STABLE THICKENINGS IN THE HOMOTOPY CATEGORY([1])

BY

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ABSTRACT. This paper extends the result that the set of stable thickenings of a simply-connected complex K are in 1-1 correspondence with $[K, BQ]$ (Q = O, PL, or TOP). which holds in the smooth, PL, and topological categories, to the homotopy category.

In this paper we consider the notion of thickening (introduced by Mazur and Wall [14]) in the homotopy category. The main result is the following:

Theorem 1. *Let K be a finite, pointed, 2-connected Poincaré complex. Then there is a natural map $\tau\colon \mathcal{T}(K) \to [K, BG]$ which is a bijection.*

Here $\mathcal{T}(K)$ denotes the set of stable thickenings of K, and $BG = \lim BG_n$, where G_n is the group of homotopy equivalences of S^{n-1}. As is well known [12], $[K, BG]$ is in 1-1 correspondence with stable fibre-homotopy equivalence classes of (stable) spherical fibre spaces over K.

A theorem of this kind has been established in the differentiable category by Mazur, in the PL category by Wall [14] and in the topological category by the author [5].

It is probable that a theorem of this kind holds without the connectivity hypothesis. Such a theorem would follow, for instance, if we had a nonsimply connected version of Browder's theorem [2, I.4.1] that the inclusion $A \subset X$ of a Poincaré pair (X, A) is a spherical fibration.([3])

1. **Preliminaries.** We work in the homotopy category of pointed topological spaces and homotopy classes of base-point preserving maps.

Let K be a finite, simply-connected CW complex of dimension k with base-point $*$, and $(M, \partial M)$ a Poincaré pair of dimension m, and base-point $* \in \partial M$.([4]) Then a (Poincaré complex-, or homotopy-) *prethickening* of K will be a base-point

preserving homotopy equivalence $\varphi: K \to (M, \partial M)$, where $m \geq k + 3$. We will denote the codimension of K in M by $n = m - k$.

Two prethickenings (M_1, φ_1) and (M_2, φ_2) will be said to be *equivalent* if there is a homotopy equivalence of pairs $h: (M_1, \partial M_1) \to (M_2, \partial M_2)$ such that the following diagram commutes:

$$
\begin{array}{ccc}
 & & (M_1, \partial M_1) \\
 & \nearrow^{\varphi_1} & \\
K & & \downarrow h \\
 & \searrow_{\varphi_2} & \\
 & & (M_2, \partial M_2)
\end{array}
$$
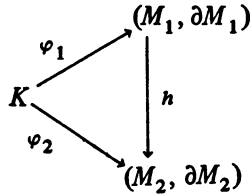
A Poincaré complex, or homotopy m-thickening, or briefly, a thickening, of K will then be an equivalence class of prethickenings. In this paper we work only in the homotopy category and all thickenings will be Poincaré complex or homotopy thickenings unless otherwise stated.

This definition is slightly different from Wall's definition for the smooth or PL cases as we have imposed the requirement of simple connectivity. This is made necessary by the lack of a suitable embedding theorem for non-simply-connected Poincaré complexes.

We now consider splittings and embeddings of Poincaré complexes. A *splitting* of a Poincaré complex M^n is a pair of Poincaré complexes $(N_1^n, \partial N_1^n)$, $(N_2^n, \partial N_2^n)$ such that $\partial N_1 = \partial N_2 = N_1 \cap N_2$ and a homotopy equivalence $\varphi: M \to N_1 \cup N_2$. This idea extends in the obvious way to splittings of Poincaré pairs.

Following [8], let K^k be a CW complex and M^n a Poincaré complex, and let $f: K^k \to M^n$. Then f is said to be *homotopic to an embedding* if there is a splitting $\{(N_1^n, \partial N_1^n), (N_2^n, \partial N_2^n), \varphi\}$ and a homotopy equivalence $h: K^k \to N_1^n$ such that

$$
\begin{array}{ccc}
K^k & \longrightarrow & M^n \\
 & \searrow_{h} \quad \nearrow_{\varphi} & \\
 & N_1^n \subseteq N_1^n \cup N_2^n &
\end{array}
$$
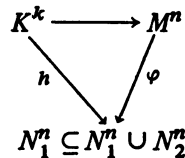
is homotopy commutative, and such that the inclusion $\partial N_1^n \subseteq N_1^n$ is $(n - k - 1)$-connected. For details, see [9, §2], or [15, Chapter 11]. Levitt [8, 3.1] gives the following embedding theorem, with the usual restrictions on Poincaré complexes (which we shall omit, preferring, as in [10], to regard them as part of the definition):

Theorem 2 (Levitt). *Let $(M^n, \partial M^n)$ be a simply-connected Poincaré pair, K^k a CW complex, $n \geq 2k + 2$, and $f: K^k \to M^n$. Then f is homotopic to an embedding.*

By using this theorem, we can construct an induced map on thickenings:

Proposition 3. *Let K^k, L^l be finite, pointed, simply-connected CW complexes and*

$f\colon K^k \to L^l$ a map. If $m \geq 2k2$, $2l + 2$, then f induces $f^*\colon \mathcal{T}^m(L^l) \to \mathcal{T}^m(K^k)$.

Proof. Let (M,φ) be a thickening of L; then $\varphi f\colon K \to M$. Applying Levitt's theorem to φf, we obtain as part of the resulting splitting of $(M^n, \partial M^n)$ a Poincaré complex $(N_1, \partial N_1)$ and a homotopy equivalence $h\colon K \to (N_1, \partial N_1)$ so that the following diagram is homotopy commutative:

$$
\begin{array}{ccc}
K^k & \xrightarrow{\quad f \quad} & L^l \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \varphi} \\
N^n \subseteq N_1^n \cup N_2^n & \xleftarrow{\quad \sim \quad} & M^m
\end{array}
$$

Then (N_1, h) is the required thickening of K.

Although the preceding construction requires a severe restriction on the dimension of M, this will not cause us any difficulty since we will be working in the stable range.

Now let $f\colon K^k \to *$. f induces $f^*\colon \mathcal{T}^m(*) \to \mathcal{T}^n(K)$. $D^m \in \mathcal{T}^m(*)$; by the trivial m-thickening we shall mean $f^*(D^m)$. We denote this thickening by K_0.

Note that if $f\colon K^k \to L^l$ is a homotopy equivalence and $m \geq 2l + 2$, $2m + 2$, then $f^*\colon \mathcal{T}^m(L) \to \mathcal{T}^l(K^l)$ is a bijection. In particular $f^*\colon \mathcal{T}^m(K^k) \to \mathcal{T}^m(K_0)$ for m sufficiently large.

Stable thickenings. As usual we have the suspension map

$$
\cdots \to \mathcal{T}^m(K) \xrightarrow{\Sigma^m} \mathcal{T}^{m+1}(K) \to \cdots.
$$

By Levitt [8], Σ^m is an isomorphism for $m \geq k + 2$. We let $\mathcal{T}(K)$ denote the common stable value; $\mathcal{T}(K)$ will be called the *set of stable homotopy thickenings* of K.

Note that in establishing the existence of $\mathcal{T}(K)$, Levitt's result enables us to dispense with a homotopy analogue of Wall's suspension sequence [1, §5]. No such analogue exists at the present time and it would be interesting to determine if one does, indeed, exist.

2. Proof of Theorem 1. We now define maps Ψ, Φ as follows, with

$$
\mathcal{T}(K) \underset{\Phi}{\overset{\Psi}{\rightleftarrows}} [K, BG].
$$

To define Ψ, let (M^m, γ) be a prethickening of K. As $(M^m, \partial M^m)$ is a Poincaré pair, there is a unique (up to stable fibre-homotopy equivalence) Spivak normal fibration ν and hence a classifying map $c_\nu\colon M \to BG$, unique up to homotopy. Furthermore, as is well known, $[K, BG]$ is a group, so inverses are defined, and $-c_\nu = c_{-\nu}$. Define $\Psi([M, \gamma])$ to be the homotopy class of the map $c_{-\nu}\gamma \in [K, BG]$. Ψ is well defined since varying (M, γ) by an equivalence only varies $\gamma c_{-\nu}$ by a homotopy, so that Ψ is well defined.

To define Φ, let (K_0, α) be a trivial thickening of K and β a homotopy inverse

for α. Let $f = f_\xi \in [f_\xi] \in [K, BG]$ be associated to the spherical fibration ξ over K. Form the associated "disc fibration" $\bar{\xi}$ over K by "coning each fibre" [1, Appendix]. This is a construction analogous to forming the disc bundle associated to a sphere bundle. Let μ be the zero section; $(E_{\bar{\xi}}, \mu)$ is then a thickening of K, and we define $\Phi([f_\xi]) = [(E_{\bar{\xi}}, \mu)]$. Note that using β we also obtain a thickening of K_0 by forming $(E', \mu') = \beta^*(E_{\bar{\xi}}, \mu)$. This will figure in the sequel. Φ is well defined, since if we vary f_ξ by a homotopy, $\bar{\xi}$ varies by a fibre homotopy equivalence mod boundaries and hence we obtain equivalent thickenings.

Note that since K and K_0 have the same homotopy type, two fibrations η_1, η_2 are fibre-homotopy equivalent over K iff their induced fibrations $\beta^*\eta_1$, $\beta^*\eta_2$ are fibre-homotopy equivalent over K_0.

For the proof of the theorem, we first prove that $\Phi\Psi = 1$. By analogy with the differentiable case we have:

Lemma 4. *Let* (X, Y) *be a Poincaré n-pair, and* ν *the normal fibration of* X, ν' *the normal fibration of* Y. *Then if* $i: Y \subset X$,

$$i^*\nu \sim_s ((i^*\nu) * 1) \sim_s \nu'.$$

Remark. Here $*$ also indicates join, and $\sim_s$ denotes stable fibre homotopy equivalence.

Proof. The first relation follows by stability. We use Browder's version of Wall's statement of the uniqueness of the Spivak normal fibration, according to which, given a Poincaré n-pair (X, Y), we obtain a normal $(k - 1)$ spherical fibration ν over X and an element $\alpha \in \pi_{n+k}(T(\nu), T(\nu|Y))$ such that $h(\alpha) \cap U = [X] \in H_n(X, Y)$, where h is the Hurewicz homomorphism, U is the Thom class of ν and T denotes the Thom space. Then ν is unique up to some fibre homotopy equivalence b, and $T(b)$ sends α_1 onto α_2. (See [2, I.4.19] for details.) The desired result then follows by checking that commutativity holds in the diagram:

$$
\begin{array}{ccccccc}
\cdots \longrightarrow & \pi_{n+k}(T(\nu)) & \longrightarrow & \pi_{n+k}(T(\nu), T(\nu\,|\,Y)) & \overset{\partial}{\longrightarrow} & \pi_{n+k-1}(T(\nu\,|\,Y)) & \longrightarrow \cdots \\
& & & \downarrow h & & \downarrow h & \\
\cdots \longrightarrow & H_{n+k}(T(\nu)) & \longrightarrow & H_{n+k}(T(\nu), T(\nu|Y)) & \overset{\partial}{\longrightarrow} & H_{n+k-1}(T(\nu\,|\,Y)) & \longrightarrow \cdots \\
& & & \downarrow \cap U & & \downarrow \cap \partial U & \\
& & & H_n(X, Y) & \overset{\partial}{\longrightarrow} & H_{n-1}(Y) &
\end{array}
$$

Note that in the bottom row $[X] \to [Y]$. Since $\nu\,|\,Y$ and ν' both satisfy

$$h(\xi) \cap U = [Y] \in H_{n-1}(Y),$$

the assertion follows from the uniqueness theorem. We now apply the result of Sutherland [13]:

Proposition 5 (Sutherland). *Let* M *be a finite 1-connected CW complex over*

which there is a reducible sphere fibering v. Let π: $B \to M$ be the projection of a q-sphere fibering ξ ($q > 1$) and let γ be a negative for ξ. Then the sphere fibering $\pi^*(\gamma * v * 1)$ over B is reducible.

Proof. By the original uniqueness theorem of Spivak, this means that $\pi^*(\eta * v * 1)$ is the normal fibration of B. We apply this to the fibration $E^{\cdot} = \beta^* E_{\xi}$ over K_0 defined above. Let π be the projection of $E^{\cdot}$. Sutherland's result then asserts that $v(\partial E^{\cdot})$ is $\pi^*(\eta * v(K_0) * 1)$, where η is an inverse for $\partial E^{\cdot} \to K_0$, and $v(K_0)$ is the normal fibration of K_0, which is trivial. Computing $\Phi(v(E^{\cdot}))$ (and hence $\Phi(E_{\xi})$) amounts to computing the classifying map of $-v(E^{\cdot})$. Since 1 and $v(K_0)$ are trivial, and using Lemma 4, this is equivalent to $c_{-\eta}$, which is homotopic to c_{ξ}, and we have the following commutative diagram:

$$(\partial E^{\cdot} \ \xi E^{\cdot})$$
$$\pi \Big/\!\!\Big/ \ \mathfrak{X} \qquad \searrow c_{-\eta}$$
$$K_0 \underset{\alpha}{\overset{\beta}{\rightleftarrows}} K \longrightarrow BG$$

where $\mathfrak{X}$ is a zero section for $E^{\cdot} \to K_0$.

Thus $\Phi\Psi = 1$, Φ is 1-1 and Ψ is onto.

It remains to prove, for instance, that Φ is onto. Let (M^m, γ) be a thickening of K. Using the path fibration c of [11, 2.8.9] (relativized) we can replace $(M^m, \partial M^m)$ by a homotopically equivalent pair $((M'^m, \partial M'^m), \gamma')$, which is a thickening, and such that the homotopy inverse π' of γ' is a fibration. Thus we may assume that $(M^m, \partial M^m)$ itself is a fibration; we have

$$(M^m, \partial M^m)$$
$$\gamma \Big\Vert\Big\uparrow \pi$$
$$K^k$$

Let $(F, \partial F)$ be the relative fibre. Applying the homotopy sequence of the fibration

$$\cdots \to \pi_n(M^m) \to \pi_n(K) \to \pi_{n-1}(F) \to \pi_{n-1}(M^m) \to \pi_{n-1}(K) \to \cdots$$

we find that all homotopy groups by F vanish and hence that F is contractible.

It remains to show that $\partial F = F \cap \partial M$ is actually a homotopy S^{n-1} ($m = n + k$). We do this using a relative Serre spectral sequence with

$$E^2_{p,q} = H_p(K, H_q(F, \partial F))$$

and $E^{\infty} = $ subquotients of $H_n(M, \partial M)$. We must show that

$$H_j(F, \partial F) = 0, \qquad j \neq n,$$
$$= Z, \qquad j = n.$$

Using the exact homology sequence of $(F, \partial F)$ it then follows that ∂F is a homology S^{n-1}.

Lemma 6. $H_j(F, \partial F) = 0, j < n.$

Proof. K is simply connected and dim $K = k$; so

$$H_j(M, \partial M) \approx H^{n+k-j}(M) = H^{n+k-j}(K) \approx 0, \quad \text{if } j < n.$$

Let s be the first index > 0 such that $H_s(F, \partial F) \neq 0$. Then

$$E^2_{0,q} = H_0(K, H_q(F, \partial F)) \cong H_q(F, \partial F)$$

(coefficients in a field). For the differentials we have, with $p = 0$,

$$E^r_{r,q-r+1} \xrightarrow{d^r} E^r_{0,q} \xrightarrow{d^r} E^r_{-r,q+r-1}.$$

The right-hand group is always zero, as E is a first quadrant spectral sequence. For $q \leq s$, $E^2_{r,q-r+1}$ is zero; hence $E^r_{r,q-r+1}$ is 0 for $r > 1$, and all differentials vanish, so that $E^r_{0,q} \cong E^\infty_{0,q}$. In particular, for $q = s$ we obtain $0 \neq E^\infty_{0,s} \subseteq H_s(M, \partial M)$ and the latter group is zero for $s < n$, a contradiction.

Next consider $H_n(F, \partial F)$. We have $E^2_{0,n} = H_n(F, \partial F)$, and differentials

$$E^r_{r,n-r+1} \xrightarrow{d^r} E^r_{0,n} \xrightarrow{d^r} E^r_{-r,n+r-1}.$$

As before, the right-hand group is always zero. For the left-hand group we look at $E^2_{r,n-r+1}$. This is always 0 for $r \neq n+1$, since in that case $H_{n-r+1}(F, \partial F) = 0$. For $r = n+1$ we have

$$E^2_{n+1,0} = H_{n+1}(K, H_0(F, \partial F))$$
$$= 0, \quad \text{since } n + 1 \gg \dim K.$$

Hence all differentials vanish and

$$H_n(M, \partial M) = E^\infty_{0,n} \cong E^2_{0,n} \cong H_n(F, \partial F).$$

But $H_n(M, \partial M) \cong H^k(K) = Z$. Hence $H_n(F, \partial F) = Z$ and $H_{n-1}(\partial F) = Z$ so that ∂F is a homology $(n-1)$-sphere.

Finally, $H_i(F, \partial F) = 0$ for $i > n$. For, by finite dimensionality, $(F, \partial F)$ has a highest dimensional nonzero homology group, say, in dimension q. Now consider the spectral sequence with $E^2_{k+2,q-1} \to E^2_{k,q} \to E^2_{k-2,q+1}$. The right-hand group is $H_{k-2}(K, H_{q+1}(F, \partial F))$ and hence is zero; the left-hand group is $H_{k+2}(K, H_q(F, \partial F))$ and is zero as K is k-dimensional. Hence, the middle term survives to E^∞,

and gives a nonzero summand for the right side of the equation

$$\dim H_r(M, \partial M) = \sum E_{p,q}^{\infty}$$

where the summation is carried out for $p + q = r$. Since by definition of q, $k + q > m$, we get the required contradiction.

It remains to prove that ∂F is simply connected; it will then follow from the Whitehead theorem that ∂F is a homotopy $(n - 1)$-sphere.

Consider the homotopy ladder of the fibration

$$(M, \partial M)$$
$$\downarrow (F, \partial F)$$
$$K$$

We obtain

$$\cdots \to \pi_2(F) \longrightarrow \pi_2(M) \xrightarrow[\cong]{\rho_\#} \pi_2(K) \longrightarrow \pi_1(F) \longrightarrow \pi_1(M) \xrightarrow[\cong]{\rho_\#} \pi_1(K) \to \cdots$$

$$\cdots \to \pi_2(\partial F) \longrightarrow \pi_2(\partial M) \longrightarrow \pi_2(K) \xrightarrow{\partial} \pi_1(\partial F) \longrightarrow \pi_1(\partial M) \longrightarrow \pi_1(K) \to \cdots$$

By hypothesis, we have $\pi_1(\partial M) = \pi_1(K) = 0$; so ∂ is onto. Then $\pi_2(K) = 0$ gives $\pi_1(\partial F) = 0$. This completes the proof.

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