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CONTINUA IN WHICH ALL CONNECTED SUBSETS ARE ARCWISE CONNECTED

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ABSTRACT. Let X be a metric continuum such that every connected subset of X is arcwise connected. Some facts concerning the distribution of local cutpoints of X are obtained. These results are used to prove that X is a regular curve.

1. Introduction. Several attempts have been made to characterize the spaces in which all connected subsets are arcwise connected, e.g. Kuratowski and Knaster [1], Whyburn [3], [5], [6] and Tymchatyn [2]. In [10] Mohler conjectured that if X is a metric continuum such that each connected subset of X is arcwise connected, then X is a regular curve. The main purpose of this paper is to resolve this conjecture in the affirmative.

The reader may consult Whyburn [8, V. 2] for a survey of the properties of hereditarily locally connected continua and [8, III. 9] for a treatment of local cutpoints. Our notation follows Whyburn [8]. We collect here some basic definitions for the convenience of the reader. A continuum is a nondegenerate, compact, connected metric space. A continuum is said to be *hereditarily locally connected* if each of its subcontinua is locally connected. A continuum is said to be *regular* if it has a basis of open sets with finite boundaries. An *arc* is a homeomorph of the closed unit interval [0, 1]. If A is an arc and $c, d \in A$ then $[c, d]$ denotes the arc in A with endpoints c and d. A subset A of a space X is said to be *arcwise connected* if every pair of points of A can be joined by an arc in A. A point p in a continuum X is said to be a *local cutpoint* or *local separating point* of X if there is a connected open set U in X such that $U - \{p\}$ is not connected. The term *neighbourhood* will always mean open neighbourhood. We denote the closure of a set A by Cl(A).

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2. In this section, we study several conditions that are satisfied by continua whose connected subsets are arcwise connected. The following theorem has played a central role in the study of these continua.

THEOREM 1 (Tymchatyn [2]). *Let X be a continuum such that $X = A_0 \cup A_1 \cup \cdots$ where*

(i) *A_0 is not empty and contains no local cutpoints of X,*

(ii) *for each $i = 1, 2, \cdots, A_i$ is a closed set,*

(iii) *for each $i \neq j$, $A_i \cap A_j$ is void.*

Then X contains a connected set that is not arcwise connected.

LEMMA 2. *If X is a locally connected continuum such that L, the set of local cutpoints of X, is totally disconnected then L can be written as the union of a countable family of pairwise disjoint closed sets.*

PROOF. It is well known (see [8, p. 63]) that L is an F_σ. Since L is totally disconnected it is easy to see that L can be written as the union of a countable family of pairwise disjoint closed sets.

By a *null sequence* of sets is meant a sequence of sets whose diameters converge to zero.

THEOREM 3. *Let X be a continuum such that every sequence of disjoint subcontinua of X is a null sequence. Then the following are equivalent:*

(a) *If $A_1, A_2, \cdots$ is any sequence of pairwise disjoint closed subsets of X and x and y are points that are separated by $X - (A_1 \cup A_2 \cup \cdots)$ then some countable subset of $X - (A_1 \cup A_2 \cup \cdots)$ separates x and y in X.*

(b) *If A is any subcontinuum of X and $x, y \in A$ then there is an arc in A which contains x and y and which contains at most countably many points that are not local cutpoints of A.*

(c) *If A is any subcontinuum of X then the set of local cutpoints of A is not contained in the union of countably many pairwise disjoint closed proper subsets of A.*

PROOF. (a) ⇒ (c). Suppose (c) fails. Then there is a subcontinuum A of X such that the set of local cutpoints of A is contained in the union of a countable family $A_1, A_2, \cdots$ of closed, proper, pairwise disjoint subsets of A. We may suppose without loss of generality that A_1 and A_2 are nonempty sets. Let $x \in A_1$ and let $y \in A_2$. Since every sequence of pairwise disjoint subcontinua of X is a null sequence we may suppose that the A_i form a null sequence. It follows that the decomposition space Y obtained from A by identifying each of the sets A_i to a point is a compact metric space. The image of the set $A_1 \cup A_2 \cup \cdots$

in Y under the natural projection π is a countable set. Hence $A-(A_1 \cup A_2 \cup \cdots)$ separates x and y in A since $Y-\pi(A_1 \cup A_2 \cup \cdots)$ separates $\pi(x)$ from $\pi(y)$ in Y. By [8, III. 9.41] every set in $A-(A_1 \cup A_2 \cup \cdots)$ which separates x and y is uncountable. In particular every set in X which separates x and y is uncountable. Thus, (a) also fails.

(b) ⇒ (c). Suppose (c) fails. Then there is a subcontinuum A of X such that the set of local cutpoints of A is contained in the union of a countable family $A_1, A_2 \cdots$ of closed proper subsets of A. By Sierpinski's theorem there exist $x, y \in A-(A_1 \cup A_2 \cup \cdots)$ and each arc in A which contains x and y contains uncountably many points of $A-(A_1 \cup A_2 \cup \cdots)$. Hence, every arc in A which contains x and y contains uncountably many points that are not local cutpoints of A. Thus, (b) also fails.

(c) ⇒ (b). Suppose (b) fails. Let L denote the set of local cutpoints of X. We may suppose without loss of generality that there exist $x, y \in X$ such that each arc in X which contains both x and y contains uncountably many points of $X-L$. For each $z \in X$ let $A(z) = \bigcup\{A \subset X | A$ is a continuum, $z \in A$ and $A-L$ is countable$\}$.

By [7, Theorem 34] each $A(z)$ is closed. Clearly, $A(z)$ is also connected. Define an equivalence relation $\sim$ on X by setting $x \sim y$ if and only if $x \in A(y)$. Since the set of nondegenerate equivalence classes of $\sim$ is a null sequence, it follows that $\sim$ is a closed relation.

Let π be the natural projection of X onto the quotient space $X/\sim$. X is hereditarily locally connected since it can contain no continuum of convergence. Hence, $X/\sim$ is also a Peano continuum. A point $z \in X/\sim$ is a local cutpoint of $X/\sim$ only if $\pi^{-1}(z)$ contains a local cutpoint of X. Also, if p is a local cutpoint of X and $A(p) = \{p\}$ then $\pi(p)$ is a local cutpoint of $X/\sim$. We shall prove that the set of local cutpoints of $X/\sim$ is totally disconnected. By Lemma 2 it will follow that the set of local cutpoints of $X/\sim$ is the union of a countable family of pairwise disjoint closed sets and hence the set of local cutpoints of X is contained in the union of a countable family of pairwise disjoint closed sets. Thus, it will have proved that (c) also fails.

Just suppose that the set of local cutpoints of $X/\sim$ is not totally disconnected. Since $X/\sim$ is a Peano continuum the set of local cutpoints of $X/\sim$ is an F_σ. It follows by the Sum Theorem for dimension zero that the set of local cutpoints of $X/\sim$ contains a continuum A. Then $\pi^{-1}(A)$ is a continuum in X which is the union of uncountably many equivalence classes of $\sim$. Let $c, d \in \pi^{-1}(A)$ such that $c \not\sim d$. Let C be an arc in $\pi^{-1}(A)$ with endpoints c and d. For convenience, we identify C with the closed unit interval $[0, 1]$ with its usual order and its usual metric. To prove the theorem, it will suffice to prove that there is an arc $D \subset \pi^{-1}(A)$ such that $c, d \in D$ and $D - L$ is at most countable.

Let $A_1, A_2, \cdots$ denote the nondegenerate equivalence classes of $\sim$. If $x \in C - L$ then $x \in A_1 \cup A_2 \cup \cdots$. Let m_1 be an integer such that diameter $(A_{m_1} \cap C) \geqslant$ diameter $(A_i \cap C)$ for each $i = 1, 2, \cdots$. Let $c_1 = \min(A_{m_1} \cap C)$ and let $d_1 = \max(A_{m_1} \cap C)$. Let B_1 be an arc in A_{m_1} with endpoints c_1 and d_1 such that $B_1 - L$ is at most countable. Notice that $A_{m_1} \cap C \subset [c_1, d_1]$ and for each $i = 1, 2, \cdots$ there do not exist $a, b \in A_i \cap C$ such that $a < c_1 < d_1 < b$.

Suppose $A_{m_1}, \cdots, A_{m_n}$ have been selected and for each $j = 1, \cdots, n$, B_j is an arc in A_{m_j} with endpoints c_j and d_j such that $B_j - L$ is at most countable and $(A_{m_j} \cap C) \subset [c_1, d_1] \cup \cdots \cup [c_j, d_j]$. Suppose also that there do not exist $k \in \{1, \cdots, n\}$ and an integer $i \in \{1, 2, \cdots\}$ with $a, b \in (C \cap A_i) - ([c_1, d_1] \cup \cdots \cup [c_n, d_n])$ such that $a < c_k < d_k < b$. Suppose the intervals $[c_i, d_i]$ are disjoint.

If, for each j, $(A_j \cap C) - ([c_1, d_1] \cup \cdots \cup [c_n, d_n])$ contains at most one point, then it is easy to check that $D = (C - ([c_1, d_1] \cup \cdots \cup [c_n, d_n])) \cup B_1 \cup \cdots \cup B_n$ is an arc in $\pi^{-1}(A)$ which contains c and d and $D - L$ is at most countable.

Let us suppose, therefore, that there exists an integer m_{n+1} such that

$$0 < \text{diameter } ((A_{m_{n+1}} \cap C) - ([c_1, d_1] \cup \cdots \cup [c_n, d_n]))$$
$$\geqslant \text{diameter } ((A_j \cap C) - ([c_1, d_1] \cup \cdots \cup [c_n, d_n]))$$

for each $j = 1, 2, \cdots$.

Let

$$c_{n+1} = \min((A_{m_{n+1}} \cap C) - ([c_1, d_1] \cup \cdots \cup [c_n, d_n]))$$

and let

$$d_{n+1} = \max((A_{m_{n+1}} \cap C) - ([c_1, d_1] \cup \cdots \cup [c_n, d_n])).$$

Let B_{n+1} be an arc in $A_{m_{n+1}}$ with endpoints c_{n+1} and d_{n+1} such that $B - L$ is at most countable.

Let $D = (C - ([c_1, d_1] \cup [c_2, d_2] \cup \cdots)) \cup B_1 \cup B_2 \cup \cdots$. For each positive integer i, $A_i \cap (D - ([c_1, d_1] \cup [c_2, d_2] \cup \cdots))$ contains at most one point. Thus, $D - L$ is at most countable.

For each positive integer j let h_j: $[c_j, d_j] \longrightarrow B_j$ be a homeomorphism such that $h_j(c_j) = c_j$ and $h_j(d_j) = d_j$. Define h: $C \longrightarrow D$ by

$$h(x) = \begin{cases} x & \text{if } x \notin [c_j, d_j] \text{ for any } j, \\ h_j(x) & \text{if } x \in [c_j, d_j]. \end{cases}$$

Then h is easily seen to be a homeomorphism of the arc C onto D. Thus D is an arc in $\pi^{-1}(A)$ which contains c and d and $D - L$ is at most countable. Thus, $\pi(c) = \pi(d)$ which is a contradiction.

(c) $\Rightarrow$ (a). Suppose (a) fails. Then there is a sequence $A_1, A_2, \cdots$ of pairwise disjoint proper closed subsets of X and points x and y that are separated by $X - (A_1 \cup A_2 \cup \cdots)$ such that no countable subset of $X - (A_1 \cup A_2 \cup \cdots)$ separates x and y in X. Let $A = A_1 \cup A_2 \cup \cdots$. We shall prove that X contains a subcontinuum B such that $x, y \in B$, no countable subset of $B - A$ separates x and y in B and $B - A$ contains only countably many local cutpoints of B. Clearly the set of local cutpoints of B is contained in the union of countably many disjoint proper closed sets so (c) fails.

We define by transfinite induction a nest of subcontinua B_α) of X as follows: Let $B_0 = X$. Let α be a countable ordinal number. Suppose that for each $n < \alpha$, B_n has been defined to be a subcontinuum of X such that $x, y \in B$ and no countable subset of $B_n - A$ separates x and y in B_n. If, for some $n < \alpha$, $B_n - A$ contains only countably many local cutpoints of B_n, we are done. Suppose, therefore, that for each $n < \alpha$, $B_n - A$ has uncountably many local cutpoints of B_n.

Case 1. *α is the successor of an ordinal number m.* By assumption $B_m - A$ contains uncountably many local cutpoints of B_m. It follows from [8, III. 9.21] that every uncountable set of local cutpoints of a continuum contains a pair of points that separate the continuum. Thus, there exist $a_\alpha, b_\alpha \in B_m - (A \cup \{x, y\})$ such that $B_m - \{a_\alpha, b_\alpha\}$ is not connected. Let B_α be the closure of the component of $B_m - \{a_\alpha, b_\alpha\}$ that contains x and y. Then, no countable set of $B_\alpha - A$ separates x and y in B_α.

Case 2. *α is a limit ordinal.* Let $B_\alpha = \bigcap_{n<\alpha} B_n$. We shall show that no countable subset of $B_\alpha - (A \cup \{x, y\})$ separates x and y in B_α. Let C be a countable subset of B_α.

Let $C' = C \cup \bigcup_{n<\alpha} \{a_{n+1}, b_{n+1}\}$. Since C' is countable x and y lie in the same component E of $X - C'$. There is an arc D in the locally connected, topologically complete, metric space E with endpoints x and y. By induction it is easy to see that $D \subset B_n$ for each $n < \alpha$. Hence $D \subset B_\alpha - C$ which implies that x and y lie in the same component of $B_\alpha - C$.

Since X does not contain uncountably many pairwise disjoint nondegenerate subcontinua it follows that for some countable ordinal α, B_α has at most countably many local cutpoints of B_α in $B_\alpha - A$. This completes the proof of Theorem 3.

Corollary III. 9.21 in Whyburn [8] asserts that if X is a metric continuum and if G is an uncountable set of local cutpoints of X then there is a countable subset G_0 of G such that every point of $G - G_0$ is of order 2 relative to $G - G_0$. In particular, there exist $a, b \in G - G_0$ such that $X - \{a, b\}$ is not connected.

It is easy to see that $X - \{a, b\}$ has either 2 or 3 components.

If X is as above and C is a connected subset of X then $\mathrm{Cl}(C) - C$ contains at most countably many local cutpoints of $\mathrm{Cl}(C)$. For let G be an uncountable set of local cutpoints of $\mathrm{Cl}(C)$. Let $a, b \in G$ such that $\mathrm{Cl}(C) - \{a, b\}$ is not connected. Since C is connected and dense in $\mathrm{Cl}(C)$ it follows that $C - \{a, b\}$ is not connected. Thus, at least one of a and b is in C and $G \not\subset \mathrm{Cl}(C) - C$.

In order that all of the connected subsets of a regular continuum should be arcwise connected it is necessary that the continuum satisfy conditions (a)–(b) of Theorem 3. If X is a regular continuum that satisfies condition (a) of Theorem 3, let Y be a connected set in X. Let D be a countable dense set in Y and let $a \in D$. By Theorem 3 for each $d \in D$ there is an arc A_d in $\mathrm{Cl}(Y)$ such that $a, d \in A_d$ and A_d contains at most countably many points that are not local cutpoints of $\mathrm{Cl}(Y)$. By the last paragraph $A_d - Y$ is at most countable. By [7, Theorem 34] $Y \cup \bigcup\{A_d \mid d \in A\}$ is arcwise connected. Thus, there is a countable set $C = \bigcup\{A_d - Y \mid d \in A\}$ such that $Y \cup C$ is arcwise connected.

Question 1. If X is a regular continuum that satisfies condition (a) of Theorem 3, is every connected subset Y of X arcwise connected?

Question 2. If X is as in Question 1, is every connected F_σ in X arcwise connected? In particular is the set of local cutpoints of X arcwise connected if it is connected?

Question 3. Let X be a regular continuum such that every pair of separated sets in X can be separated by a countable set. Is every connected subset of X arcwise connected?

THEOREM 4. *Let X be a regular continuum. If C is a connected subset of X then C cannot be decomposed into countably infinitely many pairwise disjoint sets that are closed in C.*

PROOF. Let C be a connected subset of X. Suppose $C = \bigcup_{i=1}^{\infty} A_i$ where the A_i are pairwise disjoint proper subsets of C which are closed in C. Since every sequence of pairwise disjoint connected sets in a regular space is a null sequence we may suppose the sequence $A_i)$ is a null sequence.

Define an equivalence relation $\sim$ on C by letting $x \sim y$ if and only if there is a natural number i such that $x, y \in A_i$. Let π be the natural projection of C onto $C/\!\sim$. Now $C/\!\sim$ with the quotient topology is a connected countable space. We shall obtain a contradiction by proving that $C/\!\sim$ cannot be connected because it is a countable, T_1, normal space. It is clear that $C/\!\sim$ is a T_1 space since $\pi^{-1}(p)$ is closed in C for each $p \in C/\!\sim$. It remains to prove only that $C/\!\sim$ is normal.

Let M and N be disjoint closed sets in $C/\!\sim$. Then $\pi^{-1}(M)$ and $\pi^{-1}(N)$

are disjoint closed sets in C. Since C is normal there exist disjoint open sets U and V in C such that $\pi^{-1}(M) \subset U$ and $\pi^{-1}(N) \subset V$. Since the sequence $A_i)$ is a null sequence of closed sets in X it follows that $\pi^{-1}(\pi(C - U))$ is a closed set in C which misses $\pi^{-1}(M)$. Hence, $U - \pi^{-1}(\pi(C - U))$ and $V - \pi^{-1}\pi(C - V)$ are neighbourhoods of $\pi^{-1}(M)$ and $\pi^{-1}(N)$ respectively whose images under π are disjoint neighbourhoods of M and N respectively in $C/\sim$.

COROLLARY 5. *A connected subset of a regular continuum has either one or uncountably many arc components.*

PROOF. The arc components of a connected subset of a regular continuum are closed (see [7, p. 334]); hence Theorem 4 applies.

3. **Continua that are not regular.** For the remainder of this section let X be a fixed continuum that is not regular but which contains no nonnull sequence of pairwise disjoint subcontinua.

DEFINITION. Let M be a subcontinuum of a continuum X and let $x, y \in M$. We say M satisfies property $P(x, y)$ if no finite set separates x and y in any neighbourhood of M.

THEOREM 6. *If X is a continuum that is not regular then X contains a connected subset that is not arcwise connected.*

PROOF. We may suppose as in [2] that X is a hereditarily locally connected continuum and that every sequence of pairwise disjoint subcontinua of X is a null sequence. By Lemma 2 we may suppose that for each subcontinuum P of X the set of local cutpoints of P is not totally disconnected.

Let $a \in X$ such that X is not regular at a. Let M denote the set of points of X which cannot be separated from a by a finite set in X. By Whyburn [8, V. 4.4, 4.5] M is a nondegenerate continuum. By Whyburn [8, III. 9.2] at most a countable number of points of M are local cutpoints of X.

The first three claims can be proved by contradiction. The proofs are straightforward and are omitted.

CLAIM 1. *M satisfies $P(a, b)$ for each $b \in M - \{a\}$.*

Let $b \in M - \{a\}$. Let $M_\lambda)_{\lambda\in\Lambda}$ be a maximal nest of subcontinua of M each of which satisfies property $P(a, b)$ and let $P_{ab} = \bigcap_{\lambda\in\Lambda} M_\lambda$. Then P_{ab} is a continuum which is irreducible with respect to satisfying $P(a, b)$, for if U is a neighbourhood of P_{ab} then $M_\lambda \subset U$ for some $\lambda \in \Lambda$ and hence no finite set separates a and b in U. We shall describe in great detail the structure of the continuum P_{ab}.

In [2, Theorem 6] the author considered the case where P_{ab} is an arc. To see that P_{ab} may be quite complicated let Y be the continuum constructed in the proof of Theorem 6 in [2]. Then $Y = A_0 \cup A_1 \cup A_2 \cup \cdots$ where $A_1, A, \cdots$ are pairwise disjoint arcs and A_0 is also an arc. Let Z be the decomposition space obtained from Y by contracting A_i to a point for each $i = 1, 2, \cdots$. Then $Z = P_{ab}$ where a and b are the endpoints of the arc A_0.

CLAIM 2. *If $c, d \in P_{ab}$ then P_{ab} satisfies $P(c, d)$.*

CLAIM 3. *If U is a neighbourhood of $x \in P_{ab}$ such that $b \notin U$ then there is a point c in the boundary of U and a continuum $P \subset P_{ab} - U$ such that P satisfies $P(b, c)$.*

If $x \in P_{ab}$ is not a local cutpoint of X and Q and P are continua in P_{ab} which satisfy $P(a, x)$ and $P(x, b)$ respectively then $P \cup Q$ is a continuum satisfying $P(a, b)$ and so $P \cup Q = P_{ab}$ by the irreducibility of P_{ab}.

CLAIM 4. *If $x \in P_{ab} - \{b\}$ then there exists a proper subcontinuum of P_{ab} which satisfies $P(a, x)$.*

PROOF. Suppose the claim is false. Let $x \in P_{ab} - \{b\}$ such that no proper subcontinuum of P_{ab} satisfies $P(a, x)$.

If U is any neighbourhood of b in P_{ab} then $\mathrm{Cl}(U) - U$ contains at most countably many local cutpoints of $\mathrm{Cl}(U)$ by the argument following Theorem 3. If U is connected, then by Lemma 2 there exists an arc I of local cutpoints of $\mathrm{Cl}(U)$. Now I intersects the boundary of U in a set that is compact and at most countable so U contains an arc of local cutpoints of U and hence of P_{ab}.

By the last paragraph, there exists a sequence $I_i)$ of pairwise disjoint arcs in $P_{ab} - \{b\}$ such that $\lim\sup I_i = \{b\}$ and for each $i = 1, 2, \cdots$ each point of I_i is a local cutpoint of P_{ab}.

Let c_i and d_i be the endpoints of I_i. Since P_{ab} contains only countably many local cutpoints of X we may suppose c_i and d_i are not local cutpoints of X. By the argument following Theorem 3 we may suppose that c_i and d_i are points of order 2 in P_{ab}, that c_i and d_i separate P_{ab} into either 2 or 3 components and that the component K_i of $P_{ab} - \{c_i, d_i\}$ which meets I_i contains neither a nor b. By Whyburn [3, §4] we may suppose that every point of $I_i - \{c_i, d_i\}$ disconnects K_i. Let (by [8, III. 9.21]) $z_i \in I_i - \{c_i, d_i\}$ be a point of order 2 in P_{ab} such that z_i is not a local cutpoint of X. Then z_i separates K_i into exactly two components. Finally, we may assume that the sets $K_i)$ are pairwise disjoint. For each i let U_i be a neighbourhood of K_i such that $\mathrm{Cl}(U_i) \cap \mathrm{Cl}(U_j)$ is empty for $i \neq j$. Notice that $\mathrm{Cl}(K_i) = K_i \cup \{c_i, d_i\}$.

By Claim 3 there is a component P of $P_{ab} - K_i$ such that P satisfies either $P(a, c_i)$ or $P(a, d_i)$. We may suppose without loss of generality that P satisfies $P(a, c_i)$. Since c_i is not a local cutpoint of X it follows that P does not satisfy $P(c_i, b)$ for otherwise P would be a proper subcontinuum of P_{ab} which satisfies $P(a, b)$. By Claim 3 it follows that there is a component Q of $P_{ab} - K_i$ such that Q satisfies $P(d_i, b)$. Let $P_{ac_i} \subset P$ and $P_{d_ib} \subset Q$ be continua which are irreducible with respect to satisfying $P(a, c_i)$ and $P(d_i, b)$ respectively. If $P_{ac_i} \cap P_{d_ib}$ is infinite let U be any neighbourhood of $P_{ac_i} \cup P_{ad_i}$. Let A be a finite set in U and let $x \in (P_{ac_i} \cap P_{d_ib}) - A$. By Claim 2, P_{ac_i} satisfies $P(a, x)$ and P_{ad_i} satisfies $P(x, b)$. Since U is a neighbourhood of both P_{ac_i} and P_{ad_i}, A does not separate a from x or x from b in U. Thus A does not separate a from b in U and so $P_{ac_i} \cup P_{d_ib}$ is a proper subcontinuum of P_{ab} which satisfies $P(a, b)$. With this contradiction we conclude that $P_{ac_i} \cap P_{d_ib}$ is finite. A similar argument can be used to show that every point of $P_{ac_i} \cap P_{d_ib}$ is a local cutpoint of X.

By using the facts that $P_{ab} - K_i$ does not satisfy $P(a, b)$ while P_{ab} satisfies $P(a, b)$, that $\mathrm{Cl}(K_i) - K_i = \{c_i, d_i\}$ and that c_i and d_i are not local cutpoints of X one can easily prove by contradiction that $\mathrm{Cl}(K_i)$ satisfies $P(c_i, d_i)$. Since c_i and d_i are not local cutpoints of X it follows from the remark following Claim 3 that $P_{ab} = P_{ac_i} \cup K_i \cup P_{d_ib}$.

Since z_i separates c_i and d_i in $\mathrm{Cl}(K_i)$ it follows from Claim 3 that $\mathrm{Cl}(K_i)$ satisfies $P(c_i, z_i)$. Since c_i is not a local cutpoint of X, $P_{ac_i} \cup \mathrm{Cl}(K_i)$ satisfies $P(a, z_i)$. Let $P_{az_i} \subset P_{ac_i} \cup \mathrm{Cl}(K_i)$ be a continuum which is irreducible with respect to satisfying $P(a, z_i)$.

Since $b \notin P_{ac_i} \cup \mathrm{Cl}(K_i)$ we may suppose that, for each i, $z_{i+1} \in P_{d_ib}$. Hence, $\mathrm{Cl}(K_{i+1}) \subset P_{d_ib}$. By Claim 2 we may suppose that $P_{d_{i+1}b} \subset P_{d_ib} - K_{i+1}$. It now follows that $P_{az_i} \subset P_{ac_i} \cup \mathrm{Cl}(K_i) \subset P_{ac_{i+1}} \subset P_{az_{i+1}}$.

Since P_{ab} is irreducible with respect to satisfying $P(a, x)$ it follows that $x \notin P_{az_i}$ for each i but $x \in \limsup P_{az_i} = P_{ab}$. We may suppose that for each i there is $x_i \in P_{az_i} - P_{az_{i-1}}$ such that $\lim x_i = x$. Since P_{ab} contains at most countably many local cutpoints of X we may suppose each x_i is not a local cutpoint of X.

For each $i = 2, 3, \cdots$ there is a continuum $P_{d_{i-1}c_{i+1}} \subset P_{d_{i-1}b} \cap P_{ac_{i+1}}$ such that $P_{d_{i-1}c_{i+1}}$ is irreducible with respect to satisfying $P(d_{i-1}, c_{i+1})$. Notice that $x_i, z_i \in P_{d_{i-1}c_{i+1}}$.

For each i let W_i be a neighbourhood of $P_{ab} - K_i$ and A_i a finite set such that A_i separates a and b in W_i and $W_{i+j} \subset W_i \cup U_i$ for all $i, j = 1, 2, \cdots$. Since x_i and z_i are not local cutpoints of X we may suppose that $x_i, z_i \notin A_j$ for all i and j. Note that for $j = 1, 2, \cdots, z_{i+j}$ and x_{i+j} lie in the component of $W_i - A_i$

which contains b and, for $j = 1, \cdots, n-1$, z_{i-j} and x_{i-j} lie in the component of $W_i - A_i$ which contains a.

For each $i = 2, 4, 6, \cdots$, $P_{d_{i-1}c_{i+1}} \subset W_1 \cap \cdots \cap W_i \cap W_{i+1}$ so there exists an arc C_i joining x_i to z_i in $(W_1 \cap \cdots \cap W_{i-1} \cap W_{i+1}) - (A_1 \cup \cdots \cup A_{i-1} \cup A_{i+1})$. The arcs C_i are pairwise disjoint by construction. This contradicts our assumption that every sequence of disjoint subcontinua of X is a null sequence. The claim is proved.

CLAIM 5. *If $x \in P_{ab}$ then there is a unique continuum P_{ax} (resp. P_{xb}) in P_{ab} which is irreducible with respect to satisfying $P(a, x)$ (resp. $P(x, b)$).*

PROOF. Just suppose A and B are two subsets of P_{ab} which are irreducible with respect to satisfying $P(a, x)$. Let $y \in A - B$ and let $z \in B - A$ such that y and z are not local cutpoints of X. Let $P_{yb} \subset P_{ab}$ be a continuum which is irreducible with respect to satisfying $P(y, b)$. By the remark following Claim 3 $P_{ab} = A \cup P_{yb}$ thus $z \in P_{yb}$. By Claim 4 there exists a continuum $P_{zb} \subset P_{yb} - \{y\}$ such that P_{zb} satisfies $P(z, b)$. Thus, $B \cup P_{zb}$ is a proper subcontinuum of P_{ab} which satisfies $P(a, b)$. This is a contradiction. The claim is proved.

CLAIM 6. *If $x, y \in P_{ab}$ then either $P_{ax} \subset P_{ay}$ or $P_{ay} \subset P_{ax}$.*

PROOF. Suppose $P_{ax} \not\subset P_{ay}$ and $P_{ay} \not\subset P_{ax}$. By Claims 2 and 5, $x \notin P_{ay}$ and $y \notin P_{ax}$. Since P_{ab} contains at most countably many local cutpoints of X, we may suppose x and y are not local cutpoints of X.

Since $P_{ax} \cup P_{xb} = P_{ab}$, $y \in P_{xb}$. By Claim 4, $P_{yb} \not\subseteq P_{xb}$. Thus, $x \notin P_{yb}$ and $P_{ay} \cup P_{yb}$ is a proper subcontinuum of P_{ab} which satisfies $P(a, b)$. This is a contradiction. The claim is proved.

Let $2^{P_{ab}}$ denote the space of closed subsets of P_{ab} with the Hausdorff metric topology. Let $\mathcal{C}$ denote the closure in $2^{P_{ab}}$ of $\{P_{ax} | x \in P_{ab}\}$.

CLAIM 7. *$\mathcal{C}$ is homeomorphic to the closed unit interval* $[0, 1]$.

PROOF. $2^{P_{ab}}$ is a compact metric space that is partially ordered by inclusion. This partial order is a closed relation on $2^{P_{ab}}$. Since $\mathcal{C}$ is compact, we need only prove that $\mathcal{C}$ is connected and totally ordered under inclusion (Ward [9]).

Let $C \in \mathcal{C}$. We start by showing that if $x \in C$ then $P_{ax} \subset C$. For let $x_i)$ be a sequence in P_{ab} such that $P_{ax_i})$ converges to C in $\mathcal{C}$. If, for arbitrarily large i, $x \in P_{ax_i}$ then $P_{ax} \subset P_{ax_i}$ for all such i by Claims 2 and 5 and hence $P_{ax} \subset \limsup P_{ax_i} = C$. If, on the other hand, $P_{ax_i} \subset P_{ax}$ for arbitrarily large i and if U is a neighbourhood of C such that a and x can be separated by a finite set in U, then for some sufficiently large i P_{ax_i} can be separated in U by that same finite set. This contradicts Claim 2. By Claim 6, these are the only two cases we need consider. Thus, $x \in C$ implies $P_{ax} \subset C$.

Let $C, D \in \mathcal{C}$ such that $C \not\subset D$. Let $x \in C - D$ such that x is not a local cutpoint of X and let $y \in D$. Then $x \notin P_{ay}$ since $P_{ay} \subset D$. By Claim 6 $y \in P_{ax} \subset D$. Thus $D \subset C$ and $\mathcal{C}$ is totally ordered by inclusion.

If C is not connected then there exist $C, D \in \mathcal{C}$ with $C \subsetneq D$ such that for each $E \in \mathcal{C}$ either $E \subset C$ or $D \subset E$. Now, $D - C$ contains at least two points x and y such that x and y are not local cutpoints of X. Then $P_{ax} = P_{ay} = D$. By Claim 4 $x = y$. With this contradiction we conclude that $\mathcal{C}$ is connected. The claim is proved.

CLAIM 8. *If $x_i)$ is a sequence in P_{ab} such that the sequence $P_{ax_i})$ converges in $\mathcal{C}$ then $x_i)$ converges in P_{ab}.*

PROOF. Just suppose that $x_i)$ and $y_i)$ are two sequences in P_{ab} which converge to x and y respectively such that the sequences $P_{ax_i})$ and $P_{ay_i})$ both converge to $C \in \mathcal{C}$. We may suppose that, for each i, $x_i)$ and $y_i)$ are not local cutpoints of X and neither sequence is constant.

By Claim 6 we need to consider only three cases.

Case 1. For each i, $P_{ax_i} \subset P_{ax_{i+1}}$ and $P_{ay_i} \subset P_{ay_{i+1}}$. If, for each i and j, $P_{ax_i} \subset P_{ay_j}$ then $C = P_{ay_1}$ and the sequence $y_i)$ would have to be constant. We may suppose, therefore, that for each i $P_{ax_i} \subset P_{ay_i} \subset P_{ax_{i+1}}$. As in the proof of Claim 4 if $y \in P_{ax_i} \cap P_{x_ib}$ and y is not a local cutpoint of X then $y = x_i$. Since P_{ab} contains at most countably many local cutpoints of X, $P_{ax_i} \cap P_{x_ib}$ is at most a countable set. Since $P_{ay_{i-1}} \not\subseteq P_{ax_i}$ it follows that $P_{ab} - (P_{x_ib} \cup P_{ay_{i-1}})$ is a nonempty open subset of P_{ab} which is contained in P_{ax_i}. As in Claim 4 there exists an arc I_i of local cutpoints of P_{ab} such that $I_i \subset P_{ab} - (P_{x_ib} \cup P_{ay_{i-1}}) \subset P_{ax_i} - P_{ay_{i-1}}$. Now construct as in Claim 4 a sequence $A_i)$ of pairwise disjoint arcs such that for each i $x_i, y_i \in A_i$. Since every sequence of disjoint continua in X is null, $x = y$.

Case 2. For each i $P_{ax_i} \supset P_{ax_{i+1}}$ and $P_{ay_i} \supset P_{ay_{i+1}}$.

Argue as in Case 1.

Case 3. For each i $P_{ax_i} \subset P_{ax_{i+1}}$ and $P_{ay_i} \supset P_{ay_{i+1}}$. Then $P_{ax_i} \subset P_{ay_j}$ for each i and j. Let U be a neighbourhood of x in P_{ab} such that the boundary of U contains no local cutpoints of X and $y \notin U$. For each i let $z_i \in P_{x_iy_i} \cap$ (boundary of U). Then, $P_{ax_i} \subset P_{az_i} \subset P_{ay_i}$ so that $\lim P_{az_i} = C$.

If for some subsequence $z_{i_j})$ $P_{az_{i_j}} \subsetneq P_{az_{i_{j+1}}}$ for each j, then we are in Case 1 with the sequences $x_i)$ and $z_{i_j})$. If for some subsequence $z_{i_j})$ $P_{az_{i_j}} \supsetneq P_{az_{i_{j+1}}}$

for each j then we are in Case 2 with the sequences $z_{i_j})$ and $y_i)$. If the sequence $z_i)$ contains a constant subsequence we may suppose $z_i = z$ for each i. Let $w_i)$ be a sequence in P_{az} which converges to z. Then $\lim P_{aw_i} = P_{az} = C$. We may now apply Case 1 to the sequences $w_i)$ and $x_i)$.

The claim is now proved.

We are now in a position to adapt the proof of Theorem 6 in [2] to the continuum P_{ab}.

We shall attach to P_{ab} an infinite sequence of pairwise disjoint closed sets A_i such that no pair of points of P_{ab} can be separated by a finite set in $P_{ab} \cup (\bigcup A_i)$.

Let h be a homeomorphism of the closed unit interval $[0, 1]$ onto C such that $h(0) = \{a\}$ and $h(1) = P_{ab}$. Define f: $[0, 1] \longrightarrow P_{ab}$ by letting

$$f(r) = \begin{cases} x, & \text{if } h(r) = P_{ax}, \\ \lim x_i, & \text{if } h(r) = \lim P_{ax_i}. \end{cases}$$

By Claim 8 f is a continuous function. Notice that if $x \in P_{ab}$ is not a local cutpoint of X then $f^{-1}(x)$ is a singleton.

If A is a set in X and $\epsilon > 0$ we let $S(A, \epsilon)$ denote the ϵ-neighbourhood of A in X.

By the proof of Claim 4 if $r \in [0, 1]$, then $h(r) = \{f(r)\} \cup (\bigcup\{h(s) \mid s < r\})$.

Let $0 < r < 1$. For $\epsilon > 0$ there exists, by the proof of Claim 4, $\delta > 0$ such that $r < s < \delta + r$ implies $h(s) \subset h(r) \cup S(f(r), \epsilon)$.

Let $Y_r = \bigcup\{P_{xb} \mid x \in P_{ab} - h(r)\}$. Let $r_i)$ be a sequence in $[0, 1]$ which is strictly decreasing to r such that, for each i, $f(r_i)$ is not a local cutpoint of X. It can be shown as in the proof of Claim 4 that $Y_r = \bigcup P_{f(r_i)b}$ and $Y_r \cup \{f(r)\}$ is compact.

If $0 < s < 1$ and U is any neighbourhood of $h(s)$ then no pair of points of $h(s)$ can be separated in U by a finite set. We may suppose therefore that either

(*)

(1) for each i there exist $c_i, d_i \in [0, 1]$ with $c_i < r < d_i$ such that $f(c_i) = f(d_i) \in (h(r) \cap Y_r \cap S(f(r), 1/i)) - \{f(r)\}$, or

(2) for each i there exists an arc $C_i \subset S(f(r), 1/i)$ such that $C_i \cap h(1)$ consists precisely of the two endpoints of C_i, one endpoint of C_i is in $h(r) - \{f(r)\}$ and the other is in $h(1) - h(r)$. The arcs C_i may be taken to be pairwise disjoint.

Suppose (1) holds. Let $E_i = \{f(c_i)\}$. We wish to show that $\lim c_i = r$. Just suppose that for each i $c_i \leqslant s < r$. Then $h(s) \cup Y_r$ is a continuum in P_{ab}

which satisfies $P(a, b)$. If $s < t < r$ and $f(t)$ is not a local cutpoint of X then $f(t) \notin h(s) \cup Y_r$. This contradicts the assumption that P_{ab} is an irreducible continuum with respect to satisfying $P(a, b)$. Thus, $\lim c_i = r$. Similarly, $\lim d_i = r$.

Now suppose (2) holds. Since every sequence of disjoint subcontinua of X is a null sequence, there exists a sequence $\epsilon_i)$ of positive numbers converging to zero such that if D_i is the component of $[S(h(1), \epsilon_i) \cup C_i] - h(1)$ which meets C_i then $D_i \cap D_j = \emptyset$ for $i \neq j$ and the diameters of the D_i converge to 0.

Let $M_i = \mathrm{Cl}(D_i) \cap (h(r) - \{f(r)\})$ and $N_i = \mathrm{Cl}(D_i) \cap (Y_r - \{f(r)\})$. If M_i (resp. N_i) has an isolated point let c_i (resp. d_i) be an isolated point of M_i (resp. N_i). If M_i (resp. N_i) has no isolated points let $c_i = \inf \{s \in [0, 1] \mid \mathrm{Cl}(D_i) \cap h(s)$ is uncountable$\}$ (resp. $d_i = \sup \{s \in [0, 1] \mid \mathrm{Cl}(D_i) \cap P_{f(s)b}$ is uncountable$\}$). Then $h(c_i) \cap \mathrm{Cl}(D_i)$ is at most countable.

As in (1) $\lim c_i = \lim d_i = r$. Let E_i be an arc in $D_i \cup \{c_i, d_i\}$ with endpoints c_i and d_i.

We wish to prove that for each $\epsilon > 0$ and each $s \in [0, 1]$ either

$$(**) \qquad (h(s) \cap Y_s \cap S(f(s), \epsilon)) - \{f(s)\} \neq \emptyset$$

or there exists an arc in $S(f(s), \epsilon) - E_i$ which joins $h(s)$ to $h(1) - h(s)$. We need only consider $s \in [0, 1]$ such that $f(s) = f(c_i)$ or $f(s) = f(d_i)$.

Clearly, (**) is satisfied if both M_i and N_i have an isolated point. Suppose, therefore, that M_i does not have an isolated point. Let $s \in [0, 1]$ such that $f(s) = f(c_i)$ and suppose $\epsilon > 0$ is given such that

$$(h(s) \cap Y_s \cap S(f(s), \epsilon)) - \{f(s)\} = \emptyset.$$

By (*) there is a sequence of arcs $F_j \subset S(f(s), \epsilon) - \{f(s)\}$ which join $h(s)$ to $h(1) - h(s)$ such that $\lim F_j = \{f(s)\}$. Let e_j and f_j be the endpoints of F_j. We may suppose $F_j \cap h(1) = \{e_j, f_j\}$ where $e_j \in h(s)$ and $f_j \in h(1) - h(s)$.

Just suppose that for each j $E_i \cap F_j \neq \emptyset$. Then each $e_j \in \mathrm{Cl}(D_i)$. By the choice of c_i and by the assumption that M_i is a perfect set each neighbourhood of e_j contains uncountably many points of Y_{c_i}. Since $Y_{c_i} \cup \{f(c_i)\}$ is compact $e_j \in Y_{c_i}$. If $s \leqslant c_i$ then

$$e_j \in (h(s) \cap Y_s \cap S(f(s), \epsilon)) - \{f(s)\}$$

which is a contradiction. If $c_i < s$ then for each j let $e_j' \in [0, 1]$ such that $f(e_j') = e_j$. We get as in (1) that $\lim e_j' = s$ and hence eventually $e_j \in Y_s - h(c_i)$ which is again a contradiction. We conclude that for all sufficiently large j $E_i \cap F_j = \emptyset$. Thus, (**) is satisfied.

If there exist $c < ½ < d$ such that $f(c) = f(d) \in S(f(½), 1) - \{f(½)\}$ let $C(½, 1) = \{f(c)\}$. Otherwise, let $C(½, 1)$ be an arc in $S(h(1), 1)$ with endpoints

$f(c)$ and $f(d)$ where $\frac{1}{4} < c < \frac{1}{2} < d < \frac{3}{4}$ and $C(\frac{1}{2}, 1)$ is obtained as was E_i above.

Let $A_1 = C(\frac{1}{2}, 1)$. Suppose $A_1, \cdots, A_{n-1}$ have been constructed to be pairwise disjoint closed sets such that for each $i = 1, \cdots, n-1$

(i) A_i is the union of a finite number of arcs and points each of which was obtained as was E_i above,

(ii) for each $a \in [1/2^i, 1 - 1/2^i]$ there is a component C of A_i such that C meets both $f([1/2^i, a]) - \{f(a)\}$ and $f([a, 1 - 1/2^i]) - \{f(a)\}$,

(iii) $P_{ab} \cap A_i \supset P_{ab} - f([0, 1/2^{i+1}] \cup [1 - 1/2^{i+2}, 1])$.

For each $x \in [1/2^n, 1 - 1/2^n]$ let $C(x, n) \subset S(h(1), 1/2^{n-1}) - (A_1 \cup \cdots \cup A_{n-1})$ be an (possibly degenerate) arc chosen as was E_i with endpoints $f(c(x,n))$ and $f(d(x, n))$ where $1/2^{n+1} < c(x, n) < x < d(x, n) < 1 - 1/2^{n+1}$, $C(x, n)$ minus its endpoints lies in $X - h(1)$ and $C(x, n)$ satisfies (**). The set of open intervals $]c(x, n), d(x, n)[$ such that $x \in [1/2^n, 1 - 1/2^n]$ is an open cover for the compact set $[1/2^n, 1 - 1/2^n]$ hence there exists a minimal finite set $\{x_1, \cdots, x_k\} \subset [1/2^n, 1 - 1/2^n]$ such that

$$]c(x_1, n), d(x_1, n)[\cup \cdots \cup]c(x_k, n), d(x_k, n)[$$

covers $[1/2^n, 1 - 1/2^n]$. It is clear that conditions (i)–(iii) are satisfied for n. Let $A_n = C(x_1, n) \cup \cdots \cup C(x_k, n)$.

Let $Y = P_{ab} \cup A_1 \cup A_2 \cup \cdots$. Since every sequence of disjoint subcontinua of X is a null sequence Y is a continuum.

To prove that Y contains a connected set that is not arcwise connected it suffices by Theorem 1 and the fact that P_{ab} contains only countably many local cutpoints of X to prove that if z is a local cutpoint of Y then either $z \in A_i$ for some i or z is a local cutpoint of X. Let $z \in P_{ab}$ such that z is not a local cutpoint of X and $z \notin A_i$ for any i. It follows from the fact that $f^{-1}(z)$ is a singleton that if U is any connected neighbourhood of z in P_{ab} such that $U - \{z\}$ has more than one component then $U - \{z\}$ has exactly two components one of which is contained in $\{f(y) | y < f^{-1}(z)\}$ and the other is in $\{f(y) | f^{-1}(z) < y\}$. It is now easy to see that z is not a local cutpoint of Y and the theorem is proved.

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