THREE LOCAL CONDITIONS ON A GRADED RING

BY

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ABSTRACT. Let $R = \sum_{i \in \mathbb{Z}} R_i$ be a commutative graded Noetherian ring with unit and let $A = \sum_{i \in \mathbb{Z}} A_i$ be a finitely generated graded R module. We show that if we assume that A_M is a Cohen Macaulay R_M module for each maximal graded ideal M of R , then A_P is a Cohen Macaulay R_P module for each prime ideal P of R . With $A = R$ we show that the same is true with Cohen Macaulay replaced by regular and Gorenstein, respectively.

Introduction. The following question was posed by Nagata in [9]: if $R = \sum_{i \in \mathbb{Z}} R_i$ is a commutative graded Noetherian ring with unit such that R_M is Cohen Macaulay for each maximal graded ideal M of R , then is R_P Cohen Macaulay for every prime ideal P of R ? Paul Roberts [6], M. Hochster and L. J. Ratliff [4] and the author arrived independently at affirmative answers to the question. An expanded version of the question in the case of a graded ring R , graded by the integers, admitting a finitely generated graded R module A with the property that A_M is Cohen Macaulay for each maximal graded ideal M of R is answered in Theorem 1.1 below.

In §2 the same question is answered with R graded by the integers and Cohen Macaulay replaced by regular. If R is graded by the positive integers and is a regular ring, then R admits a structure theorem in the case that R has a finite number of maximal graded ideals. In Proposition 2.3 we show that R is a direct sum of polynomial rings over semilocal regular domains.

In §3 the question is once again answered when R is graded by the integers, but this time Cohen Macaulay is replaced by Gorenstein. An example is given to show that if $R = \sum_{i \in \mathbb{Z}} R_i$ is a Gorenstein ring, R_0 need not be Gorenstein. In fact in the example R_0 is not even Cohen Macaulay.

Paul Roberts has recently proved these results using methods different from those given below.

The basic references for the ideas that follow are [5], [7] and [8].⁽¹⁾ All rings are commutative Noetherian rings with unit element. $R = \sum_{i \in \mathbb{Z}} R_i$ will

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⁽¹⁾ The notation is primarily that of [5].

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always be a graded ring and $A = \sum_{i \in \mathbb{Z}} A_i$ will always be a graded R module. $\mathbb{Z}$ will denote the group of integers and $\mathbb{Z}^+$ the set of nonnegative integers. If B is a finitely generated T module and I is an ideal of T , then $G(I, B)$ will denote the length of a maximal R sequence in I on B .

1. Graded Cohen Macaulay modules. Let E be a finitely generated module over T , a local ring with maximal ideal M . If $I = \text{ann}_T E = \{r \in T \mid rE = 0\}$, then we denote by $\dim E$ the Krull dimension of T/I . E is a Cohen Macaulay module if either $E = 0$ or if $E \neq 0$ and $\dim E = G(M, E)$.

THEOREM 1.1. *Let A be a finitely generated R module. Suppose A_M is a Cohen Macaulay R_M module for every maximal graded ideal M of R . Then A_P is a Cohen Macaulay R_P module for every prime ideal P of R .*

We say that M is a maximal graded ideal of R if M is maximal among graded ideals of R with respect to avoiding the set of units of R . If R has nonzero homogeneous elements of negative and positive degree, a maximal graded ideal need not be maximal, as simple examples show. Yet such ideals can be characterized easily enough as we show in the lemma below.

LEMMA 1.2. *Let R be a graded ring such that every nonzero homogeneous element is a unit. Then either R is a field or $R = K[u, u^{-1}]$, where K is a field and u is transcendental over K .*

PROOF. If R has no homogeneous elements of nonzero degree, then R is a field. Otherwise, at least R_0 is a field and the set $\Gamma = \{i \in \mathbb{Z} \mid R_i \neq 0\}$ is a subgroup of $\mathbb{Z}$. Hence $\Gamma \approx \mathbb{Z}^m$ for some positive integer m . Let $0 \neq u \in R_m$. If y is any nonzero homogeneous element of positive degree in R , then $y \in R_{km}$ for some $k > 0$. $yu^{-k} \in R_0 = K$; therefore $y \in K[u]$. Similarly if y is of negative degree, $y \in K[u^{-1}]$. Since any element of R is a sum of homogeneous elements, $R = K[u, u^{-1}]$. It is easily seen that u is transcendental over K .

Two rings of the type described in Lemma 1.2 are those of the form R/M where M is a maximal graded ideal and R_S , where R is a domain and S is the multiplicatively closed set of nonzero homogeneous elements of R . We shall use this latter remark to establish the relationship between a prime ideal P and P^* the prime ideal generated by the homogeneous elements contained in P .

PROPOSITION 1.3. *Let $P \supsetneq P^*$ be as described above in a graded ring R . If $\text{rank } P = n$ then $\text{rank } P^* = n - 1$, $n \geq 1$.*

PROOF. We proceed by induction on n . If $n = 1$, $P \supsetneq Q$, where Q is a minimal prime ideal of R . Hence $P^* = Q$, and $\text{rank } P^* = 0$. If $n > 1$, then there exists a prime ideal Q such that $P \supsetneq Q$ and $\text{rank } Q = n - 1$. If Q is a

graded prime ideal, we are finished since $P^* \supset Q$. So we assume Q is not a graded ideal and $Q \not\supseteq Q^*$. By induction, $\text{rank } Q^* = n - 2$. Now $Q^* \subset P^* \subset P$. Hence it will suffice to show that $Q^* \neq P^*$. Otherwise, in $\bar{R} = R/Q^*$, $\bar{P}$ is a prime ideal such that $\text{rank } \bar{P} = 2$ and $\bar{P} \cap S = \emptyset$, where S is the set of nonzero homogeneous elements of $\bar{R}$. Hence $\bar{P}$ is a proper prime ideal of $\bar{R}_S$. Yet by Lemma 1.2 the Krull dimension of $\bar{R}_S$ is no larger than 1. This contradiction finishes the proof.

In order to prove Theorem 1.1 we attempt to translate the problem to a suitable polynomial extension of R . As a result we must extend the module A . If X is an indeterminate, the graded $R[X]$ module $A[X] = R[X] \otimes_R A = \sum_{i \in \mathbb{Z}} B_i X^i$ where $B_k = \sum_{i+j=k} A_i X^j$, if we assume $\deg X = 1$. In the following proposition we take note of the properties inherited by the $R[X]$ module $A[X]$ from the R module A .

PROPOSITION 1.4. *Let R and A be as in Theorem 1.1. Then for each maximal graded ideal M of $R[X]$ the $R[X]_M$ module $A[X]_M$ is Cohen Macaulay.*

PROOF. It is clear that the $R[X]$ module $A[X]$ is finitely generated. Let M be a maximal graded ideal of $R[X]$. $M \cap R = Q$ is a graded prime ideal of R . Now A_Q is a Cohen Macaulay R_Q module and $A[X]_M$ is a localization of $A_Q[X]$. Hence we show that the $R_Q[X]_P$ module $A_Q[X]_P$ is Cohen Macaulay, where $P = M_Q$. Let $u_1, \dots, u_r$ be a maximal R -sequence in the maximal ideal of R_Q on A_Q . After reducing modulo the ideal generated by this R -sequence, we are in the case where B is a Cohen Macaulay T module, T a local ring, with maximal ideal m , and $G(m, B) = 0 = \dim B$. Now if N is a maximal ideal of $T[X]$ which contains the annihilator of $B[X]$, then $N = (m, f)$ where f is a monic polynomial in X . Since it is monic f cannot be a zero divisor on $B[X]$. Hence $G(N_N, B[X]_N) = 1 = \dim B[X]_N$ and $B[X]_N$ is a Cohen Macaulay $T[X]_N$ module for each maximal ideal N of $T[X]$.

Now suppose $U = R[X_1, \dots, X_n]$. We shall denote $A[X_1, \dots, X_n]$ by $A[U]$. We will translate the problem to such a ring U in order to find an R -sequence on $A[U]$ with the right properties. The following proposition shows how to choose the ring U and the appropriate R -sequence.

PROPOSITION 1.5. *Let I be a graded ideal of R with $G(I, A) = k$, A a finitely generated R module. There exists a polynomial extension U of R such that $G(IU, A[U]) = k$ and there exist k homogeneous elements in IU forming a maximal R -sequence on $A[U]$.*

PROOF. Choose homogeneous elements $u_1, \dots, u_r$ in I forming an R -sequence on A . Let J be the ideal generated by these elements. If $I \subset Z(A/JA)$ then $r = k$ and we take $U = R$. If $I \not\subset Z(A/JA)$, assume there exists no

homogeneous element in I that is not a zero divisor on A/JA . Let $f = f_{-m} + \cdots + f_n$ be an element of I not in $Z(A/JA)$. Pass to $R[X]$, $\deg X = +1$. Now $f^* = f_{-m}X^{m+n} + \cdots + f_0X^n + \cdots + f_n$ is a homogeneous element of $IR[X]$ and $f^* \notin Z(A[X]/JA[X])$. Otherwise one could find a nonzero $v \in A/JA$ such that $vf_i = 0$ for each $-m \leq i \leq n$. Hence $u_1, \dots, u_r, f^*$ are homogeneous elements in $IR[X]$ forming an R -sequence. If we have not obtained a maximal R -sequence at this point, repeat the above procedure. We are assured of finding a maximal R -sequence of homogeneous elements in some finite polynomial extension of R , since for any such extension U , $G(IU, A[U]) = G(I, A) = k$.

We are now in a position to prove a graded version of Theorem 1.1.

LEMMA 1.6. *With the notation and hypotheses of Theorem 1.1, if Q is any graded prime ideal of R and $Q \supset I = \text{ann}_R A$ then $G(Q, A) = \text{rank } Q/I$.*

PROOF. Pass to a polynomial extension U of R such that by Proposition 1.5 there exist homogeneous elements $u_1, \dots, u_k$ forming a maximal R -sequence in QU on $A[U]$. Let J be the ideal of U generated by the u_i . Now QU is an associated prime ideal of the graded module $A[U]/JA[U]$. Hence there exists a homogeneous element $v \in A[U]$, $v \notin JA[U]$ such that $vQU \subset JA[U]$. Let $K = \{r \in U | rv \in JA[U]\}$. K is a proper graded ideal of U . There exists a maximal graded ideal M of U such that $M \supset K$. Since $M \supset K \supset QU \supset IU = \text{ann}_U A[U]$, we have that $A[U]_M \neq 0$. Hence $v_M \notin JA[U]_M$ and $v_M QU_M \subset JA[U]_M$. So $G(QU_M, A[U]_M) \leq k$. But $G(QU_M, A[U]_M) \geq k$ is always true. Now $A[U]_M$ is a Cohen Macaulay U_M module (Proposition 1.4). Hence $k = G(QU, A[U]) = G(QU_M, A[U]_M) = \text{rank } QU_M/IU_M = \text{rank } Q/I$.

We now prove the theorem.

PROOF OF THEOREM 1.1. Let N be a nongraded prime ideal of R . If $N \not\supset I = \text{ann}_R A$, $A_N = 0$. So suppose $N \supset I$. We wish to show that $G(N, A) = \text{rank } N/I$. Now $N \not\supseteq N^*$, and $\text{rank } N = \text{rank } N^* + 1$ (Proposition 1.3). Since A is a graded module, $N^* \supset I$. Hence

$$k = \text{rank } N^*/I = G(N^*, A) \leq G(N, A) \leq \text{rank } N/I = \text{rank } N^*/I + 1,$$

by the above lemma and Theorem 27 [7, Chapter 6]. It will suffice to show that $G(N^*, A) < G(N, A)$. Otherwise, pass to a polynomial extension U of R , so that there exist homogeneous elements $u_1, \dots, u_k$ in N^*U forming a maximal R -sequence on $A[U]$. If J is the ideal generated by the u_i then NU is contained in a maximal associated ideal P of the module $A[U]/JA[U]$. P is a graded prime ideal and by the previous lemma $k = G(P, A[U]) = \text{rank } P/IU$. Yet $\text{rank } NU/IU = k + 1$. This contradiction finishes the proof.

2. Graded regular rings. In this section we prove a theorem analogous to the one proved in the previous section.

THEOREM 2.1. *Let R be such that R_M is a regular local ring for each maximal graded ideal M of R . Then R_N is a regular local ring for each maximal ideal N of R .*

Before beginning the proof we take note of localizations at multiplicative sets of homogeneous elements. If T is a graded ring and S a multiplicative set of homogeneous elements then T_S has an obvious graded structure, $T_S = \sum_{i \in \mathbb{Z}} T_i$ where $T_i = \{a/b | a, b \text{ are homogeneous elements of } T, b \in S \text{ and } \deg a - \deg b = i\}$. If S is the set of homogeneous elements that do not belong to a graded prime ideal P of T then P_S is the unique maximal graded ideal of T_S . It is easy to see that if T_P is a domain and P and S are as above then T_S is a domain. We now prove the theorem.

PROOF. In order to show that R_N is a regular local ring, we will show that T_P is a regular local ring, where $T = R_S$, $P = N_S$ and S is the set of homogeneous elements that do not belong to N^* . If we let $M = N_S^*$, then M is the unique maximal graded ideal of T . What we will actually show is that T_U is a regular local ring for each maximal ideal U of T , and we shall proceed by induction on the rank of M in T . Now T_M is a regular local ring, since it is a localization of R_Q for some maximal graded ideal Q of R . So if $\text{rank } M = 0$ then T_M is a field. It follows that T is a ring as described in Lemma 1.2. Hence T_U is a regular local ring for each maximal ideal U of T . So suppose $\text{rank } M = n > 0$ and Q is any maximal ideal of T . Now $Q^* \subset M$, and if this inclusion is proper, then by localizing at the homogeneous elements that do not belong to Q^* in T we can reduce to a graded ring with unique maximal graded ideal which has rank less than n . Since T_Q is a localization of that graded ring, T_Q is a regular local ring by induction. So suppose that $Q^* = M$. Now choose an $x \in M - M^2$, $x \neq 0$, such that x is homogeneous. Let $\bar{T} = T/(x)$, $\bar{M} = M/(x)$ and $\bar{Q} = Q/(x)$. $\bar{T}_{\bar{M}}$ is a regular local ring and $\text{rank } \bar{M} = n - 1$. By induction $\bar{T}_{\bar{Q}}$ is a regular local ring. Hence T_Q is a regular local ring.

Examples of rings with the properties described in the theorem, graded regular rings, are polynomial rings over regular rings and symmetric algebras with a natural grading of the form $S_T(I)$, where T is a regular ring and I is a nonprincipal invertible ideal of T .

We conclude this section by showing that a class of graded regular rings are polynomial rings. D. K. Harrison [3] has obtained this result in the case that R_0 is a field.

PROPOSITION 2.2. *Let $R = \sum_{i \in \mathbb{Z}} R_i$ be a graded regular ring with a unique maximal graded ideal M . Then R_0 is a regular local ring and $R = R_0[u_1, \dots, u_k]$, where the u_i are homogeneous elements of R and independent indeterminates over R_0 .*

PROOF. The proof is by induction on $n = \text{rank } M = \text{Krull dim } R$. (That $\text{rank } M = \text{Krull dim } R$ is an easy deduction from Proposition 1.3 and the fact that in this case M is a maximal ideal of R .) If $n = 0$, $R = R_0$ is a field. If $n > 0$, choose a nonzero homogeneous element x of positive degree in $M - M^2$ (if no such x exists one can show that $R = R_0$). Let $\bar{R} = R/(x)$ and $\bar{M} = M/(x)$. Now $\bar{R}$ is a graded regular ring, since $\bar{R}_{\bar{M}}$ is a regular local ring by choice of x . Now $n - 1 = \text{rank } \bar{M} = \text{Krull dim } \bar{R}$; hence by induction $\bar{R}_0$ is a regular local ring and $\bar{R} = \bar{R}_0[\bar{u}_1, \dots, \bar{u}_k]$. Now $\bar{R}_0 = R_0$, and if $\text{Krull dim } R_0 = d$ then $d + k = n - 1$. Since the mapping of R to $\bar{R}$ is a graded map choose homogeneous pre-images u_i of the $\bar{u}_i$. Letting $u_{k+1} = x$ then it is easy to see that $(u_1, \dots, u_{k+1}) = \sum_{i > 0} R_i$. Hence $R = R_0[u_1, \dots, u_{k+1}]$. Lastly let $T = R_0[X_1, \dots, X_{k+1}]$ be mapped to R in the obvious fashion, preserving degrees, where the X_i are indeterminates. Now since $\text{rank } M = \text{rank } \bar{M} + 1$, $\text{Krull dim } R = k + d + 1$. Since this is also the Krull dimension of T , the above mapping is an isomorphism and this finishes the proof.

Using the above proposition we show that a graded regular ring with a finite number of maximal graded ideals is a direct sum of polynomial rings.

PROPOSITION 2.3. *Let $R = \sum_{i \in \mathbb{Z}} R_i$ be a graded regular ring with a finite number of maximal graded ideals. Then R is the direct sum of a finite number of graded regular domains, each with a finite number of maximal graded ideals. If $T = \sum_{i \in \mathbb{Z}} T_i$ is a graded regular domain with a finite number of maximal graded ideals, then $T = T_0[u_1, \dots, u_k]$ where the u_i are homogeneous elements of T and independent indeterminates over T_0 .*

PROOF. Since R is a regular ring, R is the direct sum of a finite number of regular domains. Since the idempotents are homogeneous elements of degree zero in R , the first assertion follows. As for the second assertion we proceed by induction on $\text{Krull dim } T = n$. If $n = 0$, $T = T_0$ is a field. If $n > 0$ and $T \neq T_0$ let T_j be the nonzero T_0 module of least positive degree in T . Since by the previous proposition T_j is a locally free, finitely generated T_0 module, T_j is a free T_0 module. Let x be a free generator of T_j . Now x is in the square of no maximal graded ideal of T . Hence $\bar{T} = T/(x)$ is a graded regular ring of Krull dimension $n - 1$. $\bar{T}$ is a domain since $\bar{T}_0 = T_0$ and T_0 has no idempotents. By induction $\bar{T} = \bar{T}_0[\bar{u}_1, \dots, \bar{u}_k]$, where if the $\text{Krull dim } T_0 = d$ then $d + k = n - 1$.

Letting u_i be the homogeneous preimages of the $\bar{u}_i$ in T and $u_{k+1} = x$, then it is easy to see that $T = T_0[u_1, \dots, u_{k+1}]$. Let $U = T_0[X_1, \dots, X_{k+1}]$ and map U to T in the obvious way preserving degrees, where the X_i are independent indeterminates over T_0 . Then since $\text{Krull dim } T = d + k + 1 = \text{Krull dim } U$, the above mapping is an isomorphism and this finishes the proof.

That in the above propositions the graded rings considered are Z^+ graded is essential as the following example shows. If $R = K[x, y, z, w]$ where K is a field and x, y, z, w are independent indeterminates over K , then R is a Z graded ring by letting $\deg x = \deg y = +1$ and $\deg z = \deg w = -1$. R is clearly a regular ring, yet $R_0 = K[xz, xw, yz, yw]$ is not a regular ring and R cannot be a polynomial ring over R_0 . Notice that $m = (x, y, z, w)$ contains all homogeneous elements in R of degree different from zero. Hence, if we localize R at the set of homogeneous elements that do not belong to m and call this new ring T , then the subring of homogeneous elements of T of degree zero, T_0 , is just R_0 localized at the ideal (xz, xw, yz, yw) . T is a regular Z graded ring with a unique maximal graded ideal, which cannot be a polynomial ring over T_0 .

3. Graded Gorenstein rings. A local ring T is said to be Gorenstein if T as a module over itself has finite injective dimension. As a result T is Cohen Macaulay and any system of parameters of T generates an irreducible ideal. These conditions are also sufficient for T to be Gorenstein. The basic references for this material are [5, Chapter 4, §5] and Bass's *Ubiquity* paper [1].

THEOREM 3.1. *Let R be such that R_M is Gorenstein at every maximal graded ideal M of R . Then R_N is Gorenstein at each maximal ideal N of R .*

The plan of proof is much the same as that of the proof of the theorem in the first section. We shall translate the problem to a suitable polynomial extension of R and make use of Proposition 1.5.

LEMMA 3.2. *Let R be a graded ring with the property given in Theorem 3.1. Then $R[X]$ also has this property, X an independent indeterminate of positive degree over R .*

PROOF. If N is a maximal graded ideal of $R[X]$ and $N \cap R = Q$, then $R[X]_N$ is a localization of $R_Q[X]$. But R_Q is Gorenstein and any localization of $R_Q[X]$ is also Gorenstein.

PROPOSITION 3.3. *Let R be as in the above theorem. Suppose $I = (y_1, \dots, y_r)$ is an ideal of rank r in R , generated by r homogeneous elements.*

Then I is unmixed and each primary component of I is irreducible.

PROOF. If R has the property stated in the above theorem then R_N is Cohen Macaulay at each maximal ideal N of R . Hence I is unmixed. Let P be a prime ideal minimal over I . P is a graded ideal and the P -primary component q belonging to I is $IR_P \cap R$. Now letting $\bar{R} = R/I$ and $\bar{P} = P/I$, $\bar{R}_{\bar{P}} = R_P/IR_P$ has zero Krull dimension and is Gorenstein. Hence the zero ideal of $\bar{R}_{\bar{P}}$ is irreducible. Then since IR_P is irreducible in R_P , q is irreducible in R .

We now sketch some technical lemmas concerning homogenization in general graded rings. Let R be a graded ring and define the operation h on the elements of R : if $f = f_{-m} + \cdots + f_n \in R$ then $f^h = f_{-m}X^{m+n} + \cdots + f_0X^n + \cdots + f_n \in R[X]$, $\deg X = 1$. The corresponding operation a on the elements of $R[X]$ is as follows: if $u = u_{-k}X^{k+j} + \cdots + u_0X^j + \cdots + u_j$ is a homogeneous element of degree j in $R[X]$ then $u^a = u_{-k} + \cdots + u_0 + \cdots + u_j$. If u and v are homogeneous elements of $R[X]$, then it is easy to see that $(uv)^a = u^a v^a$, and if u and v have the same degree then $(u+v)^a = u^a + v^a$. If $f \in R$ then $(f^h)^a = f$, and if u is a homogeneous element of $R[X]$ and $k = \deg u - \deg(u^a)^h$ then $(u^a)^h X^k = u$. If f and g are elements of R define $d(f)$ and $d(g)$ to be the degrees of the constituent homogeneous elements of highest degree in a representation of f and g respectively. If $k = d(f) + d(g) - d(fg)$ then $X^k(fg)^h = f^h g^h$. If $k = \max(d(f), d(g)) - d(f+g)$ then $X^k(f+g)^h = f^h + X^l g^h$, where if $d(f) > d(g)$ then $l = d(f) - d(g)$.

The operation h can be extended to ideals so that if I is an ideal of R then I^h is a graded ideal of $R[X]$ and each homogeneous element of I^h is of the form $X^m f^h$ where $m > 0$ and $f \in I$. The usual propositions (like those listed in [10, Chapter VII, §5, Theorem 17]) are true in this more general situation. We just note the following: If I and J are ideals of R and if $I \supsetneq J$ then $I^h \supsetneq J^h$; if I is a prime ideal and J is I -primary then I^h is prime and J^h is I^h -primary: if $I = \bigcap_{i=1}^n J_i$ is an irredundant primary representation of I then $I^h = \bigcap_{i=1}^n J_i^h$ is an irredundant primary representation of I^h . One final note: if (f) is a principal ideal of R then $(f)^h$ need not be a principal ideal of $R[X]$ if R has zero divisors. But $(f)_S^h$ is a principal ideal generated by f^h in $R[X]_S$ where $S = \{1, X, X^2, \dots\}$.

We now prove Theorem 3.1.

PROOF. Let N be any nongraded maximal ideal of R . We shall ultimately show that R_N is Gorenstein. Let $N \supsetneq N^*$. By Proposition 1.5 there exists a polynomial extension T of R such that there exist homogeneous elements $u_1, \dots, u_k$ in T that form a maximal R -sequence in N^*T , where $k = G(N^*T, T) = \text{rank } N^*T$. Let I be the ideal generated by these elements in T . In $U = T/I$, let $Q = NT/I$ and $Q^* = N^*T/I$. U has the local Gorenstein property at its maximal

graded ideals and since $\text{rank } N = \text{rank } N^* + 1$, $\text{rank } Q = 1$ and $\text{rank } Q^* = 0$. Since U is locally Cohen Macaulay, $G(Q) = 1$. Hence let $y = y_{-m} + \cdots + y_n$ be a nonzero divisor in Q . In $U[X]$, $\deg X = 1$, y^h is a homogeneous nonzero divisor. In $U[X]_S$, $(y^h)_S = (y^h)_S$, $S = \{1, X, X^2, \dots\}$. By Proposition 3.3 every primary component of $(y^h)_S$ is irreducible. It follows that every primary component of (y) is irreducible. Now the zero ideal of the zero Krull dimensional local ring U_Q/yU_Q is irreducible. Hence U_Q/yU_Q is Gorenstein. Then T_{NT} is Gorenstein. Since T_{NT} is a faithfully flat R_N module it follows that R_N is Gorenstein.

We conclude with an example of a graded ring which is locally Gorenstein such that the subring of homogeneous elements of degree 0 is not Cohen Macaulay.

EXAMPLE 3.4. Let $T = k[X, Y]/(X^2, XY)$, where k is any field. If $q_1 = (X^2, Y)$ and $q_2 = (X)$, then in T the image of $q_1 \cap q_2$ is the zero ideal. Let W, V be indeterminates, each assigned degree one. Let $S = T[W]$, $Q_1 = q_1S$, $Q_2 = q_2S + WS$ and $I = Q_1 \cap Q_2$. Let $U = S/I$. Letting small letters denote the images in U , our example is $R = U[V]/(xV + yV, V^2)$. Notice that R is a graded ring of Krull dimension one, and its subring of degree zero $R_0 = T$ is not Cohen Macaulay. We claim that R_N is Gorenstein, N the unique maximal graded ideal of R . Now $R \approx k[A, B, C, D]/J$, where $J = (A^2, AB, BC, AD + BD, D^2)$, under the mapping X, Y, W, V to A, B, C, D respectively. Now $B + C$ is not a zero divisor on J . Otherwise, if $g(B + C) \in J$, one can show that $g = \beta BD$ for some β . But if $g(B + C) \in J$ then $gB = \beta B^2D \in J$. Now if $\beta \notin J$ then β is a power of B . But now $B^sD \notin J$ for any s . Finally $k[A, B, C, D]/(J, B + C)$ is a zero dimensional Gorenstein local ring, and hence R_N is Gorenstein.

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