA 3. Suppose $f: S^2 \rightarrow S^2$ is an orientation-reversing p.a.p. homeomorphism, p is a fixed point of f , and Y is the component of the set of period two points such that $p \in Y$. Then Y is nondegenerate.

PROOF. Assume $Y = \{p\}$. We shall establish a contradiction.

Denote by $P(2, f)$ the set of period two points of f .

Claim 1. There is an orientation-reversing, p.a.p. self-homeomorphism of S^2 whose set of period two points is totally disconnected and contains a fixed point.

Proof of Claim 1. Let G be the decomposition of S^2 whose elements are the points of $S^2 - P(2, f)$ and the components of $P(2, f)$. G is upper semicontinuous [7, p. 137], hence the decomposition space S^2/G is a cactoid [16, (2.2)', p. 172]. It is easily seen that the induced map $g = \pi f \pi^{-1}: S^2/G \rightarrow S^2/G$ (where $\pi: S^2 \rightarrow S^2/G$ is the decomposition map) is a p.a.p. homeomorphism, and $P(2, g) = \pi(P(2, f))$. Thus, $P(2, g)$ is totally disconnected. Now, $\pi(p)$ is not a cut point of S^2/G , since $\pi^{-1}(\pi(p)) = p$, so $\pi(p)$ is either an endpoint of S^2/G or is contained in a true cyclic element of S^2/G , [16, p. 66]. If M is a true cyclic element (2-sphere) containing $\pi(p)$, then $g(\pi(p)) = \pi(p)$, so $g(M) = M$, and $g|M$ is the required homeomorphism (it is clear that g must be orientation-reversing).

If $\pi(p)$ is an endpoint, then there is a cutpoint $\pi(q)$ fixed by g [16, 4.22, p. 247]. If $C(\pi(q), \pi(p))$ is the cyclic chain [16, p. 71] of S^2/G from $\pi(q)$ to $\pi(p)$, then every cyclic element of $C(\pi(q), \pi(p))$ is invariant under g [16, 4.3, p. 248]. Let M be any true cyclic element of $C(\pi(q), \pi(p))$. Then M is a 2-sphere and $g(M) = M$. M contains a point which separates $\pi(q)$ and $\pi(p)$ in S^2/G [16, 5.2, p. 71], and this point must be a fixed point of g [16, 4.21, p. 247]. Thus, $g|M$ is the required orientation-reversing homeomorphism.

Claim 1 has been established.

By Claim 1 we may assume without loss of generality that $P(2, f)$ is totally disconnected.

Let x, y be points of $S^2 - \{p\}$ such that $f(x) = y$, and $f(y) = x$ ($x = y$ is allowed). The existence of x, y follows from the fact that if $f^2|_{S^2 - \{p\}}$ were fixed point free, then $S^2 - \{p\}$ would contain a point converging to p under positive iterates of f^2 [2, Theorem 8, p. 45] (or see Theorem 7 of the present paper). Let K be a continuum in S^2 which is invariant under f , which contains $\{p, x, y\}$ and which is minimal with respect to containing $\{p, x, y\}$ and being closed, connected, and invariant. By Lemma 2, there is a set $U_1 \subset K$, open in K , such that the orbit of U_1 misses a neighborhood of $P(2, f) \cap K$. Let A_1 be the component of $K - \text{orbit}(U_1)$ containing p . Note that A_1 is invariant and nondegenerate. Since K is minimal, A_1 cannot contain x or y . Let D_1 be the component of $S^2 - A_1$ containing x .

Claim 2. $f(D_1) \cap D_1 = \emptyset$.

Proof of Claim 2. Suppose $f(D_1) \cap D_1 \neq \emptyset$. Then D_1 is a simply-connected, invariant domain with a nondegenerate boundary. We note that D_1 has a prime end which is fixed under f . For, let T be a C -transformation of D_1 onto the open unit disk, and extend TfT^{-1} to an orientation-reversing homeomorphism h of the closed unit disk onto itself. Then h must have two fixed points on the unit circle, and these two fixed points correspond to fixed prime ends of f .

But then the orientation-preserving homeomorphism f^2 must also have a fixed prime end in D_1 . Hence, every prime end of D_1 is either fixed under f^2 or converges to a fixed prime end under positive iterates of f^2 [3, Lemma 14].

Since $P(2, f)$ is totally disconnected and closed, there is an endcut B in D_1 such that $\text{Cl}(B) \cap \text{Bd}(D_1)$ is not in $P(2, f)$. Thus, if F is the prime end defined by B , the principal point of F is not fixed under f^2 , and so F is not a fixed prime end of f^2 [12, Lemma 1]. Thus, F converges under positive iterates of f^2 to a fixed prime end E . Since every principal point of E is fixed under f^2 [12, Lemma 1] and $P(2, f)$ is totally disconnected, E has only one principal point, [14, Corollary, p. 275]. Thus, there is an endcut A of D_1 which defines E . But then, by Lemma 1, f^2 is not recurrent on $\text{Bd}(D_1)$. This contradicts the fact that, since f is p.a.p. on $\text{Bd}(D_1)$, f^2 is p.a.p. on $\text{Bd}(D_1)$ [8, p. 31].

Claim 2 is established.

Since $x \in D_1$ and A_1 is invariant, $f^2(D_1) = D_1$, hence $\text{Bd}(D_1) \cup f(\text{Bd}(D_1))$ is an invariant subset of f .

Claim 3. There is a nondegenerate, invariant subcontinuum L of A_1 such that if O is the component of $S^2 - L$ containing x , then $p \in \text{Bd}(O)$.

Proof of Claim 3. If $p \in \text{Bd}(D_1)$, then $\text{Bd}(D_1) \cup f(\text{Bd}(D_1))$ is the required subcontinuum. Suppose $p \notin \text{Bd}(D_1)$. Let B_1 be an invariant subcontinuum of

A_1 containing $\{p\} \cup \text{Bd}(D_1) \cup f(\text{Bd}(D_1))$ and minimal with respect to these properties. Let V_2 be a ball of radius less than $\frac{1}{2}$, centered at p . By Lemma 2, there is a set $U_2 \subset B_1 \cap V_2$, open in B_1 , such that the orbit of U_2 misses a neighborhood of $[B_1 \cap P(2, f)] \cup \text{Bd}(D_1) \cup f(\text{Bd}(D_1))$. Let A_2 be the component of $B_1 - \text{orbit}(U_2)$ containing p . Then A_2 is invariant, nondegenerate; and, by the minimality of B_1 , A_2 does not intersect $\text{Bd}(D_1) \cup f(\text{Bd}(D_1))$. If D_2 is the component of $S^2 - A_2$ containing $\text{Bd}(D_1)$ (and thus containing $\text{Cl}(D_1)$), then by the same proof as in Claim 2, $f(D_2) \cap D_2 = \emptyset$. Also, $f^2(D_2) = D_2$, since $x \in D_2$. And clearly, we must have $\text{orbit}(U_2) \cap D_2 \neq \emptyset$. But then, for some integer n , $f^n(D_2) \cap U_2 \neq \emptyset$. Thus, either $D_2 \cap U_2 \neq \emptyset$ or $f(D_2) \cap U_2 \neq \emptyset$. Hence, $D_2 \cup f(D_2)$ intersects a neighborhood of p of radius less than $\frac{1}{2}$.

If $p \in \text{Bd}(D_2)$, then $\text{Bd}(D_2) \cup f(\text{Bd}(D_2))$ is the required subcontinuum. If $p \notin \text{Bd}(D_2)$, we continue the above process. Either we terminate at a finite stage or else we obtain a sequence $D_1, D_2, \dots$ of simply-connected domains such that:

- (1) $D_1 \subset \text{Cl}(D_1) \subset D_2 \subset \text{Cl}(D_2) \subset D_3 \subset \dots$;
- (2) $\text{Bd}(D_i) \subset A_1$, $f(D_i) \cap D_i = \emptyset$, $f^2(D_i) = D_i$, for each i ;
- (3) $f(D_i) \cup D_i$ intersects a neighborhood of p of radius less than $1/i$,

for each i .

If we let $O = \bigcup_{i=1}^{\infty} D_i$, and $L = \text{Bd}(O) \cup f(\text{Bd}(O))$, then L is the required subcontinuum, with $p \in \text{Bd}(O) \cap f(\text{Bd}(O))$.

Claim 3 is established.

As in Claim 2, we must have $f(O) \cap O = \emptyset$. And since $x \in O$, $f^2(O) = O$. Now let L_1 be the outer boundary of O with respect to $f(O)$. Since $p \in \text{Bd}(O) \cap f(\text{Bd}(O))$, we must have $p \in L_1$. By Lemma 2, there is a set $V \subset L_1 \cup f(L_1)$, open in $L_1 \cup f(L_1)$, such that $\text{orbit}(V)$ misses a neighborhood of $P(2, f) \cap [L_1 \cup f(L_1)]$. Let X be the component of $[L_1 \cup f(L_1)] - \text{orbit}(V)$ containing p . Then X is nondegenerate and invariant. Let W be the component of $S^2 - X$ containing x . As before, $f(W) \cap W = \emptyset$. Since $L_1 - \text{orbit}(V)$ does not separate x and $f(x)$ [13, p. 176], $\text{Bd}(W)$ must contain points of $f(L_1) - L_1$. Let q be a point of $\text{Bd}(W) \cap f(L_1)$, $q \neq p$. Since $f(L_1) \subset \text{Bd}(f(O))$, we note that $f(O) \cup \{q\} \cup W$ is a connected set containing x and $f(x)$. Now let X_1 be an invariant subcontinuum of X containing p and $\text{Cl}(\text{orbit}(q))$ and minimal with respect to these properties. By Lemma 2, there is a set $Z \subset X_1$, open in X_1 , such that $\text{orbit}(Z)$ misses a neighborhood of $[P(2, f) \cap X_1] \cup \text{Cl}(\text{orbit}(q))$. Let X_2 be the component of $X_1 - \text{orbit}(Z)$ containing p . Then X_2 is nondegenerate, invariant, and X_2 misses the set $f(O) \cup \{q\} \cup W$. But then the component of $S^2 - X_2$ containing $f(O) \cup \{q\} \cup W$ is invariant, and we arrive at exactly the same contradiction as in the proof of Claim 2.

This final contradiction establishes Lemma 3.

LEMMA 4. Suppose $f: S^2 \rightarrow S^2$ is a homeomorphism and U is a simply-connected domain such that $f(U) \cap U = \emptyset$ and $f = \text{Id}$ on $\text{Bd}(U)$. Then $\text{Bd}(U)$ is a simple closed curve.

PROOF. Fix a point $x \in \text{Bd}(U)$.

Claim. There is an endcut A of U such that $\text{Cl}(A) \cap \text{Bd}(U) = \{x\}$.

Proof of Claim. Let $R_1, R_2, \dots$ be a chain of crosscuts defining a prime end whose impression contains x , and let $U_1 \supset U_2 \supset \dots$ be the corresponding subdomains of U (so $x \in \bigcap_{i=1}^{\infty} \text{Cl}(U_i)$). For each i , let J_i be the simple closed curve $\text{Cl}(R_i) \cup f(\text{Cl}(R_i))$, and let D_i be the component of $S^2 - J_i$ containing U_i . Then $\text{Cl}(D_1) \supset \text{Cl}(D_2) \supset \dots$. Since the diameters of the R_i 's converge to zero, the diameters of the J_i 's converge to zero, and so $\bigcap_{i=1}^{\infty} \text{Cl}(U_i) \subset \bigcap_{i=1}^{\infty} \text{Cl}(D_i)$ is the single point x . Thus, if A is a half-open arc in U such that each R_i separates the endpoint of A from some terminal portion of A , we see that $\text{Cl}(A) - A \subset \bigcap_{i=1}^{\infty} \text{Cl}(U_i) = \{x\}$. Thus, A is the required endcut and the claim is proved.

By [16, p. 58], $\text{Bd}(U)$ is a simple closed curve provided any two points of $\text{Bd}(U)$ separate $\text{Bd}(U)$. Let x, y be any two points of $\text{Bd}(U)$. By the claim there is a crosscut A of U such that $\text{Cl}(A) \cap \text{Bd}(U) = \{x, y\}$. Then $J = \text{Cl}(A) \cup f(\text{Cl}(A))$ is a simple closed curve, and $\text{Bd}(U)$ must intersect both components of $S^2 - J$ (otherwise A and $f(A)$ would lie in the same component of $S^2 - \text{Bd}(U)$). But then J separates $\text{Bd}(U) - \{x, y\}$ in S^2 , hence $\{x, y\}$ separates $\text{Bd}(U)$. The proof of Lemma 4 is complete.

LEMMA 5. Suppose $f: S^2 \rightarrow S^2$ is an orientation-reversing, p.a.p. homeomorphism, K is a component of the set of period two points of f such that K contains a fixed point, and U is a component of $S^2 - K$. Then $f(U) \cap U = \emptyset$, and $f^2(U) = U$.

PROOF. First we show that $f(U) \cap U = \emptyset$. Suppose not. Then, since K is invariant, $f(U) = U$. Let G be the (upper semicontinuous) decomposition of S^2 whose only nondegenerate element is $S^2 - U$. Then the decomposition space S^2/G is homeomorphic to S^2 [16, (2.1)', p. 171], and the induced map $g = \pi f \pi^{-1}: S^2/G \rightarrow S^2/G$ (where $\pi: S^2 \rightarrow S^2/G$ is the decomposition map) is easily seen to be a p.a.p. orientation-reversing homeomorphism. If we denote the set of period two points of f by $P(2, f)$, then $\pi(P(2, f)) = P(2, g)$, and $\pi(S^2 - U)$ is a degenerate component of $P(2, g)$. This contradicts Lemma 3. Hence $f(U) \cap U = \emptyset$.

Now suppose $f^2(U) \neq U$. Then $f^2(U) \cap U = \emptyset$. And $f^2 = \text{Id}$ on $\text{Bd}(U)$, since $K \subset P(2, f)$. Hence, by Lemma 4, $\text{Bd}(U)$ is a simple closed

curve. But then U must be one component of $S^2 - \text{Bd}(U)$ and $f^2(U)$ must be the other component. This is impossible because f^2 is orientation-preserving. Hence $f^2(U) = U$, and the proof of Lemma 5 is complete.

THEOREM 6. *Suppose $f: S^2 \rightarrow S^2$ is a p.a.p. orientation-reversing homeomorphism which has a fixed point. Then the set of fixed points of f is a simple closed curve.*

PROOF. Let K be a component of the set of period two points of f such that K contains a fixed point of f . Let $V_1, V_2, \dots$ be a list of the components of $S^2 - K$. By Lemma 5, $f(V_i) \cap V_i = \emptyset$ and $f^2(V_i) = V_i$, for each i . Let $A_1 = V_1$ and $B_1 = f(V_1)$, and suppose we have defined sets A_n, B_n which are unions of components of $S^2 - K$ such that:

- (1) $A_n \cap B_n = \emptyset$;
- (2) $V_1 \cup V_2 \cup \dots \cup V_n \subset A_n \cup B_n$;
- (3) for each i , V_i intersects A_n if and only if $f(V_i)$ intersects B_n .

Form A_{n+1}, B_{n+1} as follows: if V_{n+1} intersects $A_n \cup B_n$, let $A_{n+1} = A_n$ and $B_{n+1} = B_n$; if V_{n+1} does not intersect $A_n \cup B_n$, let $A_{n+1} = A_n \cup V_{n+1}$, and $B_{n+1} = B_n \cup f(V_{n+1})$.

Let $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcup_{n=1}^{\infty} B_n$. Then $A \cap B = \emptyset$, $f(A) = B$, $f(B) = A$, and $S^2 = A \cup B \cup K$.

Define a map $g: S^2 \rightarrow S^2$ by

$$g(x) = \begin{cases} f(x) (= f^{-1}(x)), & \text{if } x \in K, \\ f(x), & \text{if } x \in A, \\ f^{-1}(x), & \text{if } x \in B. \end{cases}$$

It is easily checked that g is a periodic, orientation-reversing homeomorphism, and that the set of fixed points of g is identical with the set of fixed points of f . The set of fixed points of g is a simple closed curve by [5]. The proof of Theorem 6 is complete.

6. Orientation-preserving homeomorphisms. In the orientation-preserving case, the similarity between the fixed point sets of p.a.p. homeomorphisms and the fixed point sets of periodic (or weakly almost periodic, see [12]) homeomorphisms no longer holds.

EXAMPLE. Let $D_1, D_2, \dots$ be a (finite or infinite) collection of closed disks in S^2 such that the union of the D_i 's is compact and locally connected, and if $i \neq j$, then $D_i \cap D_j$ is the south pole $p_0 \in S^2$. For each i , let g_i be a homeomorphism of D_i onto the disk $\{(r, \theta) \in R^2: r \leq 1\}$ ((r, θ) polar coordinates). Define $g: S^2 \rightarrow S^2$ by setting $g = \text{Id}$ outside the union of the D_i 's and setting $g|_{D_i} = g_i^{-1} f g_i$ where $f(r, \theta) = (r, \theta + 1 - r)$. Then g is

orientation-preserving and p.a.p., and the number of isolated fixed points of g is equal to the number of disks D_i .

We do, however, have the following partial result.

THEOREM 7. *Suppose $f: S^2 \rightarrow S^2$ is a recurrent, orientation-preserving homeomorphism with a finite number of fixed points. Then f has exactly two fixed points.*

PROOF. The proof consists mostly of combining known results. Let p be a fixed point of f , and let U be a neighborhood of p which contains no other fixed points.

Claim. The fixed point index $i(f, U)$ of f on U is equal to $+1$ (see [12, §4] for a short discussion of the local fixed point index $i(f, U)$ or [4] for a more comprehensive treatment).

Proof of Claim. We may assume $U \neq S^2$ so that U may be identified with a subset of the plane R^2 . Using the construction in the proof of [6, Lemma 1], we obtain an orientation-preserving homeomorphism $h: R^2 \rightarrow R^2$ such that p is the only fixed point of h , and $h = f$ on some neighborhood $V \subset U$, with $p \in V$. (In the proof of Lemma 1 of [6], choose the set D_1 of that proof to be any neighborhood of p such that $\text{Cl}(D_1)$ is a disk contained in U . The construction then easily yields the required homeomorphism h .) Since no point of V converges to p under positive or negative iterates of h , there is a point $x \in V - \{p\}$ whose orbit under h is contained in V [8, 10.28, p. 85]. If x is not a period two point, the construction given in [10, p. 89] or [2, p. 45], yields an arc A in R^2 (a so-called *translation arc*) such that $x \in A$, one of the endpoints of A is the image under h of the other endpoint, and $A \cap h(A)$ is this common endpoint. Since h is recurrent at x it is easy to see that A can be chosen so that $h^n(x) \in A$ for some $n > 1$. If x has period two then the construction of [10, p. 89] yields either a translation arc as in the previous sentence or an arc A joining x and $h(x)$ such that $A \cap h(A) = \{x, h(x)\}$. If $A \cap h(A) = \{x, h(x)\}$, let J be the simple closed curve $A \cup h(A)$. In the case $A \cap h(A)$ is a single point, let J be a simple closed curve constructed from $A \cup h(A) \cup \cdots \cup h^n(A)$, where n is the least integer greater than one such that $A \cap h^n(A) \neq \emptyset$. Then if D is the bounded component of $R^2 - J$, D must contain a fixed point of h [10, Lemma 1.1, p. 89 and Proof of Theorem 1, p. 90]. Thus, $p \in D$. And in fact, the fixed point index $i(h, D)$ is equal to 1, for, Lemma 1.1 of [10, p. 89], shows that as a point t makes one positive circuit of J , the vector from t to $h(t)$ turns through an angle of 2π ; thus, the map from J to the unit circle which takes t to $(h(t) - t)/|h(t) - t|$ has degree 1; thus, the fundamental class of $H_2(D, D - \{p\})$ is mapped to the fundamental

class of $H_2(R^2, R^2 - \{0\})$ by the homomorphism induced by $\text{Id} - h$, and thus $i(h, D) = 1$ [4]. But $D - V$, $V - D$ contain no fixed point of h , and $U - V$ contains no fixed point of f , hence:

$$1 = i(h, D) = i(h, V) = i(f, V) = i(f, U).$$

The claim is established.

Thus, if $U_1, \dots, U_m$ is a collection of pairwise disjoint open subsets of S^2 whose union contains all fixed points of f and such that each U_i , $1 \leq i \leq m$, contains exactly one fixed point of f , then:

$$\sum_{j=1}^m i(f, U_j) = L(f),$$

where $L(f)$ is the Lefschetz number of f [12, §4]. But $L(f) = 2$, since f is an orientation-preserving homeomorphism. By our claim, $i(f, U_j) = 1$, $1 \leq j \leq m$, hence $m = 2$, and the proof of Theorem 7 is complete.

We conclude with two questions.

Question 1. Is there an example of a homeomorphism $f: S^2 \rightarrow S^2$ such that f is recurrent but not p.a.p.?

Question 2. Suppose $f: S^2 \rightarrow S^2$ is an orientation-preserving p.a.p. homeomorphism such that no component of the set of fixed points separates S^2 . Must the set of fixed points have exactly two components?

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