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ω-COHESIVE SETS

BY

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ABSTRACT. We define and investigate ω-cohesiveness, a strong notion of indecomposability for subsets of the integers and their isols. This notion says, for example, that if X is the isol of an ω-cohesive set then, for any integer n, $Y + Z = \binom{X}{n}$ implies that, for some integer k, $\binom{X-k}{n} \leq Y$ or Z. From this it follows that if $f(x) \in T_1$, the collection of almost recursive combinatorial polynomials, then the predecessors of $f_\Lambda(X)$ are limited to isols $g_\Lambda(X)$ where $g(x) \in T_1$. We show existence of ω-cohesive sets. And we show that the isol of an ω-cohesive set is an n-order indecomposable isol as defined by Manaster. This gives an alternate proof to one half of Ellentuck's theorem showing a simple algebraic difference between the isols and cosimple isols. In the last section we study functions of several variables when applied to isols of ω-cohesive sets.

1. **Introduction.** Let E denote the nonnegative integers, $P(E)$ all subsets of E, Λ the isols, and $\langle\alpha\rangle$ the recursive equivalence type of $\alpha \subseteq E$. Let J be a fully effective map from $\bigcup E^n$ $(n \in E)$, one-one onto E. For $\alpha \in P(E)$, $n > 0$, define

$$\alpha^{(n)} = \{(a_1, \cdots, a_n) \in \alpha^n | a_1 > \cdots > a_n\} \quad \text{and} \quad \binom{\alpha}{n} = J(\alpha^{(n)}).$$

For $X \in \Lambda$, let $\binom{X}{n}$ denote $\langle J(\binom{\alpha}{n})\rangle$ where $\langle\alpha\rangle = X$. An isolated set α is called cohesive if for all r.e. sets ω, there is a finite set $\beta \subseteq \alpha$ such that either $(\alpha - \beta) \subseteq \omega$ or $(\alpha - \beta) \subseteq E - \omega$. In this paper, we investigate the following stronger notions of cohesiveness.

DEFINITION. For $n > 0$, an infinite set $\alpha \subseteq E$ is n-cohesive if for all r.e. sets ω, there is a finite set $\beta \subseteq \alpha$ such that either $\binom{\alpha-\beta}{n} \subseteq \omega$ or $\binom{\alpha-\beta}{n} \subseteq E - \omega$.

DEFINITION. A set $\alpha \subseteq E$ is ω-cohesive if α is n-cohesive for every integer $n \in E$.

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It is clear that if $\alpha \subseteq E$ is n-cohesive (ω-cohesive) and β is recursively isomorphic to α then β is also n-cohesive (ω-cohesive). Let Λ_c be the collection of all isols of ω-cohesive sets.

In §2, we show the existence of ω-cohesive sets, and in particular show that any infinite set of integers has an ω-cohesive subset. We also show that if α is n-cohesive then $\binom{\langle\alpha\rangle}{n} \in P_n - S_n$, the n-order indecomposables (Manaster's sets P_n, S_n are defined in §2). Thus we have a very simple method of constructing n-order indecomposable isols, where n is a finite ordinal. Unfortunately our techniques do not seem to extend to the transfinite case. Now if we take $X \in \Lambda_c$ then $\binom{X}{n} \in P_n - S_n$ for all $n > 0$. So we have a "uniform" procedure for constructing finite indecomposable isols. Using a more general technique than ours, Ellentuck in [3] proves the following restatement of the last result: There is an $X \in \Lambda$ such that $\binom{X}{n} \in P_n - S_n$ for all $n > 0$. By showing that the same statement is false in the cosimple isols, he has a simple algebraic difference between the two theories.

We also investigate how a function of an isol in Λ_c can be decomposed. Let T_1 = the collection of all almost recursive combinatorial polynomials of one variable, i.e. a function $f\colon E \to E$ is in T_1 iff there is an integer $k \geqslant 0$ such that, for some finite string of nonnegative integers $c_0, c_1, \cdots, c_n$, $f(x + k) = \Sigma\, c_i\binom{x}{i}$. Let f_Λ be the canonical extension of f to the isols. (For a discussion of combinatorial functions, almost recursive combinatorial functions and extension procedures see [5] and [6].) For $f, g \in T_1$, define $f \leqslant g$ if there is $h \in T_1$ such that $f + h = g$. And define an equivalence relation on T_1 by $f \sim g$ if there is an integer $k \geqslant 0$ such that $f(x + k) = g(x + k)$. Then Theorem 3 says that the only predecessors of $f_\Lambda(X)$, where $f \in T_1$ and $X \in \Lambda_c$, are isols of the form $g_\Lambda(X)$ for some $g \in T_1$ with $g \leqslant f$. Since $g \leqslant f$ implies that $g_\Lambda(X) \leqslant f_\Lambda(X)$ for all $X \in \Lambda$, it is always the case that an isol of the form $g_\Lambda(X)$ is a predecessor of $f_\Lambda(X)$. Theorem 3 says these are the only predecessors of $f_\Lambda(X)$ if $X \in \Lambda_c$.

It is easy to show that any ω-cohesive set has a subset whose isol is universal. This allows us (in Theorem 4) to define a map $\theta_1\colon T_1 \to \Lambda$ which preserves addition and composition, and such that $f \sim g$ iff $\theta_1(f) = \theta_1(g)$, and $\theta_1(T_1)$ is an ideal in Λ.

Let T_i = the collection of all almost recursive combinatorial polynomials $f(x_i)$ of the variable x_i. Let T_∞ = the collection of all finite sums of functions in $\bigcup T_i$ ($i \in E$). T_∞ is closed under addition, composition and predecessor. In §3, we extend Theorems 3 and 4 of §2 to functions in T_∞.

Theorem 3 does not hold if we allow product terms in our functions. For example if $f(x_1, x_2) = x_1 \cdot x_2$ and $X_1, X_2 \in \Lambda_c$ then in general the predecessors of $f_\Lambda(X_1, X_2)$ are not restricted to isols of the form $g_\Lambda(X_1, X_2)$ for some $g \leqslant f$. §4 gives a characterization of the predecessors of $f_\Lambda(X_1, \cdots, X_n)$ where f is an arbitrary almost recursive combinatorial polynomial and $X_1, X_2, X_3, \cdots$ is a "universal sequence" of isols (see §§3 and 4 for definitions).

2. Basic results.

LEMMA 1. *For any n and any infinite subset α of E, α has an n-cohesive subset β.*

PROOF. Let $\omega_1, \omega_2, \cdots$ be a list of all r.e. sets. We will construct a sequence $\alpha_0 \supseteq \alpha_1 \supseteq \alpha_2 \supseteq \cdots$ of infinite subsets of α such that, for any $i > 0$,

$$\binom{\alpha_i}{n} \subseteq \omega_i \quad \text{or} \quad \binom{\alpha_i}{n} \subseteq E - \omega_i.$$

Let $\alpha_0 = \alpha$. Suppose we have α_i. Then

$$J^{-1}\left[\binom{\alpha_i}{n} \cap \omega_{i+1}\right] \text{ and } J^{-1}\left[\binom{\alpha_i}{n} \cap (E - \omega_{i+1})\right]$$

are partitions of $\alpha_i^{(n)}$ into two disjoint sets. By Ramsey's theorem [7] α_i has an infinite subset γ such that either

$$\gamma^{(n)} \subseteq J^{-1}\left[\binom{\alpha_i}{n} \cap \omega_{i+1}\right] \quad \text{or} \quad \gamma^{(n)} \subseteq J^{-1}\left[\binom{\alpha_i}{n} \cap (E - \omega_{i+1})\right].$$

Therefore $\binom{\gamma}{n} \subseteq \omega_{i+1}$ or $\binom{\gamma}{n} \subseteq (E - \omega_{i+1})$. Let $\alpha_{i+1} = \gamma$. Now, for each $i \in E$, choose $x_i \in \alpha_i$ different from $x_0, \cdots, x_{i-1}$ (since each α_i is infinite this is always possible). Let $\beta = \{x_i | i \in E\}$. Then β is an infinite subset of α.

We claim β is n-cohesive. Suppose ω is any r.e. set. Then there is an i such that $\omega = \omega_i$. Let $\beta' = \{x_j | j \geqslant i\}$. Then $\beta' \subseteq \alpha_i$. We chose α_i such that either

$$\binom{\alpha_i}{n} \subseteq \omega_i \quad \text{or} \quad \binom{\alpha_i}{n} \subseteq E - \omega_i.$$

Therefore, $\binom{\beta'}{n} \subseteq \omega_i$ or $\binom{\beta'}{n} \subseteq E - \omega_i$. Since $\beta' = \beta - \{x_1, \cdots, x_{i-1}\}$, β is n-cohesive.

LEMMA 2. *Any infinite subset of an n-cohesive set is n-cohesive.*

PROOF. Follows directly from the definition.

LEMMA 3. *If α is n-cohesive and $m \leqslant n$, then α is m-cohesive.*

PROOF. We need to show only that α n-cohesive implies α $(n-1)$-cohesive. Suppose α is n-cohesive but not $(n-1)$-cohesive. Then there is an r.e. set ω such that, for all finite subsets β of α, $\binom{\alpha-\beta}{n-1} \nsubseteq \omega$ and $\binom{\alpha-\beta}{n-1} \nsubseteq E - \omega$. Let $A =$

$J^{-1}(\omega) \times E$. Let $\omega_1 = J(A)$. Then ω_1 is r.e. Suppose there is a finite subset β of α such that $\binom{\alpha-\beta}{n} \subseteq \omega_1$. Then $(\alpha - \beta)^{(n)} \subseteq A$. Let x_0 be the smallest element of $(\alpha - \beta)$. Then it follows that $[\alpha - (\beta \cup \{x_0\})]^{(n-1)} \subseteq J^{-1}(\omega)$. Therefore,

$$\binom{\alpha - (\beta - \{x_0\})}{n-1} \subseteq \omega.$$

Similarly if $\binom{\alpha-\beta}{n} \subseteq E - \omega_1$.

COROLLARY 1. *If* α *is n-cohesive then* $\langle\alpha\rangle \in \Lambda$.

PROOF. If α is n-cohesive then α is 1-cohesive or cohesive and hence isolated.

LEMMA 4. *If* X *is the isol of an n-cohesive set then* $Y + Z = \binom{X}{n}$ *implies that there is a* $k \in E$ *such that* $\binom{X-k}{n} \leqslant Y$ *or* $\binom{X-k}{n} \leqslant Z$.

PROOF. Clear from the definition of n-cohesive.

In [4] Manaster defines a sequence of triples $(P_\alpha, S_\alpha, I_\alpha)$ as follows: $I_0 = \{X | X \text{ is finite}\}$. For ordinals $\alpha > 0$,

$$P_\alpha = \left\{X | X = Y + Z \Rightarrow Y \in \bigcup_{\alpha'<\alpha} I_{\alpha'} \vee Z \in \bigcup_{\alpha'<\alpha} I_{\alpha'}\right\}$$

$$S_\alpha = \left\{X | X = Y + Z \wedge Z \notin \bigcup_{\alpha'<\alpha} I_{\alpha'} \right.$$
$$\left. \Rightarrow (\exists V)(\exists W)\left[Z = V + W \wedge V \notin \bigcup_{\alpha'<\alpha} I_{\alpha'} \wedge W \notin \bigcup_{\alpha'<\alpha} I_{\alpha'}\right]\right\}.$$

I_α is the ideal generated by $P_\alpha \cup S_\alpha$. It can be shown that $\alpha < \beta \Rightarrow P_\alpha \subseteq P_\beta$, $S_\alpha \subseteq S_\beta$, $I_\alpha \subseteq I_\beta$; $\bigcup_{\alpha'<\alpha+1} I_{\alpha'} = I_\alpha$; $P_\alpha \cap S_\alpha = \bigcup_{\alpha'<\alpha} I_{\alpha'}$.

In the following theorem we consider only $(P_\alpha, S_\alpha, I_\alpha)$ where α is finite. Elements of $P_n - S_n$ are called n-order indecomposables.

THEOREM 1. *If* X *is the isol of an n-cohesive set then* $\binom{X}{n} \in P_n - S_n$ *and* $X^n \in I_n - S_n$.

PROOF. We prove the first part of the theorem by induction on n. If $n = 1$ then X is the isol of a cohesive set and therefore $X \in P_1 - S_1$. Assume the theorem is true for $m \leqslant n - 1$. Suppose $\binom{X}{n} = Y + Z$. Then by Lemma 4, there is a $k \in E$ such that $\binom{X-k}{n} \leqslant Y$ or $\binom{X-k}{n} \leqslant Z$. Since

$$\binom{X}{n} = \binom{X-k}{n} + k\binom{X-k}{n-1} + \binom{k}{2}\binom{X-k}{n-2} + \cdots + \binom{k}{n-1}(X-k) + \binom{k}{n},$$

either

$$Z \leqslant \binom{k}{1}\binom{X-k}{n-1} + \binom{k}{2}\binom{X-k}{n-2} + \cdots + \binom{k}{n-1}(X-k) + \binom{k}{n}$$

or

$$Y \leqslant \binom{k}{1}\binom{X-k}{n-1} + \binom{k}{2}\binom{X-k}{n-2} + \cdots + \binom{k}{n-1}(X-k) + \binom{k}{n}.$$

Since X is the isol of an n-cohesive set, $X - k$ is the isol of an n-cohesive set. Then, by Lemma 3, $X - k$ is the isol of an m-cohesive set for $1 \leqslant m \leqslant n - 1$. By the inductive hypothesis, for $1 \leqslant m \leqslant n - 1, \binom{X-k}{m} \in P_m - S_m \subseteq P_m \subseteq P_{n-1} \subseteq I_{n-1}$. Since I_{n-1} is an ideal

$$k\binom{X-k}{n-1} + \cdots + \binom{k}{n-1}(X-k) + \binom{k}{n} \in I_{n-1}.$$

Therefore, $Y \in I_{n-1}$ or $Z \in I_{n-1}$. Therefore, by definition of P_n, $\binom{X}{n} \in P_n$. By inductive hypothesis $\binom{X}{n-1} \in P_{n-1} - I_{n-1}$. And, for any integer r, $r\binom{X}{n-1} \leqslant \binom{X}{n}$. It is shown in Theorem 8.1 of [4] that for any $Z \in I_\alpha$, there is a finite m such that any linear decomposition of Z includes at most m isols of $P_\alpha - S_\alpha$. Hence $\binom{X}{n} \notin I_{n-1}$. Since

$$P_n - S_n = P_n - \bigcup_{\alpha < n} I_\alpha = P_n - I_{n-1}, \quad \binom{X}{n} \in P_n - S_n.$$

It is clear that the function x^n is an almost recursive combinatorial polynomial of the form, $x^n = c_n\binom{x}{n} + c_{n-1}\binom{x}{n-1} + \cdots + c_1 x + c_0$. By the first part of the theorem, for $1 \leqslant i \leqslant n$, $\binom{X}{i} \in P_i - S_i \subseteq I_n$. Therefore, $X^n \in I_n - S_n$.

THEOREM 2. *If α is an infinite subset of E, then α has a subset β which is ω-cohesive.*

PROOF. By Lemma 1, there is a sequence $\alpha \supseteq \alpha_1 \supseteq \cdots$ of infinite sets such that α_n is n-cohesive. For each n, choose $x_n \in \alpha_n$ different from $x_1, \cdots, x_{n-1}$. Let $\beta = \{x_n | n > 0\}$. Claim β is ω-cohesive. For any n, $\beta = \{x_1, \cdots, x_{n-1}\} \cup \beta'$ where $\beta' \subseteq \alpha_n$. Thus β' is n-cohesive. Therefore β is n-cohesive.

COROLLARY 2. *There is an isol X such that $\binom{X}{n} \in P_n - S_n$ for all $n > 0$.*

LEMMA 5. *Any infinite subset of an ω-cohesive set is ω-cohesive.*

PROOF. Follows from the definition of ω-cohesive and Lemma 2.

THEOREM 3. *If* $f(x) \in T_1$, $X \in \Lambda_c$ *and* $Y \leqslant f_\Lambda(X)$, *then there is a* $g(x) \in T_1$ *such that* $g_\Lambda(X) = Y$ *and* $g \leqslant f$.

PROOF. First suppose $f(x) = \binom{x}{i}$. Induct on i. If $i = 0$, then $f(x) = 1$ and the theorem is true. Assume the theorem is true for $i - 1$. Suppose $f(x) = \binom{x}{i}$. Suppose $Y \leqslant f_\Lambda(X) = \binom{X}{i}$. Then by Lemma 4, there is a $k \in E$ such that either $\binom{X-k}{i} \leqslant Y$ or $\binom{X-k}{i} \leqslant \binom{X}{i} - Y$. In the first case, $Y = \binom{X-k}{i} + Z$ where $Z \leqslant \binom{X}{i} - \binom{X-k}{i} \leqslant r\binom{X}{i-1}$ for some integer r. Then $Z = Z_1 + \cdots + Z_r$ where $Z_j \leqslant \binom{X}{i-1}$ for each $1 \leqslant j \leqslant r$. By the inductive hypothesis, for each $1 \leqslant j \leqslant r$ there is a $g_j(x) \leqslant \binom{x}{i-1}$ such that $g_{j\Lambda}(X) = Z_j$. Let $g_0(x) = \binom{x-k}{i}$ and $g = g_0 + g_1 + \cdots + g_r$. Then $g(x) \leqslant f(x)$ and $g(X) = Y$. In the second case, $\binom{X-k}{i} \leqslant \binom{X}{i} - Y$ implies that $Y \leqslant \binom{X}{i} - \binom{X-k}{i}$ and then the rest follows as above.

Suppose $f(x) \in T_1$ and $Y \leqslant f_\Lambda(X)$ then there exists $k \in E$ such that $f(x + k) = \Sigma_{i=0}^n c_i\binom{x}{i}$ for some $n, c_0, \cdots, c_n \in E$ and $Y \leqslant f_\Lambda(X) \leqslant f_\Lambda(X + k) = \Sigma_{i=0}^n c_i\binom{X}{i}$. Then

$$Y = \sum_i \sum_j Y_{ij} \qquad (0 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant c_i)$$

where $Y_{ij} \leqslant \binom{X}{i}$ for $0 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant c_i$. Therefore, for $0 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant c_i$, there exists $g_{ij}(x) \in T_1$ such that $g_{ij\Lambda}(X) = Y_{ij}$. Let

$$g = \sum_i \sum_j g_{ij} \qquad (0 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant c_i).$$

Then $g_\Lambda(X) = Y$.

In [2], Ellentuck defines the notion of a universal isol. It is easy to see that his definition is equivalent to the following: an isol X is universal if for any pair of almost recursive combinatorial functions f and g, $f_\Lambda(X) = g_\Lambda(X)$ implies that there is an integer k such that for any $x \geqslant k$, $f(x) = g(x)$. We want to show that any ω-cohesive set has a subset whose isol is universal. Modify Ellentuck's method of showing the existence of universal isols [2] by replacing the set of integers E with an arbitrary infinite set $\alpha \subseteq E$ and topologize $P(\alpha)$ as he topologizes $P(E)$. Then Lemmas 1, 2 and Theorem 1 of [2] go through routinely to give the following lemma.

LEMMA 6. *Any infinite set of integers has a subset whose isol is universal.*

THEOREM 4. *There is a map* $\theta_1 \colon T_1 \to \Lambda$ *such that for any* $f, g \in T_1$
(1) $f \sim g$ *iff* $\theta_1(f) = \theta_1(g)$,
(2) $\theta_1(f(g)) = f_\Lambda(\theta_1(g))$,

(3) $\theta_1(f+g) = \theta_1(f) + \theta_1(g)$,
(4) $\theta_1(T_1)$ *is an ideal in* Λ.

PROOF. Choose a set α which is ω-cohesive. By Lemma 6, α has a subset β such that $X = \langle\beta\rangle$ is universal. Define $\theta_1 : T_1 \to \Lambda$ by $\theta_1(f) = f_\Lambda(X)$. If $f \sim g$, then $f_\Lambda(X) = g_\Lambda(X)$ since X is infinite. Hence $\theta_1(f) = \theta_1(g)$. If $\theta_1(f) = \theta_1(g)$ then $f \sim g$ since X is universal. Parts (2) and (3) follow directly from the fact that the extension procedure preserves both composition and addition of functions. Part (4) follows from part (3) and Theorem 3.

Let T_1^* be the set of all equivalence classes in T_1. Since "$\sim$" preserves addition in T_1, we can define addition in T_1^* by $[f] + [g] = [f+g]$, and $[f] \leqslant [g]$ if there is $[h] \in T_1^*$ such that $[f] + [h] = [g]$. Then T_1^* is an ideal.

COROLLARY 3. T_1^* *under addition is isomorphic to an ideal in* Λ.

3. **Functions in T_∞.** We want to extend Theorem 3 to functions of more than one variable. Let S be the collection of all almost recursive combinatorial polynomials, i.e. $f(x_1, \cdots, x_n) \in S$ iff f is an almost recursive combinatorial function such that for some integer k,

$$f(x_1 + k, \cdots, x_n + k) = \sum c_{i_1,\cdots,i_n}\binom{x_1}{i_1}\cdots\binom{x_n}{i_n}$$

with all $c_{i_1,\cdots,i_n} \geqslant 0$ and only a finite number of them nonzero and, for all $j < k$,

$$f(j, x_2, \cdots, x_n), \cdots, f(x_1, \cdots, x_{n-1}, j) \in S.$$

Then T_∞ is the subset of S consisting of all those functions which do not involve product terms. Theorem 3 easily generalizes to functions in T_∞.

THEOREM 5. *If* $f \in T_\infty$ *and* $X_1, \cdots, X_n \in \Lambda_c$, *and* $Y \leqslant f_\Lambda(X_1, \cdots, X_n)$ *then there is a function* $g \in T_\infty$ *such that* $Y = f_\Lambda(X_1, \cdots, X_n)$ *and* $g \leqslant f$.

PROOF. Since $f \in T_\infty$, f is of the form $f(x_1, \cdots, x_n) = f_1(x_1) + \cdots + f_n(x_n)$ where $f_1, \cdots, f_n \in T_1$. $Y \leqslant f_\Lambda(X_1, \cdots, X_n)$ implies $Y = Y_1 + \cdots + Y_n$ where each $Y_i \leqslant f_{i\Lambda}(X_i)$. By Theorem 3, $Y_i = g_i(X_i)$ for some $g_i \in T_1$ and $g_i \leqslant f_i$. Hence $g = g_1 + \cdots + g_n$.

We can define an equivalence relation on T_∞ (or S) just as we did on T_1. That is, for $f, g \in S$, $f \sim g$ iff $f(x_1 + k, \cdots, x_n + k) = g(x_1 + k, \cdots, x_n + k)$ for some integer k. Let T_∞^* be the set of equivalence classes in T_∞ and define $+$ and $\leqslant$ as on T_1^*. In order to extend Theorem 4, we need an analogue of Lemma 6, to give us a "universal sequence" of isols.

Let $X = (P(\mu))^{\omega}$ where μ is an arbitrary infinite subset of E. Denote a vector $(\alpha_1, \alpha_2, \cdots) \in X$ by $\underline{\alpha}$. Let Q = collection of all vectors $\underline{\alpha} = (\alpha_1, \alpha_2, \cdots) \in X$ such that (1) all the sets α_i are finite and (2) except for a finite number of coordinates, the sets $\alpha_i = \emptyset$. For $\delta \subseteq E$, let $\|\delta\|$ denote the cardinality of δ. If $\underline{\alpha}, \underline{\beta} \in Q$, define $N(\underline{\alpha}, \underline{\beta}) = \{\underline{\xi}: \underline{\alpha} \subseteq \underline{\xi}, \underline{\xi} \cap \underline{\beta} = \emptyset\}$. The collection $N = \{N(\underline{\alpha}, \underline{\beta}): \underline{\alpha}, \underline{\beta} \in Q\}$ forms a basis, and X with the topology induced by this basis is easily shown to be a complete metric space. Then by the Baire Category Theorem, if a set $A \subseteq X$ is of the first categroy (i.e. the countable union of nowhere dense sets) then $X - A$ is nonempty. We will use a straightforward generalization of Ellentuck's techniques [2] to prove the following lemma.

LEMMA 7. *Any infinite set of integers μ has a sequence of subsets $\delta_1, \delta_2, \cdots$ such that for any $f, g \in S$, if $f_\Lambda(\langle\delta_1\rangle, \cdots, \langle\delta_n\rangle) = g_\Lambda(\langle\delta_1\rangle, \cdots, \langle\delta_n\rangle)$ then $f \sim g$.*

PROOF. Let (f_i, g_i) be a list of all pairs of recursive combinatorial functions (of any number of variables) such that $f_i \not\sim g_i$. For each i, let φ_i, ψ_i be recursive operators inducing f_i, g_i respectively. Let p_i be a list of all one-one partial recursive functions from E to E. Define $H(\varphi, \psi, p) = \{\underline{\xi} \in X: \varphi(\underline{\xi}) \subseteq$ domain of p, and $p[\varphi(\underline{\xi})] = \psi(\underline{\xi})\}$. By a straightforward generalization of Lemma 2 of [2] to vectors we can show that each set $H(\varphi_i, \psi_i, p_i)$ is nowhere dense.

Let $\underline{\alpha}_k$ be a list of all vectors in Q. Define $H(\varphi, \psi, p, \underline{\alpha}_k) = \{\underline{\xi} \cup \underline{\alpha}_k: \underline{\xi} \in H(\varphi, \psi, p)\}$. Since $H(\varphi, \psi, p, \underline{\alpha}_k)$ is just a translation of $H(\varphi, \psi, p)$, it is also a nowhere dense set. Let $A = \bigcup H(\varphi_i, \psi_i, p_j, \underline{\alpha}_k)$ union over all integers i, j, k. Then A is the countable union of nowhere dense sets. Hence $X - A \neq \emptyset$. Let $\delta \in X - A$. First notice that all coordinates of $\underline{\delta}$ are infinite. For suppose not, i.e. suppose δ_i is finite for some i. Let $f(x_i) = x_i$ and $g(x_i) = \|\delta_i\|$. Then $f \not\sim g$, but $\varphi(\underline{\delta})$ is recursively isomorphic to $\psi(\underline{\delta})$ where φ, ψ induce f, g respectively.

Now we show that $\underline{\delta}$ satisfies our lemma. Suppose $f, g \in S$, $f \not\sim g$, but $f_\Lambda(\langle\delta_1\rangle, \cdots, \langle\delta_n\rangle) = g_\Lambda(\langle\delta_1\rangle, \cdots, \langle\delta_n\rangle)$. Since $f, g \in S$, there is an integer k such that $f(x_1 + k, \cdots, x_n + k) = f_i(x_1, \cdots, x_n)$ and $g(x_1 + k, \cdots, x_n + k) = g_i(x_1, \cdots, x_n)$ for some pair of recursive combinatorial functions (f_i, g_i). Since $f \not\sim g$, $f_i \not\sim g_i$. For $i = 1, \cdots, n$, let α_i be a subset of δ_i with $\|\alpha_i\| = k$. Then

$$f_{i\Lambda}(\langle\delta_1 - \alpha_1\rangle, \cdots, \langle\delta_n - \alpha_n\rangle) = g_{i\Lambda}(\langle\delta_1 - \alpha_1\rangle, \cdots, \langle\delta_n - \alpha_n\rangle).$$

Hence there is a p_j such that $(\delta_1 - \alpha_1, \cdots, \delta_n - \alpha_n, \delta_{n+1}, \cdots) \in H(\varphi_i, \psi_i, p_j)$. But then $\underline{\delta} \in A$, which contradicts the fact that $\underline{\delta} \in X - A$.

THEOREM 6. *There is a map* $\theta\colon T_\infty \to \Lambda$ *such that*

(1) $f \sim g$ *iff* $\theta(f) = \theta(g)$,

(2) $\theta(f(g_1, \cdots, g_n)) = f_\Lambda(\theta(g_1), \theta(g_2), \cdots, \theta(g_n))$,

(3) $\theta(f + g) = \theta(f) + \theta(g)$

(4) $\theta(T_\infty)$ *is an ideal in* Λ.

PROOF. Let μ be ω-cohesive. Let $\delta_1, \delta_2, \cdots$ be subsets of μ, such that $X_i = \langle\delta_i\rangle$, $i = 1, 2, \cdots$ form a "universal sequence" of isols. For $f(x_1, \cdots, x_n) \in T_\infty$ define $\theta(f) = f_\Lambda(X_1, \cdots, X_n)$. Then properties (1)–(4) follow as in the proof of Theorem 4.

COROLLARY 4. T_∞^* *under addition is isomorphic to an ideal in* Λ.

4. Functions with product terms. In Theorem 5, we defined a map, θ, such that $\theta(T_\infty)$ is closed under predecessors. In this section we want to investigate the predecessors of isols in $\theta(S)$. That is, let μ be a fixed, ω-cohesive set. By Lemma 7, μ has a sequence of subsets, $\delta_1, \delta_2, \delta_3, \cdots$ such that $(X_1, X_2, X_3, \cdots) = (\langle\delta_1\rangle, \langle\delta_2\rangle, \langle\delta_3\rangle, \cdots)$ is a "universal" sequence of isols. Use this sequence to define a map $\theta\colon S \to \Lambda$ by $\theta(f(x_1, \cdots, x_n)) = f_\Lambda(X_1, \cdots, X_n)$. The following lemma shows that $\theta(S)$ is not closed under predecessors.

LEMMA 8. *If* $f \in S - T_\infty$ *then* $\theta(f)$ *has a predecessor* $U \notin \theta(S)$.

PROOF. Since $f(x_1, \cdots, x_n) \in S$, there is an integer k such that $f(x_1 + k, \cdots, x_n + k)$ is a recursive combinatorial function. Since $f(x_1, \cdots, x_n) \notin T_\infty$, $f(x_1 + k, \cdots, x_n + k)$ contains a term of the form

$$\binom{x_1}{i_1} \cdots \binom{x_n}{i_n}$$

where for some $r \neq s$, $i_r \geqslant 1$ and $i_s \geqslant 1$. Then the function

$$x_r x_s \leqslant \binom{x_1}{i_1} \cdots \binom{x_n}{i_n} \leqslant f(x_1 + k, \cdots, x_n + k)$$

and $(x_r - k)(x_s - k) \leqslant f(x_1, \cdots, x_n)$. So we can restrict ourselves to predecessors of $(X_r - k)(X_s - k)$. Choose $\alpha_r \subseteq \delta_r$ and $\alpha_s \subseteq \delta_s$ such that $\|\alpha_r\| = \|\alpha_s\| = k$. Let

$$\beta_1 = J[((\delta_r - \alpha_r) \times (\delta_s - \alpha_s)) \cap \{(x, y) \in E^2 | x < y\}]$$

and

$$\beta_2 = J[((\delta_r - \alpha_r) \times (\delta_s - \alpha_s)) \cap \{(x, y) \in E^2 | x \geqslant y\}].$$

Let $U = \langle\beta_1\rangle$, $V = \langle\beta_2\rangle$. Then $U + V = (X_r - k)(X_s - k)$. Suppose $U, V \in \theta(S)$. Then there exist $g, h \in S$ such that $\theta(g) = U$, $\theta(h) = V$. Since $\theta((x_r - k)(x_s - k)) = (X_r - k)(X_s - k) = U + V = \theta(g) + \theta(h) = \theta(g + h)$, $(x_r - k)(x_s - k) \sim g + h$. Thus there is a k' such that

$$(x_r + k')(x_s + k') = g(x_r + k', x_s + k') + h(x_r + k', x_s + k').$$

It is easy to check that if $g(x_r + k', x_s + k') + h(x_r + k', x_s + k') = (x_r + k')(x_s + k')$ then either g or h is of the form $ax_r + bx_s \pm c$, a, b, $c \in E$. Suppose g is. Then $U = aX_r + bX_s \pm c$. From the definition of the set β_1, it is easily seen that $mX_s \leqslant U$ for all $m \in E$. Thus, in particular, $(a + b)X_s \leqslant U = aX_r + bX_s \pm c$. Therefore, $X_s \leqslant X_r + c$. Since X_s is infinite and X_r is ω-cohesive, $X_s = X_r \pm d$ for some $d \in E$. However, $x_s \not\sim x_r \pm d$. Thus $U \notin \theta(S)$.

Essentially what we did to construct the predecessor in Lemma 8 was to divide the "rectangle" $X_r \cdot X_s$ into two "triangles", U and V. This is really the only way we can get a predecessor of $X_r \cdot X_s$ which is not an isol in $\theta(S)$. Consider an arbitrary $f \in S$. Then there is a k such that $f_\Lambda(X_1, \cdots, X_m) \leqslant (X_1 \cdots X_m)^k$. So we can restrict ourselves to predecessors of the form $X_{i_1} \cdots X_{i_n}$ where $X_{i_1}, \cdots, X_{i_n}$ are from the fixed "universal" sequence of isols X_1, $X_2, \cdots$ and need not be distinct. We will show that essentially the only predecessors of $X_{i_1} \cdots X_{i_n}$ which are not isols in $\theta(S)$ are obtained by taking the "n-dimensional rectangle" $X_{i_1} \cdots X_{i_n}$ and dividing it into $n!$ "n-simplexes". Let $p_1, p_2, \cdots, p_{n!}$ be the permutations on $\{1, \cdots, n\}$, with p_1 the identity permutation. Let $E_{n,k} = \{(x_1, \cdots, x_n) | x_{p_k(1)} > \cdots > x_{p_k(n)}\}$ for $k = 1, \cdots, n!$ and $E_{n,0} = E^n - \bigcup_{k=1}^{n!} E_{n,k}$. Let $\beta_k = (\delta_{i_1} \times \cdots \times \delta_{i_n}) \cap E_{n,k}$ and $Y_k = \langle\beta_k\rangle$. Then

$$X_{i_1} \cdots X_{i_k} = Y_0 + Y_1 + \cdots + Y_{n!}.$$

Suppose $Z \leqslant X_{i_1} \cdots X_{i_n}$. Then $Z = Y'_0 + Y'_1 + \cdots + Y'_{n!}$ where each $Y'_k \leqslant Y_k$. In the following lemma, we show that the only predecessors of Y_k are isols U such that either U or $Y_k - U$ is of "lower degree" than n.

LEMMA 9. *Suppose $U + V = Y_k$. Then either U or V is of the form $c_1 Z_1 + \cdots + c_n Z_n$, where $c_1, \cdots, c_n \in E$ and, for each $j = 1, \cdots, n$, $Z_j \leqslant X_{i_1} \cdots X_{i_{j-1}} \cdot X_{i_{j+1}} \cdots X_{i_n}$.*

PROOF. First consider $Y_1 = \langle\beta_1\rangle$. Notice $\beta_1 \subseteq \mu^{(n)}$. If $U + V = Y_1$, then there exist disjoint sets $\omega_1, \omega_2 \subseteq E^n$ such that $\langle J(\omega_1 \cap \beta_1)\rangle = U$, $\langle J(\omega_2 \cap \beta_1)\rangle = V$ and $J(\omega_1)$, $J(\omega_2)$ are r.e. Since μ is ω-cohesive, it easily follows that there is a finite subset ν of μ such that either $(\mu - \nu)^{(n)} \subseteq \omega_1$ or $(\mu - \nu)^{(n)} \subseteq \omega_2$. Assume the first case holds. Then $\mu^{(n)} - (\mu - \nu)^{(n)} \supseteq \mu^{(n)} - \omega_1 \supseteq \beta_1 \cap \omega_2$. Therefore

$$\beta_1 \cap \omega_2 \subseteq \beta_1 \cap [\mu^{(n)} - (\mu - \nu)^{(n)}]$$

$$\doteq \beta_1 - (\delta_{i_1} - \nu) \times \cdots \times (\delta_{i_n} - \nu)$$

$$\subseteq \delta_{i_1} \times \cdots \times \delta_{i_n} - (\delta_{i_1} - \nu) \times \cdots \times (\delta_{i_n} - \nu),$$

and clearly $\beta_1 \cap \omega_2$ is recursively separated from its complement in $\delta_{i_1} \times \cdots \times \delta_{i_n} - (\delta_{i_1} - \nu) \times \cdots \times (\delta_{i_n} - \nu)$. Hence

$$V = \langle \beta_1 \cap \omega_2 \rangle \leqslant X_{i_1} \cdots X_{i_n} - (X_{i_1} - k_1) \cdots (X_{i_n} - k_n).$$

where $k_j = \|\delta_{i_k} \cap \nu\|$. Then clearly $X_{i_1} \cdots X_{i_n} - (X_{i_1} - k_1) \cdots (X_{i_n} - k_n)$ and hence V is of the desired form $c_1 Z_1 + \cdots + c_n Z_n$.

Now consider Y_k, for $k > 1$. Define $q\colon E^n \to E^n$ by $q((x_1, \cdots, x_n)) = (x_{p_k(1)}, \cdots, x_{p_k(n)})$. Then $q(\beta_k) \subseteq \mu^{(n)}$ and predecessors of Y_k can be mapped into predecessors of Y_1 and the above argument for Y used.

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