THE GEOMETRIC DIMENSION OF SOME VECTOR BUNDLES OVER PROJECTIVE SPACES⁽¹⁾

BY

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ABSTRACT. We prove that in many cases the geometric dimension of the p -fold Whitney sum pH_k of the Hopf bundle H_k over quaternionic projective space QP^k is the smallest n such that for all $i \leq k$ the reduction of the i th symplectic Pontryagin class of pH_k to coefficients $\pi_{4i-1}((RP^\infty/RP^{n-1}) \wedge bo)$ is zero, where bo is the spectrum for connective KO -theory localized at 2. We immediately obtain new immersions of real projective space RP^{4k+3} in Euclidean space if the number of 1's in the binary expansion of k is between 5 and 8.

1. Introduction. In this paper we determine the geometric dimension (gd) of the stable class of many of the multiples pH_k of the Hopf bundle H_k over quaternionic projective space QP^k . This enables us immediately to obtain some new immersions of real projective spaces RP^{4k+3} in Euclidean space.

Let $v(2^a(2b+1)) = a$, $v(p, i) = v_i^{(p)}$, and let $\alpha(n)$ equal the number of 1's in the binary expansion of n . Let $P_n^m = RP^m/RP^{n-1}$ and $P_n = P_n^\infty$. bo denotes the spectrum for real connective K -theory localized at 2 [3]. $\pi_{4i-1}(P_n \wedge bo)$ is a finite cyclic group ($[3]$ or $[7]$).

THEOREM 1.1. (a) If $\text{gd}(pH_k) \leq n$, then $v(p, i) \geq v(\pi_{4i-1}(P_n \wedge bo))$ for all $i \leq k$.

(b) The converse of (a) is true if $n \geq 2k$ and $\pi_{4i-1}^s(P_n) \rightarrow \pi_{4i-1}(P_n \wedge bo)$ is injective for all $i \leq k$.

REMARK 1.2. $\text{gd}(pH_k) \geq 2k$, because if the stable class of pH_k contains a $2k$ -plane bundle, it has nonzero Euler class, since $e^2 = p_k \neq 0$.

By using the computations of [7] or [3] and [11], the observation that $v_i^{(p)}$ is the coefficient of the symplectic Pontryagin class $e_i(pH_k)$, and an argument which is presented in §4 to eliminate one extraneous obstruction, Theorem 1.1 may be restated:

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THEOREM 1.3. (a) If $\text{gd}(pH_k) \leq 4m + \epsilon$ ($\epsilon = 1, 2, 3$), then for all $m + j \leq k$, $e_{m+j}(pH_k)$ is divisible by 2^{2j} if j even, $2^{2j+2-\epsilon}$ if j odd.

(b) The converse is true if $4m + \epsilon \geq 2k$ and

$$m \geq k - \begin{cases} 3 & \text{if } \epsilon = 1, \text{ and } v(p, m) \geq 3 \text{ if } m \text{ even,} \\ 4 & \text{if } \epsilon = 2, 3, \text{ and } v(p, m + 1) \geq 3 \text{ if } m \text{ odd.} \end{cases}$$

This enables us to state the precise geometric dimension of many pH_k , for example $\text{gd}(16H_4) = \text{gd}(16H_5) = 14$ and $\text{gd}(18H_{10}) = \text{gd}(18H_{11}) = 39$. Part (a) is slightly stronger than the main theorem of [14]. We conjecture that part (b) is true without the condition $m \geq k - \begin{cases} 3 \\ 4 \end{cases}$.

We obtain new immersions of $\mathbf{RP}^{4k+3}$ by combining Theorem 1.3(b) with the map $\mathbf{RP}^{4k+3} \rightarrow \mathbf{QP}^k$.

THEOREM 1.4. If $K \equiv 3 \pmod{4}$, then $\mathbf{RP}^K \subseteq \mathbf{R}^{2K-D}$, where D is given by the following table:

$\alpha(K)$	6	7	8	9	10
D	10	12	13	14	17

These results for $\alpha(K) \geq 7$ are slightly stronger than those of Milgram [12]. The first new immersion result that we obtain is $\mathbf{RP}^{191} \subseteq \mathbf{R}^{370}$. These results are not quite as good as the conjectured best possible immersion dimension:

CONJECTURE 1.5. [7]. If $K \equiv 7 \pmod{8}$ the smallest Euclidean space in which $\mathbf{RP}^K$ can be immersed as dimension

$$2K - 2\alpha(K) + \begin{cases} 0 & \text{if } \alpha(K) \equiv 0 \pmod{4}, \\ 1 & \text{if } \alpha(K) \equiv 1, 2 \pmod{4}, \\ -1 & \text{if } \alpha(K) \equiv 3 \pmod{4}. \end{cases}$$

We feel that the conjectured immersions for $\alpha(K) \leq 10$ will follow quite readily from Theorem 1.3(b) together with computations of the indeterminacies in the modified Postnikov tower [6]. As these computations are extremely detailed, they will be deferred until a later paper. The negative part of this conjecture has been announced and retracted [4], [7]. It is hoped that the methods of this paper together with those of the last sections of [5] will produce a proof of the negative part of this conjecture.

The rationale behind this conjecture is the belief that only bo -primary homotopy of V_n should obstruct multiples of the line bundle over projective spaces and that these obstructions should be given in terms of the symplectic Pontryagin classes. Some attempts have been made to show the latter directly using K -theoretic e_i -classes. In this paper we have shown that as far as divisibility by 2 is concerned, the e_i could be the obstructions.

The argument can be sketched as follows.

THEOREM 1.6. *There are bundles*

$$V_n \wedge bo \rightarrow E_n^o \rightarrow BSp, \quad V_n \wedge bu \rightarrow E_n^u \rightarrow BSp$$

such that

(i) *there are pairings of bundles*

$$E_n^o \times E_m^o \rightarrow E_{n+m}^o, \quad E_n^u \times E_m^u \rightarrow E_{n+m}^u;$$

(ii) *if $QP^k \xrightarrow{f} BSp$ classifies a stable real vector bundle θ , then $\text{gd}(\theta) \leq n$ implies f lifts to E_n^o , and if f lifts to E_n^o and $n \geq 2k$ and $\pi_{4i-1}(P_n) \rightarrow \pi_{4i-1}(P_n \wedge bo)$ is injective for all $i \leq k$, then $\text{gd}(\theta) \leq n$.*

DEFINITION 1.7. $N(p, k)$ is the smallest N such that

$$\nu(p, i) \geq \nu(\pi_{4i-1}(P_N \wedge bo)) \text{ for all } i \leq k.$$

$M(p, k)$ is the smallest M such that $\nu(p, i) \geq \nu(\pi_{4i-1}(P_M \wedge bu))$ for all $i \leq k$.

Theorem 1.1 then follows from Theorem 1.6 and

THEOREM 1.8. pH_k lifts to $E_{\max(2k, N(p, k))}^o$ but not to $E_{\max(2k, N(p, k))-1}^o$.

This is proved inductively by noting that if $p = \sum_{i=1}^r 2^{e_i}$, with all e_i distinct if p is not a power of 2, and $e_1 = e_2 = i$ if $p = 2^{i+1}$, then pH_k is classified by the composite

$$\begin{aligned} QP^k &\xrightarrow{\Delta} (QP^k \times \dots \times QP^k)^{(4k)} \\ &= \bigcup QP^{k_1} \times \dots \times QP^{k_r} \xrightarrow{\chi 2^{e_i} H} BSp \times \dots \times BSp \rightarrow BSp. \end{aligned}$$

where Δ is a skeletal map homotopic to the iterated diagonal, and the union is taken over all ordered r -tuples $\langle k_1, \dots, k_r \rangle$ such that $\sum k_i = k$. Liftings of $\chi 2^{e_i} H_{k_i}$ to $E_{M(p, k)}^u$ are found by using the liftings of the factors and the pairing of Theorem 1.6(i). These fit together to give a lifting of pH_k to $E_{M(p, k)}^u$. The results for lifting to E_N^o are obtained by using the maps $E_N^o \rightarrow E_N^u$.

2. Construction of some spaces and maps. Let $\widetilde{BSp}_n$ be the fibered product defined by the diagram

$$\begin{array}{ccc} \widetilde{BSp}_n & \longrightarrow & BO_n \\ \downarrow & & \downarrow \\ BSp & \longrightarrow & BO. \end{array}$$

We wish to form the fiberwise smash product [10] of $\widetilde{BS}p_n$ with the bo -spectrum. In order to form the fiberwise smash, the original bundle must have a section, which we obtain by applying fiberwise unreduced suspension, but then we must be careful in order to have a fiberwise map from the original bundle into the new one.

Let b_l indicate the l th space in a connective ringed Ω -spectrum b [17]. Then there is a map $S^l \rightarrow b_l$, the unit of the ring, which commutes with $\epsilon: b_l \rightarrow \Omega b_{l+1}$ and with the pairing $\mu: b_{l_1} \wedge b_{l_2} \rightarrow b_{l_1+l_2}$. Let $x_l \in \Omega^l b_l$ correspond to this map. Let $F \rightarrow E \xrightarrow{p} B$ be a fiber bundle. Form

$$\Omega_B^l((S_B^n E) \wedge_B b_l) \xrightarrow{\bar{p}_n} B$$

as in [10]. Let $P_B^n \Omega_B^l((S_B^n E) \wedge_B b_l)$ be the space of maps

$$\sigma: I^n \rightarrow \Omega_B^l((S_B^n E) \wedge_B b_l)$$

such that $\bar{p}_n \sigma$ is constant and for $\bar{s} \in I^n$, $\bar{t} \in I^l$, $\sigma(\bar{s})(\bar{t}) = [\bar{s}, p^{-1} \bar{p}_n(\sigma(I^n)), x_l(\bar{t})]$. Then there is a fiberwise map $i_E: E \rightarrow P_B^n \Omega_B^l((S_B^n E) \wedge_B b_l)$ given by $i_E(e)(\bar{s}, \bar{t}) = [\bar{s}, e, x_l(\bar{t})]$. Also, there are obvious maps

$$P_B^n \Omega_B^l((S_B^n E) \wedge_B b_l) \rightarrow P_B^n \Omega_B^{l+1}((S_B^n E) \wedge_B b_{l+1})$$

so that we can form

$$(E \wedge_B b)_n = \lim_I P_B^n \Omega_B^l((S_B^n E) \wedge_B b_l).$$

Note that

$$\begin{aligned} \text{fiber}((E \wedge_B b)_n \rightarrow B) &= \lim P^n \Omega^l(S^n F \wedge b_l) \simeq \lim \Omega^{l+n}(\Sigma^n F \wedge b_l) \\ &\simeq \lim_I \Omega^{l+n}(F \wedge b_{l+n}) = F \wedge b, \end{aligned}$$

since $\Sigma^n b_l \rightarrow b_{l+n}$ is a $(2l-n)$ -equivalence. (Here $P^n \Omega^l(S^n F \wedge b_l)$ is the set of maps $\sigma: I^n \times I^l \rightarrow S^n F \wedge b_l$ such that $\bar{s} \in I^n$ implies $\sigma(\bar{s}, \bar{t}) = [\bar{s}, F, x_l(\bar{t})]$ and $\bar{t} \in I^l$ implies $\sigma(\bar{s}, \bar{t}) = *$. The equivalence with $\Omega^{n+l}(\Sigma^n F \wedge b_l)$ is obtained by reducing the suspensions.)

Given a pairing of fiber bundles

$$\begin{array}{ccc} E_1 \times E_2 & \xrightarrow{m} & E_3 \\ \downarrow p_1 \times p_2 & & \downarrow p_3 \\ B_1 \times B_2 & \longrightarrow & B_3 \end{array}$$

we define maps

$$\bar{m}: ((S_{B_1}^{n_1} E_1) \wedge_{B_1} b_{l_1}) \times ((S_{B_2}^{n_2} E_2) \wedge_{B_2} b_{l_2}) \rightarrow ((S_{B_3}^{n_1+n_2} E_3) \wedge_{B_3} b_{l_1+l_2}),$$

$$P_{B_1}^{n_1} \Omega_{B_1}^{l_1} ((S_{B_1}^{n_1} E_1) \wedge_{B_1} b_{l_1}) \times P_{B_2}^{n_2} \Omega_{B_2}^{l_2} ((S_{B_2}^{n_2} E_2) \wedge_{B_2} b_{l_2})$$

$$\downarrow \bar{m}$$

$$P_{B_3}^{n_1+n_2} \Omega_{B_3}^{l_1+l_2} ((S_{B_3}^{n_1+n_2} E_3) \wedge_{B_3} b_{l_1+l_2})$$

by

$$\bar{m}([\bar{s}_1, e_1, y_1], [\bar{s}_2, e_2, y_2]) = [(\bar{s}_1, \bar{s}_2), m(e_1, e_2), \mu(y_1, y_2)],$$

$$\bar{m}(\sigma_1, \sigma_2)(\bar{s}_1, \bar{s}_2, \bar{t}_1, \bar{t}_2) = \bar{m}(\sigma_1(\bar{s}_1, \bar{t}_1), \sigma_2(\bar{s}_2, \bar{t}_2)).$$

Since $P_B^n \Omega_B^l((S_B^n E) \wedge_B b) \rightarrow (E \wedge_B b)_n$ is an $(l-n)$ -equivalence, by choosing l_1 and l_2 large, $\bar{m}$ may be regarded as defining a bundle pairing

$$(E_1 \wedge_{B_1} b)_{n_1} \times (E_2 \wedge_{B_2} b)_{n_2} \rightarrow (E_3 \wedge_{B_3} b)_{n_1+n_2}$$

through any finite skeleta, which is compatible with m , since $i_{E_3} m = \bar{m}(i_{E_1} \times i_{E_2})$.

Let

$$E_n^o = (\widetilde{BSp}_n \wedge_{BSp} bo)_n \quad \text{and} \quad E_n^u = (\widetilde{BSp}_n \wedge_{BSp} bu)_n.$$

Thus, there are commutative diagrams of fiber bundles:

$$\begin{array}{ccc} V_n & \longrightarrow & V_n \wedge bo \\ \downarrow & & \downarrow \\ \widetilde{BSp}_n & \longrightarrow & E_n^o \\ \downarrow & & \downarrow \\ BSp & & BSp \end{array} \quad \begin{array}{ccc} V_n & \longrightarrow & V_n \wedge bu \\ \downarrow & & \downarrow \\ \widetilde{BSp}_n & \longrightarrow & E_n^u \\ \downarrow & & \downarrow \\ BSp & & BSp \end{array}$$

where $V_n = \lim_k V_{n+k, k}$. The pairings of Theorem 1.6(i) now follow by the construction of the previous paragraph, since the Whitney sum pairing $BO_n \times BO_m \rightarrow BO_{n+m}$ induces a map $\widetilde{BSp}_n \times \widetilde{BSp}_m \rightarrow \widetilde{BSp}_{n+m}$. Theorem 1.6(ii) follows easily since $\text{gd}(\theta) \leq n$ if and only if f lifts to $\widetilde{BSp}_n$. There is a well-known map [9] $P_n \rightarrow V_n$ which is a $2n$ -equivalence. Thus in the stable range the only

obstructions to lifting f from E_n^o to $\widetilde{BSp}_n$ are due to the non- bo -primary homotopy of P_n in dimensions $4i - 1$. Note that for n odd all the homotopy of P_n is 2-primary, while for n even and p an odd prime, the p -primary homotopy of P_n cannot cause an obstruction by naturality from the lifting problem for E_{n-1}^o .

THEOREM 2.1. *For $\Delta \leq 4$ there are equivalences of skeleta*

$$(E_n^o/E_{n-\Delta}^o)^{(2(n-\Delta)+1)} \rightarrow \bigvee_I (\Sigma^{4|I|+1} P_{n-\Delta}^{n-1} \wedge bo)^{(2(n-\Delta)+1)},$$

$$(E_n^u/E_{n-\Delta}^u)^{(2(n-\Delta)+1)} \rightarrow \bigvee_I (\Sigma^{4|I|+1} P_{n-\Delta}^{n-1} \wedge bu)^{(2(n-\Delta)+1)}$$

where I ranges over $\{0\}$ and all sets of positive integers and $|I|$ is the sum of all elements of I .

REMARK. This theorem seems to be true for arbitrary Δ but we shall just need it for $\Delta \leq 4$.

PROOF. As noted by Moore [13], there is a spectral sequence converging to $H^*(\widetilde{BSp}_n, \widetilde{BSp}_{n-\Delta}; \mathbb{Z}_2)$ with $E_2^{p,q} = H^p(BSp; H^q(V_n, V_{n-\Delta}))$, which in the stable range is $H^p(BSp; H^q(\Sigma P_{n-\Delta}^{n-1}))$. The map $(\widetilde{BSp}_n, \widetilde{BSp}_{n-\Delta}) \xrightarrow{f} (BO_n, BO_{n-\Delta})$ induces a map of spectral sequences. Since there are no differentials in the spectral sequence of $(BO_n, BO_{n-\Delta})$ in the stable range, there are none in that of $(\widetilde{BSp}_n, \widetilde{BSp}_{n-\Delta})$. Moreover, it follows from naturality of the external cup product that in the stable range $H^*(\widetilde{BSp}_n, \widetilde{BSp}_{n-\Delta}) \approx H^*(BSp) \otimes H^*(\Sigma P_{n-\Delta}^{n-1})$ as modules over the subalgebra A_1 of the mod 2 Steenrod algebra generated by Sq^1 and Sq^2 . Since $\Delta \leq 4$ each $H^q(\widetilde{BSp}_n, \widetilde{BSp}_{n-\Delta})$, $q \leq 2(n-\Delta)+1$, contains elements of only one filtration, so there is no ambiguity in writing an element as an element of $H^*(BSp) \otimes H^*(\Sigma P_{n-\Delta}^{n-1})$. Then, for example, in $H^*(\widetilde{BSp}_{4m+5}, \widetilde{BSp}_{4m+1})$,

$$Sq^2(e_1 \otimes sa_{4m+3}) = Sq^2(f^*(w_4 \cup w_{4m+4})) = f^*((w_6 + w_2 w_4) \cup w_{4m+4})$$

$$= \bar{f}^*(w_6 + w_2 w_4) \cup f^*(w_{4m+4}) = 0,$$

where $\bar{f}: \widetilde{BSp}_n \rightarrow BO_n$.

The E_2 -term of the Adams spectral sequence for $\pi_*((\widetilde{BSp}_n/\widetilde{BSp}_{n-\Delta}) \wedge bo)$ is

$$\text{Ext}_{A_1}^{s,t}(H^*(\widetilde{BSp}_n, \widetilde{BSp}_{n-\Delta}), \mathbb{Z}_2) \approx \text{Ext}_{A_1}^{s,t}(H^*(\bigvee \Sigma^{4|I|+1} P_{n-\Delta}^{n-1}), \mathbb{Z}_2).$$

These groups can be read off from [3]. There can be no differentials by naturality, since there are no possible differentials in the Adams spectral sequence for $\widetilde{BSp}_{n-\epsilon+1}/\widetilde{BSp}_{n-\epsilon} \wedge bo$, $1 \leq \epsilon \leq \Delta$.

We can use our determination of $\pi_*(\widetilde{BSp}_n/\widetilde{BSp}_{n-\Delta} \wedge bo)$ to map

$$\vee \Sigma^{4|I|+1} P_{n-\Delta}^{n-1} \xrightarrow{\vee f_I} \widetilde{BSp}_n/\widetilde{BSp}_{n-\Delta} \wedge bo$$

by using the Barratt-Puppe sequence to extend the map cell-by-cell. For example, consider the case $\widetilde{BSp}_{4m+5}/\widetilde{BSp}_{4m+1} \cdot \pi_q(\widetilde{BSp}_{4m+5}/\widetilde{BSp}_{4m+1} \wedge bo)$ ($q \leq 4m+14$) can be pictured as below, where dots indicate nontrivial classes and vertical lines indicate multiplication by 2. (See [3].)

Thus for each I there is a map

$$S^{4|I|+4m+2} \vee S^{4|I|+4m+4} \xrightarrow{f_I} \widetilde{BSp}_{4m+5}/\widetilde{BSp}_{4m+1} \wedge bo$$

such that $f_I^*(e_I \otimes sa_n \otimes 1) = g_{4|I|+r}$ for $r = 4m+2, 4$. $2f_I|S^{4|I|+4m+2} \simeq 0$ so that f_I extends over $S^{4|I|+4m+2} \cup_2 e^{4|I|+4m+3} \vee S^{4|I|+4m+4}$. The map

$$S^{4|I|+4m+4} \xrightarrow{\bar{\eta}} S^{4|I|+4m+2} \cup_2 e^{4|I|+4m+3}$$

generates $\pi_{4|I|+4m+4}(S^{4|I|+4m+2} \cup_2 e^{4|I|+4m+3}) \approx \mathbb{Z}_2$. $f_I \bar{\eta}$ is thus twice the generator of the appropriate summand of $\pi_{4|I|+4m+4}(\widetilde{BSp}_{4m+5}/\widetilde{BSp}_{4m+1} \wedge bo)$ and hence equals $2f_I|S^{4|I|+4m+4}$. Thus f_I extends over

$$S^{4|I|+4m+2} \cup_2 e^{4|I|+4m+3} \vee S^{4|I|+4m+4} \cup_{2, \bar{\eta}} e^{4|I|+4m+5} \approx \Sigma^{4|I|+1} P_{4m+1}^{4m+4}.$$

The composition

$$\begin{aligned} \vee \Sigma^{4|I|+1} P_{4m+1}^{4m+4} \wedge bo &\xrightarrow{\vee f_I \wedge I} \widetilde{BSp}_{4m+5}/\widetilde{BSp}_{4m+1} \wedge bo \wedge bo \\ &\xrightarrow{1 \wedge \mu} \widetilde{BSp}_{4m+5}/\widetilde{BSp}_{4m+1} \wedge bo \end{aligned}$$

induces an isomorphism in $\mathbb{Z}_2$ -cohomology through the stable range, and since bo is localized at 2, it is a $(8m + 3)$ -equivalence. But

$$E_{4m+5}^o/E_{4m+1}^o \approx \widetilde{BSp}_{4m+5}/\widetilde{BSp}_{4m+1} \wedge bo.$$

A similar argument works when bo is replaced by bu .

THEOREM 2.2. (i) For $\Delta \leq 4$ there are maps $(E_n^o)^{(2(n-\Delta)+1)} \rightarrow \Sigma P_{n-\Delta}^{n-1} \wedge bo$ whose fiber has the same $2(n-\Delta)$ -type as $E_{n-\Delta}^o$, and similarly for E_n^u .

(ii) The following diagram is homotopy commutative, where the horizontal maps are induced by complexification $bo \rightarrow bu$.

$$\begin{array}{ccc} (E_n^o)^{(2(n-\Delta)+1)} & \longrightarrow & (E_n^u)^{(2(n-\Delta)+1)} \\ \downarrow & & \downarrow \\ \Sigma P_{n-\Delta}^{n-1} \wedge bo & \longrightarrow & \Sigma P_{n-\Delta}^{n-1} \wedge bu \end{array}$$

(iii) For $\Delta = 1$, the maps $(E_n^u)^{2n-1} \xrightarrow{h_n} S^n \wedge bu$ can be chosen so that the diagrams

$$\begin{array}{ccc} (E_{2n}^u)^{4n-1} \times (E_{2m}^u)^{4m-1} & \longrightarrow & (E_{2(n+m)}^u)^{4n+4m-1} \\ \downarrow h_{2n} \times h_{2m} & & \downarrow h_{2(n+m)} \\ S^{2n} \wedge bu \wedge S^{2m} \wedge bu & \longrightarrow & S^{2n+2m} \wedge bu \end{array}$$

are homotopy commutative.

PROOF. Let E be the fiber of the composite

$$E_n^{o(2(n-\Delta)+1)} \rightarrow E_n^o/E_{n-\Delta}^{o(2(n-\Delta)+1)} \xrightarrow{h} \bigvee \Sigma^{4l+1} P_{n-\Delta}^{n-1} \rightarrow \Sigma P_{n-\Delta}^{n-1} \wedge bo,$$

where h is a homotopy inverse of the map constructed in Theorem 2.1. Through dimension $2(n-\Delta)$ there is a commutative diagram of fibrations:

$$\begin{array}{ccc} P_{n-\Delta}^{n-1} \wedge bo & \longrightarrow & P_{n-\Delta}^{n-1} \wedge bu \\ \downarrow & & \downarrow \\ E_{n-\Delta}^o & \longrightarrow & E \\ & \searrow & \swarrow \\ & E_n^o & \end{array}$$

In both fibrations the fundamental class $a_{n-\Delta} \otimes 1$ transgresses nontrivially, and since it is the only nonzero cohomology class in its dimension it must be mapped to itself. But all classes in $H^*(P_{n-\Delta}^{n-1} \wedge bo; \mathbb{Z}_2)$ are related to one another by the action of the Steenrod algebra, and hence the map of fibers must be a $2(n-\Delta)$ -equivalence. Thus $E_{n-\Delta}^o \rightarrow E$ is a $2(n-\Delta)$ -equivalence.

Part (ii) follows by functoriality of complexification; i.e., a map

$$\widetilde{BSp}_n / \widetilde{BSp}_{n-\Delta} \wedge bo \rightarrow \Sigma P_{n-\Delta}^{n-1} \wedge bo$$

induces a compatible map

$$\widetilde{BSp}_n / \widetilde{BSp}_{n-\Delta} \wedge bu \rightarrow \Sigma P_{n-\Delta}^{n-1} \wedge bu,$$

which can be thought of as smashing with the identity map of $S^o \cup_\eta e^2$.

(iii) It suffices to find elements $x_{2n} \in K_\cup^o(\widetilde{BSp}_{2n})$ satisfying

(a) the restriction to $K_\cup^o(\widetilde{BSp}_{2n-1})$ of x_{2n} is zero,

(b) $m^*(x_{2(n+m)}) = x_{2n} \otimes x_{2m}$, where m is the natural pairing $\widetilde{BSp}_{2n} \times \widetilde{BSp}_{2m} \rightarrow \widetilde{BSp}_{2n+2m}$,

(c) x_{2n} has filtration $2n$ and is represented by a map $\widetilde{BSp}_{2n} \rightarrow S^{2n} \wedge bu$ which is nontrivial in $\mathbb{Z}_2$ -cohomology.

We then let h_n be the composite

$$E_{2n}^u \rightarrow \widetilde{BSp}_{2n} \wedge bu \xrightarrow{x_{2n} \wedge 1} S^{2n} \wedge bu \wedge bu \xrightarrow{1 \wedge \mu} S^{2n} \wedge bu.$$

The elements $y_{2n} \in K_\cup^o(B \text{Spin}_{2n})$ corresponding to the representation $\Delta_{2n}^+ - \Delta_{2n}^-$ [8] under the isomorphism $K_\cup^o(B \text{Spin}_{2n}) \approx R(\text{Spin}_{2n})^\wedge$ [2] satisfy the analogues of (a), (b), (c), so we choose $x_{2n} = g^* y_{2n}$, where g is the map $\widetilde{BSp}_{2n} \rightarrow B \text{Spin}_{2n}$ obtained by lifting the map $\widetilde{BSp}_{2n} \rightarrow BO_{2n}$.

THEOREM 2.3. For $b = bo$ or bu and any space X ,

$$\left[\bigvee_{i=1}^r QP^{k_i}, X \wedge b \right] \approx \bigoplus_{i_i \leq k_i} \pi_{4\Sigma i_i}(X \wedge b).$$

PROOF. Let $D_q(\bigvee QP^{k_i})$ be a Spanier-Whitehead dual [16]. Then by [17]

$$\left[\bigvee QP^{k_i}, X \wedge b \right] \approx \pi_q \left(D \left(\bigvee QP^{k_i} \right) \wedge X \wedge b \right).$$

Since $H^q(D(\bigvee QP^{k_i}); \mathbb{Z}_2)$ is nonzero only every fourth dimension, as in the proof of Theorem 2.1, $D(\bigvee QP^{k_i}) \wedge b \approx \bigvee S^{q-4\Sigma i_i} \wedge b$. Thus

$$\pi_q \left(D \left(\bigvee QP^{k_i} \right) \wedge X \wedge b \right) \approx \pi_q \left(\bigvee S^{q-4\Sigma i_i} \wedge X \wedge b \right) \approx \bigoplus \pi_{4\Sigma i_i}(X \wedge b).$$

THEOREM 2.4. *There exists a generator $e_L(X, b)$ of the $\pi_{4\Sigma l_i}(X \wedge b)$ -component of $[\bigvee QP^{k_i}, X \wedge b]$ for $b = bu$, $X = S^{2n}$, ΣP_{2n-1}^{2n} , or ΣP_{2n-1}^{2n+1} and for $b = bo$, $X = \Sigma P_n^{n+\Delta}$, $\Delta \leq 4$, $n + \Delta \not\equiv 3 \pmod{4}$ if $\Delta > 0$ satisfying:*

- (i) *they are natural with respect to inclusions $\bigvee QP^{k_i} \rightarrow \bigvee QP^{k_i}$;*
- (ii) *they are natural with respect to maps $X \rightarrow X'$ which are inclusions or collapsings and with respect to complexification $bo \rightarrow bu$ in the sense that if $\pi_{4i}(X \wedge b) \rightarrow \pi_{4i}(X' \wedge b')$ sends generator to 2^e -generator, then $[\bigvee QP^{k_i}, X \wedge b] \rightarrow [\bigvee QP^{k_i}, X' \wedge b']$ sends $e_L(X, b)$ to $2^e e_L(X', b')$;*
- (iii) *if $\Delta: QP^k \rightarrow (QP^k \times \cdots \times QP^k)^{(4k)}$ is a skeletal map homotopic to the diagonal, then*

$$\Delta^*: [(QP^k \times \cdots \times QP^k)^{(4k)}, X \wedge bu] \rightarrow [QP^k, X \wedge bu]$$

satisfies $\Delta^(e_L) = e_{|L|}$, where $|L| = \Sigma l_i$ if $L = \langle l_i \rangle$;*

- (iv) *the natural pairing $\bigotimes [QP^{k_i}, S^{2n_i} \wedge bu] \rightarrow [\bigvee QP^{k_i}, S^{\Sigma 2n_i} \wedge bu]$ sends $e_{i_1} \otimes \cdots \otimes e_{i_r} \mapsto e_{i_1, \dots, i_r}$.*

PROOF. As in [1, Lemma 2.5] $k\hat{\alpha}(\bigvee_{i=1}^r QP^{k_i})$ is the truncated polynomial algebra $k\hat{\alpha}(pr) [\alpha_1, \dots, \alpha_r] / \alpha_i^{k_i+1} = 0$, where $\alpha_i \in k\hat{\alpha}(\bigvee QP^{k_i})$, and similarly for $k\hat{u}$. Thus

$$\left[\bigvee_{i=1}^r QP^{k_i}, S^n \wedge b \right] = k^n \left(\bigvee QP^{k_i} \right) \approx \bigoplus_{l_i \leq k_i} \alpha^{l_1} \cdots \alpha^{l_r} \cdot \pi_{4\Sigma l_i}(S^n \wedge b).$$

Let $e_L(S^n, b) \in [\bigvee QP^{k_i}, S^n \wedge b]$ correspond to $\alpha^{l_1} \cdots \alpha^{l_r}$ times the canonical generator of $\pi_{4\Sigma l_i}(S^n \wedge b)$. If

$$\pi_{4|L|}(S^n \wedge b) \xrightarrow{i_{\#}} \pi_{4|L|}(\Sigma P_{n-1}^{n-1+\Delta} \wedge b)$$

is surjective, let $e_L(\Sigma P_{n-1}^{n-1+\Delta}, b) = i_{\#}(e_L(S^n, b))$. This takes care of all $b = bu$ cases and some bo -cases. For other cases $e_L(\Sigma P_{n-1}^{n-1+\Delta}, bo)$ can be chosen to satisfy (ii) with respect to complexification and an appropriate collapsing map $P_{n-1}^{n-1+\Delta} \rightarrow P_{n-1+\epsilon}^{n-1+\Delta}$. Thus we have defined e_L which satisfy (ii). They clearly satisfy (i) and (iv). (iii) is clearly true for $e_L(S^{2n}, bu)$ and follows for other $e_L(X, bu)$ by naturality. Indeed,

$$\begin{aligned} \Delta^*(e_L(X, bu)) &= \Delta^* i_{\#}(e_L(S, bu)) = i_{\#} \Delta^*(e_L(S, bu)) \\ &= i_{\#}(e_{|L|}(S, bu)) = e_{|L|}(X, bu). \end{aligned}$$

DEFINITION 2.5. A class $[f] \in [\bigvee QP^{k_i}, X \wedge b]$ can be written uniquely as $\Sigma n_L e_L(X, b)$. We will call $n_{\langle k_i \rangle}$ the *top component* of $[f]$, or the *top obstruction* if $f: \bigvee QP^{k_i} \rightarrow E_n \rightarrow \Sigma P_{n-\Delta}^{n-1} \wedge b$.

COROLLARY 2.6. (i) Let k, r , and $Y = QP^{l_1} \times \cdots \times QP^{l_s}$ be fixed. Suppose that for all $K = \langle k_i: i = 1, \dots, r \rangle$ such that $\sum k_i = k$ we have compatible maps

$$f_K: QP^{k_1} \times \cdots \times QP^{k_r} \times Y \longrightarrow X \wedge bu.$$

These induce a map

$$QP^k \times Y \xrightarrow{\Delta} QP^{k_1} \times \cdots \times QP^{k_r} \times Y \xrightarrow{f} X \wedge bu$$

such that the top component of $[f\Delta]$ equals the sum of the top components of the $[f_K]$.

(ii) If $f_i: QP^{k_i} \rightarrow S^{2n_i} \wedge bu$ and $f = \bigvee f_i: \bigvee QP^{k_i} \rightarrow S^{2\sum n_i} \wedge bu$, then the top component of $[f]$ is the product of the top components of the $[f_i]$.

3. Proof of Theorem 1.8. It will be convenient to first prove

THEOREM 3.1. pH_k lifts to E_M^u , where $M = \max(2k, M(p, k))$ and $M(p, k)$ is defined in Definition 1.7. The class in $\bigoplus_{i=1}^k \pi_{4i}(\Sigma P_{M-1}^M \wedge bu)$ corresponding to the map $QP^k \xrightarrow{pH} E_{M+1}^u \rightarrow \Sigma P_{M-1}^M \wedge bu$ under the isomorphism of Theorem 2.3 and Definition 2.5 has a nonzero π_{4i} -component if and only if $\nu(p, i) < \nu(\pi_{4i-1}(P_{M-1} \wedge bu))$.

As outlined in the Introduction, Theorem 3.1 is proved by writing pH_k as a Whitney sum. The combinatorial result which makes the induction work is the following theorem, whose proof is given in §4.

THEOREM 3.2. If e_i are distinct and $0 \leq k_i \leq 2^{e_i}$ and $\sum k_i = k$, then $\sum \nu(2^{e_i}, k_i) \geq \nu(\Sigma 2^{e_i}, k)$. For fixed $\langle e_i \rangle$ and k there are an odd number of $\langle k_i \rangle$ as above for which equality is obtained.

Because we can only conclude the liftings of the factors in the stable range, the induction is made somewhat more complicated, requiring the following definitions.

DEFINITION 3.3. A triple $(2^e, k_1, k_2)$ with $0 \leq k_i \leq 2^e$ is *special* if exactly one of the k_i equals 2^e or exactly one of the k_i satisfies, for all $l \leq k_i$, $2l + \nu(l) < e$.

DEFINITION 3.4. Let $S = \{(2^{e_i}, k_i)\}$ be a finite set of ordered pairs such that $0 \leq k_i \leq 2^{e_i}$. If for two of the elements of S we have $e_\alpha = e_\beta$, let

$$S' = \{(2^{e_\alpha+1}, k_\alpha + k_\beta)\} \cup \{(2^{e_i}, k_i) \in S: i \neq \alpha, \beta\},$$

and write $S \mapsto S'$ and let

$$e(S, S') = \begin{cases} 1 & \text{if } (2^{e_\alpha}, k_\alpha, k_\beta) \text{ is special,} \\ 0 & \text{if not.} \end{cases}$$

There is a (not necessarily unique) sequence $\xi: S = S_0 \mapsto S_1 \mapsto \cdots \mapsto S_n$, where S_n has distinct exponents e_i . Let

$$\gamma'(\xi) = \sum_{i=0}^{n-1} \epsilon(S_i, S_{i+1}) \quad \text{and} \quad \gamma(S) = \min_{\xi} (\gamma'(\xi)),$$

where ξ ranges over all such sequences.

LEMMA 3.5. $\nu(\Sigma 2^{e_i}, \Sigma k_i) \leq \Sigma \nu(2^{e_i}, k_i) + \gamma(\{(2^{e_i}, k_i)\})$.

PROOF. Let $\xi: S_0 \mapsto \cdots \mapsto S_n$ with $S_0 = \{(2^{e_i}, k_i)\}$ and $S_n = \{(2^{e'_i}, k'_i)\}$, e'_i distinct, be a sequence having minimal γ' . If $S = \{(2^{e''_i}, k''_i)\}$, let $\nu(S) = \Sigma \nu(2^{e''_i}, k''_i)$. Then

$$\nu(S_i) - \nu(S_{i+1}) = \nu(2^{e_i}, k_i) + \nu(2^{e_i}, k'_i) - \nu(2^{e_{i+1}}, k + k') \geq -\epsilon(S_i, S_{i+1}).$$

This follows immediately from the observations that $\nu(2^e, k) = e - \nu(k)$ if $k > 0$, and $\nu(k + k') \geq \min(\nu(k), \nu(k'))$. Thus

$$\begin{aligned} \Sigma \nu(2^{e_i}, k_i) &= \nu(S_0) \geq \nu(S_n) - \Sigma \epsilon(S_i, S_{i+1}) \\ &= \Sigma \nu(2^{e'_i}, k'_i) - \gamma(S) \geq \nu\left(\Sigma 2^{e'_i}, \Sigma k'_i\right) - \gamma(S) \\ &= \nu\left(\Sigma 2^{e_i}, \Sigma k_i\right) - \gamma(S). \end{aligned}$$

The last inequality is due to Theorem 3.2.

COROLLARY 3.6. $M(\Sigma 2^{e_i}, \Sigma k_i) \geq \Sigma M(2^{e_i}, k_i) - 2\gamma(\{(2^{e_i}, k_i)\})$.

PROOF. Since $\nu(\pi_{2i-1}(P_{2n} \wedge bu)) = \nu(\pi_{2i-1}(P_{2n+1} \wedge bu)) = i - n$, we have $M(p, k) = \max_{i \leq k, p} (4i - 2\nu(p, i))$. Choose $l_i \leq k_i$ such that $M(2^{e_i}, k_i) = 4l_i - 2\nu(2^{e_i}, l_i)$. Note $l_i = 0$ if and only if for all $l \leq k_i$, $2l + \nu(l) < e$. Then

$$\begin{aligned} \Sigma M(2^{e_i}, k_i) - 2\gamma(\{(2^{e_i}, k_i)\}) &= 4\Sigma l_i - 2\left(\Sigma \nu(2^{e_i}, l_i) + \gamma(\{(2^{e_i}, l_i)\})\right) \\ &\leq 4\Sigma l_i - 2\nu\left(\Sigma 2^{e_i}, \Sigma l_i\right) \leq M\left(\Sigma 2^{e_i}, \Sigma k_i\right). \end{aligned}$$

Theorem 3.1 follows immediately from Theorems 3.7(b), 3.8(b) and 2.4(i).

THEOREM 3.7. (a) $\times 2^{e_i} H_{k_i}$ lifts to E_M^u , where

$$M = \max \left(2\Sigma k_i, M\left(\Sigma 2^{e_i}, \Sigma k_i\right) \right) + 2\gamma(\{(2^{e_i}, k_i)\}),$$

if the product contains two or more factors.

(b) pH_k lifts to $E_{\max(2k, M(p, k))}^u$.

PROOF. It suffices to prove:

(i) If (a) is true whenever $\Sigma 2^{e_i} \leq p$ and (b) is true for pH_k , whenever $k' < k$, then (b) is true.

(ii) If (b) is true whenever $p \leq \max\{2^{e_i}\}$ and (a) is true for $\times 2^{e_i}H_{k_i}$, whenever $\Sigma 2^{e_i} = \Sigma 2^{e_i}$ and $\max\{2^{e_i} + k_i': M(2^{e_i}, k_i') < 2k_i'\} < \max\{2^{e_i} + k_i: M(2^{e_i}, k_i) < 2k_i\}$, then (a) is true.

Proof of (i) if p is not a power of 2. Write $p = \sum_{i=1}^r 2^{e_i}$ with e_i distinct. If $0 \leq k_i \leq 2^{e_i}$ and $\Sigma k_i = k$, then by hypothesis $\times 2^{e_i}H_{k_i}$ lifts to $E_{\max(2k, M(p, k))}^u$. Thus

$$\bigcup_{\langle k_i \rangle} QP^{k_1} \times \dots \times QP^{k_r} \xrightarrow{\times 2^{e_i}H} BSp$$

lifts to $E_{\max(2k, M(p, k))+1}^u$ since the difference obstruction for matching the liftings on their intersections is an element of $H^q(\times QP; \pi_q(P_{\max(2k, M(p, k))+1} \wedge bu)) = 0$, and hence the union lifts to $E_{\max(2k, M(p, k))}^u$, since there is no obstruction for lifting a space with cohomology only in even dimensions from E_{2M+1}^u to E_{2M}^u . As noted in the Introduction pH_k is classified by

$$QP^k \xrightarrow{\Delta} \bigcup QP^{k_1} \times \dots \times QP^{k_r} \rightarrow BSp$$

and hence it lifts to $E_{\max(2k, M(p, k))}^u$.

Proof of (i) if $p = 2^{i+1}$. Let $M = M(p, k)$. By hypothesis $2^iH_l \times 2^iH_{k-l}$ lifts to $E_{\max(2k, M+2)}^u$. Assume $M \geq 2k$, for otherwise we are done. As in the proof of the first case the liftings to E_{M+3}^u are compatible, and by Theorem 2.2 there is a map $E_{M+3}^u(4k) \rightarrow \Sigma P_{M+1}^{M+2} \wedge bu$ whose fiber has the $4k$ -type of E_{M+1}^u . Thus the lifting to E_{M+1}^u (and hence to E_M^u) exists if and only if the top component of the composite

$$QP^k \xrightarrow{\Delta} \bigcup QP^l \times QP^{k-l} \rightarrow E_{M+3}^u \rightarrow \Sigma P_{M+1}^{M+2} \wedge bu$$

is zero, since the lower components are zero by the induction hypothesis. By Corollary 2.6(i) this is zero if and only if the top obstruction is nonzero for an even number of $2^iH_l \times 2^iH_{k-l}$. But the top obstruction for $2^iH_l \times 2^iH_{k-l}$ equals that of $2^iH_{k-l} \times 2^iH_l$, since the map $QP^{k-l} \times QP^l \rightarrow E_{M+3}^u$ can be chosen to be the composite

$$QP^{k-l} \times QP^l \xrightarrow{T} QP^l \times QP^{k-l} \xrightarrow{g} E_{M+3}^u,$$

where g is a lifting for $2^iH_l \times 2^iH_{k-l}$. The only unpaired obstruction is that of $2^iH_{k/2} \times 2^iH_{k/2}$ if k is even, but this is zero by the hypothesis since $\gamma(\{(2^i, k/2), (2^i, k/2)\}) = 0$.

Proof of (ii). If $M(2^{e_i}, k_i) \geq 2k_i$ for all i , then by hypothesis $\times 2^{e_i} H_{k_i}$ lifts to E_S^u , where

$$S = \sum M(2^{e_i}, k_i) \leq M\left(\sum 2^{e_i}, \sum k_i\right) + 2\gamma(\{(2^{e_i}, k_i)\}) = M$$

by Corollary 3.6.

Otherwise, choose the maximal value of $2^{e_i} + k_i$ such that $M(2^{e_i}, k_i) < 2k_i$. After renumbering we may assume that $i = 1$. Then $\times 2^{e_1} H_{k_1}$ is classified by

$$\times QP^{k_1} \xrightarrow{\Delta \times 1} \bigcup_{k_0=0}^{k_1} QP^{k_0} \times QP^{k_1-k_0} \times \bigtimes_{i \geq 2} QP^{k_i} \rightarrow BSp,$$

where the second map classifies $2^{e_1-1}H \times 2^{e_1-1}H \times \bigtimes_{i \geq 2} 2^{e_i}H$. By hypothesis $2^{e_1-1}H_{k_0} \times 2^{e_1-1}H_{k_1-k_0} \times \bigtimes_{i \geq 2} 2^{e_i}H_{k_i}$ lifts to E_M^u , where

$$\begin{aligned} M &= \max \left(2 \sum k_i, M\left(\sum 2^{e_i}, \sum k_i\right) \right. \\ &\quad \left. + 2\gamma(\{(2^{e_1-1}, k_0), (2^{e_1-1}, k_1 - k_0), (2^{e_2}, k_2), \dots\}) \right) \\ &\leq \max \left(2 \sum k_i, M\left(\sum 2^{e_i}, \sum k_i\right) \right. \\ &\quad \left. + 2\gamma(\{(2^{e_1}, k_1), (2^{e_2}, k_2), \dots\}) + 2 \right) = M'. \end{aligned}$$

As in the proof of (i), all obstructions (except possibly one which has zero obstruction) occur in pairs so that the lifting of $\times 2^{e_i} H_{k_i}$ from $E_{M'+1}^u$ to $E_{M'-1}^u$ exists.

THEOREM 3.8. (a) Let $M = M(\sum 2^{e_i}, \sum k_i) + 2\gamma(\{(2^{e_i}, k_i)\})$. If $M > 2\sum k_i$, then the top obstruction for lifting $\times 2^{e_i} H_{k_i}$ to E_{M-1}^u is nonzero if and only if $M(\sum 2^{e_i}, \sum k_i) = \sum M(2^{e_i}, k_i) - 2\gamma(\{(2^{e_i}, k_i)\})$ and for all i , $M(2^{e_i}, k_i) = 4k_i - 2\nu(2^{e_i}, k_i)$.

(b) If $M(p, k) > 2k$, then the top obstruction for lifting pH_k to $E_{M(p,k)-1}^u$ is nonzero if and only if $M(p, k) = 4k - 2\nu(p, k)$.

PROOF. As in Theorem 3.7 it suffices to prove:

- (i) If (a) is true whenever $\sum 2^{e_i} \leq p$, then (b) is true.
- (ii) If (b) is true whenever $p \leq \max\{2^{e_i}\}$ and (a) is true for $\times 2^{e_i} H_{k_i}$ whenever $\sum 2^{e_i} = \sum 2^{e_i}$ and $\max\{2^{e_i} + k_i' : M(2^{e_i}, k_i') \leq 2k_i'\} < \max\{2^{e_i} + k_i : M(2^{e_i}, k_i) \leq 2k_i\}$, then (a) is true.

Proof of (i) if p is not a power of 2. Write $p = \sum 2^{e_i}$ with e_i distinct. The top obstruction for pH_k is nonzero if and only if there is an odd number of $\langle k_i \rangle$ such that $\times 2^{e_i} H_{k_i}$ has nonzero top obstruction (for lifting to $E_{M(p,k)-1}^u$). The top obstruction is certainly zero for those $\langle k_i \rangle$ for which $\sum M(2^{e_i}, k_i) < M(\sum 2^{e_i}, \sum k_i)$,

so we need consider only those $\langle k_i \rangle$ for which $\Sigma M(2^{e_i}, k_i) = M(\Sigma 2^{e_i}, \Sigma k_i)$. Then

$$\Sigma M(2^{e_i}, k_i) = M(p, k) \geq 4k - 2\nu(p, k) \geq \Sigma (4k_i - 2\nu(2^{e_i}, k_i)).$$

If $M(p, k) > 4k - 2\nu(p, k)$, then $\Sigma(M(2^{e_i}, k_i) - (4k_i - 2\nu(2^{e_i}, k_i))) > 0$, so for some i , $M(2^{e_i}, k_i) > 4k_i - 2\nu(2^{e_i}, k_i)$ and hence by hypothesis $\times 2^{e_i}H_{k_i}$ has top obstruction zero. Since this is true for all $\langle k_i \rangle$, pH_k has top obstruction zero by Corollary 2.6(i).

If $M(p, k) = 4k - 2\nu(p, k)$, then $\times 2^{e_i}H_{k_i}$ has nonzero top obstruction if and only if for all i , $M(2^{e_i}, k_i) = 4k_i - 2\nu(2^{e_i}, k_i)$ if and only if $\nu(p, k) = \Sigma \nu(2^{e_i}, k_i)$, and by Theorem 3.2 this happens for an odd number of $\langle k_i \rangle$. (Note that $\{\langle k_i \rangle: \nu(p, k) = \Sigma \nu(2^{e_i}, k_i)\} \subset \{\langle k_i \rangle: M(p, k) = \Sigma M(2^{e_i}, k_i)\}$.) Thus $\times 2^{e_i}H_{k_i}$ has nonzero top obstruction.

Proof of (i) if $p = 2^{i+1}$. Let $M = M(p, k)$. There is a commutative diagram

$$\begin{array}{ccccc} & & E_{M+1}^u & \longrightarrow & \Sigma P_{M-1}^M \wedge bu \\ & \nearrow & \downarrow & & \downarrow \\ QP^k & \xrightarrow{\Delta} & \bigcup QP^l \times QP^{k-l} & \xrightarrow{2^i H \times 2^i H} & E_{M+3}^u \\ & & & \nearrow & \downarrow \\ & & & \Sigma P_{M-1}^{M+2} \wedge bu & \downarrow \\ & & & \Sigma P_{M+1}^{M+2} \wedge bu. & \end{array}$$

By Theorem 2.4(ii) and Corollary 2.6(i), $2^{i+1}H_k$ has nonzero top obstruction in $\pi_{4k}(\Sigma P_{M-1}^M \wedge bu)$ if and only if it has nonzero top obstruction in $\pi_{4k}(\Sigma P_{M-1}^{M+2} \wedge bu)$ if and only if the sum of the top obstructions for the $2^i H_l \times 2^i H_{k-l}$ in $\pi_{4k}(\Sigma P_{M-1}^{M+2} \wedge bu)$ is nonzero. If $(2^i, l, k-l)$ is not special, then $2^i H_l \times 2^i H_{k-l}$ lifts to E_M^u and hence its top obstruction in $\pi_{4k}(\Sigma P_{M-1}^{M+2} \wedge bu)$ is divisible by 2, so that the sum of the top obstructions for $2^i H_l \times 2^i H_{k-l}$ and $2^i H_{k-l} \times 2^i H_l$ is zero. (If k is even, $2^i H_{k/2} \times 2^i H_{k/2}$ lifts to $E_{M(p,k)-1}^u$.) If $(2^i, l, k-l)$ is special but $\{l, k-l\} \cap \{0, 2^i\}$ is empty, then by the induction hypothesis $2^i H_l \times 2^i H_{k-l}$ has zero top obstruction for lifting to E_{M+1}^u , so that the sum of the obstructions in $\pi_{4k}(\Sigma P_{M-1}^{M+2} \wedge bu)$ for $2^i H_l \times 2^i H_{k-l}$ and $2^i H_{k-l} \times 2^i H_l$ is zero. The only case remaining is $2^i H_{\epsilon 2^i} \times 2^i H_{k-\epsilon 2^i}$, where $\epsilon = 0$ if $k \leq 2^i$ and $\epsilon = 1$ if $k > 2^i$. The sum of the obstructions in $\pi_{4k}(\Sigma P_{M-1}^{M+2} \wedge bu)$ from this and its transpose is zero if and only if the top obstruction for lifting $2^i H_{\epsilon 2^i} \times 2^i H_{k-\epsilon 2^i}$ to E_{M+1}^u is zero if and only if $M(2^i, k - \epsilon 2^i) > 4(k - \epsilon 2^i) - 2\nu(2^i, k - \epsilon 2^i)$ if and only if $M(2^{i+1}, k) > 4k - 2\nu(2^{i+1}, k)$.

Proof of (ii). If $M(2^{e_i}, k_i) > 2k_i$ for all i , this follows from Corollary 2.6(ii). Otherwise, the proof follows by combining the methods of Theorem 3.6(ii) and the case $p = 2^{i+1}$ above.

In order to deduce Theorem 1.8, we need the following lemma, whose proof is similar to that which will be used in Theorem 1.8.

LEMMA 3.9. For M even, if a map $QP \rightarrow BSp$ lifts to E_M^o and to E_{M-4}^u , then it lifts to E_{M-2}^o .

PROOF. In the remainder of this section we shall denote $O_k^n = \Sigma P_k^n \wedge bo$ and $U_k^n = \Sigma P_k^n \wedge bu$. Suppose $M = 4m$. Consider the following commutative diagram:

$$\begin{array}{ccc} QP^k & \xrightarrow{l} & E_{M+1}^o \longrightarrow E_{M+1}^u \\ & & \downarrow i_1 \\ & & O_{M-3}^M \xrightarrow{i_3} U_{M-3}^M \\ & & \downarrow i_2 \\ & & O_{M-2}^M \end{array}$$

The maps of homotopy groups

$$\pi_{4i}(O_{M-2}^M) \leftarrow \pi_{4i}(O_{M-3}^M) \rightarrow \pi_{4i}(U_{M-3}^M)$$

are $Z_4 \xleftarrow{g} Z_8 \xrightarrow{g} Z_4$ if $i - m$ is even, and $Z_2 \xleftarrow{g} Z_2 \xrightarrow{2g} Z_4$ if $i - m$ is odd. $[i_3 i_1 l] = 0 \in \bigoplus_{i=m}^k \pi_{4i}(U_{M-3}^M)$. Thus by Theorem 2.4(ii) $[i_1 l]$ is divisible by 4 in $\bigoplus_{i=m}^k \pi_{4i}(O_{M-3}^M)$ and hence $[i_2 i_1 l] = 0$. Thus the lifting to E_{M-2}^o exists. A similar proof works if $M = 4m + 2$.

LEMMA 3.10. (i) If $M(p, k) = 4m - 2$, then $N(p, k) = 4m$ if there is an $i \leq k$ such that $i - m$ is odd and $\nu(p, i) < \nu(\pi_{4i}(\Sigma P_{4m-3} \wedge bu))$, and otherwise $N(p, k) = 4m - 1$.

(ii) If $M(p, k) = 4m$, then $N(p, k) = 4m + 2$ if there is an $i \leq k$ such that $i - m$ is odd and $\nu(p, i) < \nu(\pi_{4i}(\Sigma P_{4m-1} \wedge bu))$, and otherwise $N(p, k) = 4m$.

PROOF. If $0 \leq \epsilon \leq 3$, then

$$\pi_{4i}(\Sigma P_{4m+\epsilon} \wedge bo) = 2(i - m) + \begin{cases} 1 & \text{if } \epsilon = 0 \text{ or } 1 \text{ and } i - m \text{ is odd,} \\ -1 & \text{if } \epsilon = 3 \text{ and } i - m \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\nu(\pi_{4i}(\Sigma P_{4m-2} \wedge bu)) \geq (\pi_{4i}(\Sigma P_{4m} \wedge bo))$ and

$$\begin{aligned} \nu(\pi_{4i}(\Sigma P_{4m-3} \wedge bu)) &= \nu(\pi_{4i}(\Sigma P_{4m-2} \wedge bo)) \\ &= \nu(\pi_{4i}(\Sigma P_{4m-1} \wedge bo)) + \begin{cases} 1 & \text{if } i - m \text{ even,} \\ 0 & \text{if } i - m \text{ odd.} \end{cases} \end{aligned}$$

(i) follows from this, and (ii) is proved similarly.

PROOF OF THEOREM 1.8. pH_k lifts to $\widetilde{BSp}_{4k}$ and hence to E_{4k}^o . Thus by using Lemma 3.9, it lifts to $E_{M(p,k)+2}^o$.

If $M(p, k) = 4m - 2$, the obstruction for lifting pH_k from E_{4m-1}^u to E_{4m-3}^u is an element of $\bigoplus_{i=m}^k \pi_{4i}(\Sigma P_{4m-3}^{4m-2} \wedge bu)$. We will show below that if this class has a nonzero component for $i - m$ odd, then pH_k does not lift to E_{4m-1}^o , and if its π_{4i} -component is zero for all i such that $i - m$ is odd, then pH_k lifts to E_{4m-1}^o but not to E_{4m-2}^o . Combining this with Theorem 3.1 and Lemma 3.10 proves Theorem 1.8 when $M(p, k) = 4m - 2$.

The claim follows from the commutative diagram:

The induced maps in π_{4i} of the O - and U -spaces are

$i - m$ even,

$i - m$ odd.

Since pH_k lifts to E_{4m-1}^u , the class in $\pi_{4i}(U_{4m-1}^{4m})$ is zero. Thus the class in $\pi_{4i}(O_{4m-1}^{4m})$ is zero if $i - m$ is even, while for $i - m$ odd the class in $\pi_{4i}(O_{4m-1}^{4m})$ is zero if and only if the class in $\pi_{4i}(U_{4m-3}^{4m-2})$ is zero. The claims about lifting to E_{4m-1}^o follow, since pH_k lifts to E_{4m-1}^o if and only if the class in $\bigoplus \pi_{4i}(O_{4m-1}^{4m})$ is zero. Since pH_k does not lift to E_{4m-1}^u , the class in $\pi_{4i}(U_{4m-3}^{4m-2})$ is nonzero for some i , and for either parity this shows the class in $\pi_{4i}(O_{4m-2}^{4m})$ is nonzero. Hence pH_k does not lift to E_{4m-2}^o .

If $M(p, k) = 4m$, the theorem is proved similarly by showing that pH_k lifts exactly to E_{4m+2}^o if the obstruction in $\pi_{4i}(U_{4m-1}^{4m})$ is nonzero for some i such that $i - m$ is odd, and otherwise it lifts exactly to E_{4m}^o . This follows as before from the diagram

where the homomorphisms of π_{4i} are

4. Proof of Theorems 1.3(b) 1.4, and 3.2. The following proposition is easily verified.

- PROPOSITION 4.1. (i) $\nu(m, n) = \alpha(n) + \alpha(m - n) - \alpha(m)$;
 (ii) $\alpha(2^i - n) = i - \alpha(n - 1)$;
 (iii) $\alpha(n - \Delta) = \alpha(n) - \Delta + \sum_{j=0}^{\Delta-1} \nu(n - j)$.

THEOREM 4.2. If $p < 2^i$, $0 \leq l \leq 2^i$, and $0 \leq k - l \leq p$, then $\nu(2^i, l) + \nu(p, k - l) \geq \nu(2^i + p, k)$. For fixed i , p , and k , there is an odd number of l for which equality is achieved.

PROOF (M. G. BARRATT). By using Proposition 4.1(i) it is immediately verified that

$$\begin{aligned}
 \nu(2^i, l) + \nu(p, k - l) &= \nu(2^i + p, k) + \nu(k, l) + \nu(2^i + p - k, 2^i - l) \\
 &\geq \nu(2^i + p, k).
 \end{aligned}$$

By comparing coefficients of x^k in $(1 + x)^{2^i}(1 + x)^p = (1 + x)^{2^i + p}$, we see that

$$\sum_l \binom{2^l}{l} \binom{p}{k-l} = \binom{2^l + p}{k},$$

and hence

$$\nu \left(\sum_l \binom{2^l}{l} \binom{p}{k-l} \right) = \nu(2^l + p, k).$$

The result then follows from the observation that $\nu(\Sigma A_l) \geq \min(\nu(A_l))$ with equality if and only if the minimum is attained for an odd number of l .

Theorem 3.2 follows immediately from Theorem 4.2 by induction on the number of e_i .

PROPOSITION 4.3 [15]. *Let ξ_k denote the Hopf bundle over $\mathbb{R}P^k$. Then*

- (i) $\text{gd}(4p\xi_{4k+3}) \leq \text{gd}(mH_k)$;
- (ii) $\mathbb{R}P^k \subseteq \mathbb{R}^{k+l}$ if and only if $\text{gd}((2^l - k - 1)\xi_k) \leq l$ for large l .

PROOF OF THEOREM 1.4. Let $K = 8k + 7$. Then it suffices to show $\text{gd}((2^l - 2k - 2)H_{2k+1}) \leq 8k + 7 - D$. $\nu(\pi_{4l-1}(P_n \wedge bo))$ is easily seen to be [3, 7] given by the table

		i			
		$2k-2$	$2k-1$	$2k$	$2k+1$
n	$8k-3$	0	0	3	4
	$8k-5$	0	1	4	5
	$8k-6$	0	2	4	6
	$8k-7$	0	3	4	7
	$8k-10$	2	4	6	8

Proposition 4.1 easily implies

$$\nu(2^l - 2k - 2, i) = \begin{cases} \alpha(k) - 1 & \text{if } i = 2k - 2, \\ \alpha(k) + \nu(k) + 1 & \text{if } i = 2k - 1, \\ \alpha(k) & \text{if } i = 2k, \\ \alpha(k) + 1 & \text{if } i = 2k + 1. \end{cases}$$

One can now verify that

$$\nu(2^l - 2k - 2, i) \geq \nu(\pi_{4l-1}(P_{8k+7-D} \wedge bo)) \quad \text{for all } i \leq 2k + 1.$$

The condition of Theorem 1.3(b) is easily verified. A similar argument can be made when $K = 8k + 3$.

PROOF OF THEOREM 1.3(b). By observing the tables of [11] and [7], we see that the condition of Theorem 1.1(b) is satisfied for $m \geq k - \binom{3}{4}$ unless m

even, $\epsilon = 1$ or m odd, $\epsilon = 2$ or 3. In these cases we note there is a class h_2^2 at height 2 in dimension

$$n + \begin{cases} 6 \\ 9 \\ 8 \end{cases} \text{ if } n \equiv \begin{cases} 1 (8) \\ 6 (8), \\ 7 (8) \end{cases}$$

which is not *bo*-primary. (Here we use n to correspond to k in [11] and to our $4m + \epsilon$.) This class is not present in the homotopy of $P_{n-\delta}$, where

$$\delta = \begin{cases} 4 \\ 1 \\ 2 \end{cases} \text{ if } n \equiv \begin{cases} 1 \\ 6 \\ 7 \end{cases}.$$

We consider the maps of modified Postnikov towers [6] induced by the maps of fibrations

$$\begin{array}{ccccc} V_{n-\delta} & \longrightarrow & V_n & \longrightarrow & V_n \wedge bo \\ \searrow & & \searrow & & \searrow \\ \widetilde{BSp}_{n-\delta} & \longrightarrow & \widetilde{BSp}_n & \longrightarrow & E_n^o \\ \downarrow & & \downarrow & & \downarrow \\ BSp & \equiv & BSp & \equiv & BSp. \end{array}$$

Let E_i , E'_i , E''_i indicate the spaces obtained in modified Postnikov towers through the dimension $4k$ of QP^k ; this will be $\leq$

$$\begin{array}{c} n + \begin{cases} 11 & 1 \\ 14 & \text{if } n \equiv 6. \\ 13 & 7 \end{cases} \\ \begin{array}{ccccc} \widetilde{BSp}_{n-\delta} & \longrightarrow & \widetilde{BSp}_n & \longrightarrow & E_n^o \\ \vdots & & \vdots & & \vdots \\ E_3 & \longrightarrow & E'_3 & \longrightarrow & E''_3 \\ \downarrow & & \downarrow & & \downarrow \\ E_2 & \longrightarrow & E'_2 & \longrightarrow & E''_2 \\ \downarrow & & \downarrow & & \downarrow \\ E_1 & \longrightarrow & E'_1 & \longrightarrow & E''_1 \\ \downarrow & & \downarrow & & \downarrow \\ QP^k & \longrightarrow & BSp & \equiv & BSp & \equiv & BSp \end{array} \end{array}$$

By the assumptions on $\nu(p, i)$, QP^k lifts to E_3 and to E_n^o . But all k -invariants of E_p' , $i \geq 3$, come from those of E_i'' and hence map to zero in QP^k . Thus QP^k lifts to $\overline{BS}p_n$.

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