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# BOUNDARY BEHAVIOR OF THE CARATHEODORY AND KOBAYASHI METRICS ON STRONGLY PSEUDOCONVEX DOMAINS IN $\mathbf{C}^n$ WITH SMOOTH BOUNDARY

BY

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ABSTRACT. The Carathéodory and Kobayashi distance functions on a bounded domain $G$ in $\mathbf{C}^n$ have related infinitesimal forms. These are the Carathéodory and Kobayashi metrics. They are denoted by $F(z, \xi)$ (length of the tangent vector $\xi$ at the point $z$). They are defined in terms of holomorphic mappings, from $G$ to the unit disk for the Carathéodory metric, and from the unit disk to $G$ for the Kobayashi metric.

We consider the boundary behavior of these metrics on strongly pseudoconvex domains in $\mathbf{C}^n$ with $C^2$ boundary. $\xi$ is fixed and $z$ is allowed to approach a boundary point $z_0$. The quantity $F(z, \xi)\, d(z, \partial G)$ is shown to have a finite limit. In addition, if $\xi$ belongs to the complex tangent space to the boundary at $z_0$, then this first limit is zero, and $(F(z, \xi))^2 d(z, \partial G)$ has a (nontangential) limit in which the Levi form appears.

We prove an approximation theorem for bounded holomorphic functions which uses peak functions in a novel way. The proof was suggested by N. Kerzman. This theorem is used here in studying the boundary behavior of the Carathéodory metric.

**1. Introduction and statement of results.** This paper is based on the author's doctoral dissertation [7]. The main results were announced in [8]. At the suggestion of N. Kerzman we have included an approximation theorem which is of independent interest (Theorem 2).

By a metric we shall mean a differential metric in the sense of Grauert and Reckziegel [10]. A topological metric is always referred to as a distance.

Let $G$ be a dounded domain in $\mathbf{C}^n$. Let $\Delta$ be the unit disk in $\mathbf{C}$. Let $\Delta(G)$ be the set of holomorphic mappings from $G$ to $\Delta$, and $G(\Delta)$ the set of holomorphic mappings from $\Delta$ to $G$. Let $\rho$ be the Poincaré distance on $\Delta$.

DEFINITION 1. The Carathéodory metric on $G$ is the function $F_C$: $G \times \mathbf{C}^n \longrightarrow \mathbf{R}^+$ defined by [21]

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$$F_C(z, \xi) = \sup_{f \in \Delta(G); f(z)=0} |f_*(\xi)| = \sup_{f \in \Delta(G); f(z)=0} \left| \sum_{i=1}^{n} \frac{\partial f}{\partial z_i}(z)\xi_i \right|.$$

(The condition $f(z) = 0$ is superfluous [21].) The Carathéodory distance is defined by [2]

$$C(z_1 z_2) = \sup_{f \in \Delta(G)} \rho(f(z_1), f(z_2)), \qquad z_1, z_2 \in G.$$

It was shown by Reiffen [21] that if $\gamma$: $[0, 1] \longrightarrow G$ is a $C^1$ curve in $G$ with Carathéodory length defined by

$$L_C(\gamma) = \sup_{0=t_0<t_1<\cdots<t_k=1} \sum_{i=1}^{k} C(\gamma(t_{i-1}), \gamma(t_i)),$$

then

$$L_C(\gamma) = \int_0^1 F_C(\gamma(t), \gamma'(t))\,dt.$$

DEFINITION 2. The Kobayashi metric on $G$ is the function $F_K$: $G \times \mathbf{C}^n \longrightarrow \mathbf{R}^+$ defined by [22]

$$F_K(z, \xi) = \inf\{\alpha \mid \alpha > 0,\ \exists f \in G(\Delta) \text{ with } f(0) = z,\ f'(0) = \xi/\alpha\}.$$

The Kobayashi distance is defined as follows [18]: Let $z_1, z_2 \in G$. Consider all finite sequences of points $z_1 = p_0, p_1, \cdots, p_{k-1}, p_k = z_2$ of $G$ such that there exist points $x_1, \cdots, x_k$, $y_1, \cdots, y_k$ of $\Delta$ and mappings $f_1, \cdots, f_k \in G(\Delta)$ satisfying $f_i(x_i) = p_{i-1}$ and $f_i(y_i) = p_i$, $i = 1, \cdots, k$. Set

$$D(z_1, z_2) = \inf \sum_{i=1}^{k} \rho(x_i, y_i)$$

where the infimum is taken over all choices of points and mappings. Royden [22] showed that $D$ is actually the integrated form of $F_K$. That is, given $z_1, z_2 \in G$,

$$D(z_1, z_2) = \inf_{\gamma} \int_0^1 F_K(\gamma(t), \gamma'(t))\,dt$$

where the infimum is taken over all differentiable curves $\gamma$: $[0, 1] \longrightarrow G$ joining $z_1$ to $z_2$.

*Note.* The Carathéodory distance and metric do not satisfy such a relationship.

For fixed $\xi$ we investigate the behavior of the metrics $F(z, \xi)$ as $z$ approaches a point $z_0 \in \partial G$.

THEOREM 1. (The notation is explained below.) *Let $G$ be a (bounded*[1]*) strongly pseudoconvex domain in $\mathbf{C}^n$ with $C^2$ boundary. Let $F(z, \xi)$ be either the Carathéodory or Kobayashi metric on $G$. Let $z_0 \in \partial G$. Let $\phi$ be a $C^2$ defining function for $\partial G$ such that $\|\nabla_z \phi(z_0)\| = 1$. Then*

(i) $$\lim_{z \to z_0} F(z, \xi) d(z, \partial G) = \tfrac{1}{2} \| \xi_N(z_0) \|.$$

*If $\xi_N(z_0) = 0$, i.e. $\xi$ is a holomorphic tangent vector to $\partial G$ at $z_0$, then*

(ii) $$\lim_{z \to z_0; z \in \Lambda} (F(z, \xi))^2 d(z, \partial G) = \tfrac{1}{2} L_{\phi, z_0}(\xi) = \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j}(z_0) \xi_i \bar{\xi}_j .$$

$d(z, \partial G)$ is the Euclidean distance to the boundary. $\nabla_z \phi$ is the vector $(\partial\phi/\partial z_1, \cdots, \partial\phi/\partial z_n)$. $\xi_N(z_0)$ is the (complex) normal component of $\xi$ at $z_0$. $\Lambda$ in (ii) denotes a cone of arbitrary aperture with vertex at $z_0$ and axis the interior normal to $\partial G$ at $z_0$.

Fix the aperture of the cone $\Lambda$. It is shown that the limits are approached uniformly in vectors $\xi$ of unit length and in the boundary point $z_0$. It is also possible to reformulate Theorem 1 in such a way that the restriction $z \in \Lambda$ in the second limit is not needed (§5.4).

The significance of Theorem 1 lies in (i) the different limiting behavior in (complex) tangential and normal directions (cf. Stein [23]), and (ii) the appearance of the Levi form as the limiting value of a quantity defined inside the domain.

The approximation theorem is the following:

THEOREM 2. *Let $G$ be a (bounded) strongly pseudoconvex domain in $\mathbf{C}^n$ with $C^2$ boundary. Let $z_0 \in \partial G$. Let $\psi$ be a peak function for $G$ at $z_0$. Let $0 < a < b < 1$, and let*

$$A = \{z \in G \mid |\psi(z)| > a\}, \qquad B = \{z \in G \mid |\psi(z)| > b\}.$$

*Also choose an integer $m \geqslant 0$ and an arbitrary $\eta > 0$. Then there exists a positive constant $L = L(G, a, b, m, \eta)$ such that the following holds: given $f \in H^\infty(A)$, there exists $\hat{f} \in H^\infty(G)$ such that*

(i) $\|D^\alpha \hat{f} - D^\alpha f\|_{L^\infty(B)} \leqslant \eta \|f\|_{L^\infty(A)}$, $0 \leqslant |\alpha| \leqslant m$.

(ii) $\|\hat{f}\|_{L^\infty(G)} \leqslant L \|f\|_{L^\infty(A)}$.

(1) A strongly pseudoconvex domain is by assumption bounded (Definition 4).

Given a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ of length $n$, we write $D^\alpha = (\partial/\partial z_1)^{\alpha_1} \cdots (\partial/\partial z_n)^{\alpha_n}$. Also $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

The constant $L$ can be chosen independently of $z_0$ if the peak function $\psi$ depends continuously on $z_0$. Since the uniform statement in Theorem 1 also depends on choosing peak functions in a continuous manner, we discuss this question in §3.2.

The significance of Theorem 2 is that it gives approximation of holomorphic functions up to the boundary. It is used here in studying the boundary behavior of the Carathéodory metric.

*Note.* One cannot hope to obtain such a theorem with a bound independent of $\eta$.

G. Henkin [14] has independently obtained an estimate for the Carathéodory metric which suffices to show that biholomorphic maps between strongly pseudoconvex domains satisfy a Hölder-½ condition.

The boundary behavior of the Bergman kernel and metric is a somewhat older question. Bergman [1] obtained results for domains in $C^2$ which, at a given boundary point, admit both interior and exterior 'domains of comparison'. By no means do all strongly pseudoconvex domains have this property. However in [15] Hörmander showed that the boundary behavior of the kernel function on such domains is a local question. Suitable local domains of comparison could be found for any strongly pseudoconvex domain with smooth boundary. This idea is the basis of our approach.

With a suitable choice of coordinates, analytic ellipsoids (§2.3) provide local domains of comparison. The metrics and the limits in Theorem 1 can be computed explicitly for these domains. (They are complex-linearly equivalent to the unit ball.) The reduction to local questions is made possible by the monotonicity property of the metrics (§2.1), together with (a) for the Carathéodory metric, Theorem 2, and (b) for the Kobayashi metric, and estimate of Royden (Lemma 4).

Related work on the Bergman metric has been done by Diederich ([4], [5]). In [4] he obtains statement (i) in our Theorem 1 for the Bergman metric with a factor of $(n+1)^{1/2}$ on the right-hand side. Statement (ii) with a factor of $n+1$ on the right-hand side is contained in [5].

Recent work on the boundary behavior of the Bergman kernel has been done by Fefferman [6].

The author is indebted to N. Kerzman for advice and encouragement as well as for the idea for the proof of Theorem 2. He also wishes to thank H. L. Royden for pointing out the relevance of his estimate for the Kobayashi metric.

## 2. The Carathéodory and Kobayashi metrics.

2.1. *General properties.* See §1 for definitions of the metrics and corresponding distance functions. These definitions can be given for arbitrary complex manifolds, but the metrics may be zero in some directions, and the distances zero for distinct pairs of points, in this case. $F$ without a subscript $C$ or $K$ refers to either metric unless specified otherwise. We have

(i) *Regularity properties.*

(a) Since $\Delta(G)$ is a normal family, the supremum in the definition of $F_C(z, \xi)$ is assumed by some $f \in \Delta(G)$. Necessarily $f(z) = 0$ [21]. $F_C$ is continuous in $(z, \xi)$ by a normal families argument [21, pp. 29–30]. It is not in general Hermitian [21, Satz 10].

(b) The Kobayashi metric is in general only upper semicontinuous [22]. It is continuous in case $G(\Delta)$ is a normal family by the same argument as in (a). Kerzman [16] has shown that this is the case if $G$ is a bounded domain of holomorphy in $\mathbf{C}^n$ with $\mathbf{C}^1$ boundary.

(ii) *Homogeneity in* $\xi$. If $c \in \mathbf{C}$, $F(z, c\xi) = |c|F(z, \xi)$.

(iii) The Carathéodory metric is subadditive in $\xi$. For if $f \in \Delta(G)$ and $\xi, \eta \in \mathbf{C}^n$,

$$\left|\sum_{i=1}^{n} \frac{\partial f}{\partial z_i}(z)(\xi_i + \eta_i)\right| \leqslant \left|\sum_{i=1}^{n} \frac{\partial f}{\partial z_i}(z)\xi_i\right| + \left|\sum_{i=1}^{n} \frac{\partial f}{\partial z_i}(z)\eta_i\right|.$$

(iv) If $G \subset \mathbf{C}^n$, $G' \subset \mathbf{C}^m$ are bounded domains, and $\Phi\colon G \longrightarrow G'$ is a holomorphic mapping, then $F_{G'}(\Phi(z), \Phi_*\xi) \leqslant F_G(z, \xi)$. Hence

(v) $F$ is preserved by biholomorphic mappings; and

(vi) *Monotonicity.* If $G \subset G'$ and $\xi$ lies in the tangent space of $G$ then $F_G(z, \xi) \geqslant F_{G'}(z, \xi)$. For (iv) applies to the inclusion mapping.

(vii) On the unit disk $\Delta$ both metrics coincide with the Poincaré metric, i.e. $F(z, \xi) = |\xi|/(1 - |z|^2)$. Because the automorphism group of the disk is transitive it suffices to check the equality at the origin. Here it follows from the Schwarz lemma.

2.2. *The metrics on the unit ball in* $\mathbf{C}^n$. Let $B_n$ denote the unit ball in $\mathbf{C}^n$. If $w, \eta \in \mathbf{C}^n$ we write $\langle w, \eta\rangle = \Sigma_{i=1}^{n} w_i \bar{\eta}_i$.

PROPOSITION 1. *The Carathéodory and Kobayashi metrics on $B_n$ are given by*

$$(F(w, \eta))^2 = \frac{\|\eta\|^2}{1 - \|w\|^2} + \frac{|\langle w, \eta\rangle|^2}{(1 - \|w\|^2)^2}. \tag{1}$$

PROOF. (This is the Bergman metric on $B_n$ except that the factor $(n + 1)$

is missing. Cf. [23, p. 23].) This expression is invariant under unitary transformations and under the automorphism

$$w_1' = \frac{w_1 - a}{1 - aw_1}, \quad w_j' = \frac{\sqrt{1-a^2} \cdot w_j}{1 - aw_1}, \qquad j = 2, \cdots, n,$$

which takes the point $(a, 0, \cdots, 0)$ on the real $w_1$ axis to the origin.

Hence it is invariant under the full automorphism group of the ball. Since this group is transitive on directions as well as points, the metrics must coincide with (1) except possibly for constant factors.

We claim that both metrics must reduce to the Poincaré metric on the unit disk $\Delta_1$ in (say) the $w_1$ plane. Thus no constant factors appear. For, by monotonicity, $F_{B_n}(w, \xi) \leqslant F_{\Delta_1}(w, \xi)$ for $w \in \Delta_1$ and $\xi$ in the $w_1$ plane. To prove the opposite inequality let $\pi$: $\mathbf{C}^n \longrightarrow \mathbf{C}$ be the projection onto the first coordinate.

(a) *Carathéodory metric.* The function $f \in \Delta(\Delta_1)$ which maximizes $|f_*(\xi)|$ extends to a function $\tilde{f} \in \Delta(B_n)$ by composition with $\pi$.

(b) *Kobayashi metric.* Suppose there were a vector $\xi$ in the $w_1$ plane and a mapping $f \in B_n(\Delta)$ such that $f(0) = 0$, $f'(0) = \xi/\alpha$, with $\alpha < F_{\Delta_1}(0, \xi)$. Then $\pi \circ f \in \Delta_1(\Delta)$ and $(\pi \circ f)'(0) = \xi/\alpha$, a contradiction.

2.3. *The metrics on analytic ellipsoids.* Let $(a_{ij})_{i,j=1}^n$ be a Hermitian positive definite matrix. We write $H(\xi, z) = \Sigma_{i,j=1}^n a_{ij}\xi_i\bar{z}_j$. In the terminology of [4] we introduce

DEFINITION 3. An analytic ellipsoid is a domain

$$E = \{z \in \mathbf{C}^n \mid \phi_E(z) = -z_1 - \bar{z}_1 + H(z, z) < 0\}.$$

Given positive constants $0 < c < C$ we denote by $\mathcal{E}(c, C)$ the set of analytic ellipsoids $E$ for which $c\|z\|^2 \leqslant H(z, z) \leqslant C\|z\|^2$. Notice that $H(\xi, \xi) = L_{\phi_E}(\xi)$.

PROPOSITION 2. *The Carathéodory and Kobayashi metrics on E are both given by*

$$(F_E(z, \xi))^2 = \frac{H(\xi, \xi)}{-\phi_E(z)} + \left| \frac{H(\xi, z) - \xi_1}{-\phi_E(z)} \right|^2. \tag{2}$$

PROOF. We follow the computation of the Bergman kernel function for such domains in [4] and [15]. First diagonalize $\Sigma_{i,j=2}^n a_{ij}z_i\bar{z}_j$ by a unitary transformation in the variables $z_2, \cdots, z_n$. (2) is invariant. The linear transformation

$$v_1 = z_1, \quad v_k = z_k + a_{1k}z_1/a_{kk}, \qquad k = 2, \cdots, n,$$

removes the remaining off-diagonal terms. $E$ is transformed into

$$\left\{ v \in C^n \middle| a_0|v_1 - 1/a_0|^2 + \sum_{k=2}^{n} a_{kk}|v_k|^2 < 1/a_0 \right\}$$

where $a_0 = a_{11} - \Sigma_{k=2}^{n} |a_{1k}|^2/a_{kk}$. Now if we set

$$w_1 = a_0(v_1 - 1/a_0), \quad w_k = (a_0 a_{kk})^{1/2} v_k, \qquad k = 2, \cdots, n,$$

the transformed domain becomes the unit ball. So if in (1) we make the substitutions (still assuming $(a_{ij})_{i,j=2}^{n}$ is diagonal)

$$w_1 = a_0 z_1 - 1,$$

$$w_k = (a_0 a_{kk})^{1/2}(z_k + a_{1k} z_1/a_{kk}), \qquad k = 2, \cdots, n,$$

$$\eta_1 = a_0 \xi_1,$$

$$\eta_k = (a_0 a_{kk})^{1/2}(\xi_k + a_{1k}\xi_1/a_{kk}), \qquad k = 2, \cdots, n,$$

$$\begin{aligned} 1 - \|w\|^2 &= a_0 \left( v_1 + \bar{v}_1 - a_0|v_1|^2 - \sum_{k=2}^{n} a_{11}|v_k|^2 \right) \\ &= a_0(z_1 + \bar{z}_1 - H(z, z)), \end{aligned}$$

we obtain (2).

2.4. *Evaluation of limits for analytic ellipsoids.* We shall evaluate the limits in Theorem 1 at the point $z_0 = 0$ of $\partial E$. We impose the restriction $z \in \Lambda$ in both limits. $\Lambda$, a cone with vertex at 0 and axis the positive real $z_1$ axis, is given by $\Lambda = \{z \in \mathbf{C}^n | \operatorname{Re} z_1 > k\|z\|\}$ for some $k \in (0, 1)$. We replace $d(z, \partial G)$ by $\operatorname{Re} z_1$ for the moment.

PROPOSITION 3. *Let $E = \{z \in \mathbf{C}^n | \phi_E(z) = -z_1 - \bar{z}_1 + H(z, z) < 0\}$ be an analytic ellipsoid. Let $F_E(z, \xi)$ be either the Carathéodory or Kobayashi metric on $E$. Then*

$$\lim_{z \to 0; z \in \Lambda} F_E(z, \xi) \operatorname{Re} z_1 = \tfrac{1}{2} |\xi_1|. \tag{3}$$

*In addition if $\xi_1 = 0$ then*

$$\lim_{z \to 0; z \in \Lambda} (F_E(z, \xi))^2 \operatorname{Re} z_1 = \tfrac{1}{2} H(\xi, \xi) = \tfrac{1}{2} L_{\phi_E}(\xi). \tag{4}$$

REMARK. Suppose $\Lambda$ and constants $0 < c < C$ are fixed. Then the limits (3) and (4) are approached uniformly in $E \in \mathcal{E}(c, C)$ and in unit vectors $\xi$. (This

means for (3) that given $\epsilon > 0$ there is a neighborhood $W$ of 0 such that $|F_E(z, \xi)\mathrm{Re}\, z_1 - \frac{1}{2}|\xi_1|| < \epsilon$ when $z \in E \cap W$ for all $E$ and all unit vectors $\xi$.)

PROOF. Since $\Sigma_{i,j=1}^n a_{ij} z_i \bar{z}_j = O(\|z\|^2)$,

$$\lim_{z \to 0; z \in \Lambda} \frac{\mathrm{Re}\, z_1}{-\phi_E(z)} = \frac{1}{2}. \tag{5}$$

From (2) we obtain

$$\lim_{z \to 0; z \in \Lambda} (F_E(z, \xi)\ \mathrm{Re}\ z_1)^2 = \tfrac{1}{4} \lim_{z \to 0} |H(\xi, z) - \xi_1|^2 = \tfrac{1}{4}|\xi_1|^2 \tag{6}$$

and if $\xi_1 = 0$,

$$\lim_{z \to 0; z \in \Lambda} (F_E(z, \xi))^2 \mathrm{Re}\ z_1 = \tfrac{1}{2} H(\xi, \xi). \tag{7}$$

Note that $-\xi_1$ is the normal component of $\xi$ here. The uniform statement is clear.

## 3. Strongly pseudoconvex domains.

### 3.1. *General remarks.*

DEFINITION 4. A strongly pseudoconvex domain in $\mathbf{C}^n$ with $C^2$ boundary is a bounded domain $G$ in $\mathbf{C}^n$ for which there exist a neighborhood $U$ of $\partial G$ and a real-valued function $\phi \in C^2(U)$ such that

(i) $G \cap U = \{z \in U | \phi(z) < 0\}$;

(ii) $\phi$ is strictly plurisubharmonic in $U$, i.e.

$$\sum_{i,j=1}^n \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j}(z) \xi_i \bar{\xi}_j > 0 \quad \text{for } \xi \in \mathbf{C}^n - \{0\} \text{ and } z \in U;$$

for $\xi \in \mathbf{C}^n - \{0\}$ and $z \in U$;

(iii) $\nabla \phi(z) \neq 0$ in $U$.

That this is equivalent to the usual definition of strong pseudoconvexity is shown in [11, pp. 263–264]. Thus we may take the defining function $\phi$ for $\partial G$ in Theorem 1 to be strictly plurisubharmonic. We may simultaneously assume $\|\nabla_z \phi(z_0)\| = 1$ for all $z_0 \in \partial G$, where $\nabla_z \phi = (\partial\phi/\partial z_1, \cdots, \partial\phi/\partial z_n)$. (Replace $\phi$ by $\chi(\phi/|\nabla_z \phi|)$, where $\chi$ satisfies $\chi'(0) = 1$ and is otherwise suitably chosen [11, pp. 263–264].)

The complex tangent space to the boundary at a point $z_0 \in \partial G$ is the set $\{\xi \in \mathbf{C}^n | \Sigma_{i=1}^n\ \partial\phi/\partial z_i\ (z_0)\xi_i = 0\}$. One can decompose a vector $\xi \in \mathbf{C}^n$ into complex normal and tangential components at $z_0$: $\xi = \xi_N(z_0) + \xi_T(z_0)$. This decomposition extends to points $z \in G$ which are sufficiently near $\partial G$ that there is a unique boundary point $z_0$ at minimum distance from $z$.

We refer to [11, Chapter 9], for elementary properties of the Levi form

$$L_{\phi,z_0}(\xi) = \sum_{i,j=1}^{n} \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j}(z_0)\xi_i\bar{\xi}_j.$$

Since $G$ is bounded there exist positive constants $k$, $K$ such that $k\|\xi\|^2 \leqslant L_{\phi,z_0}(\xi) \leqslant K\|\xi\|^2$ for all $z_0 \in \partial G$ and $\xi \in \mathbf{C}^n$. Also suppose $\tilde{\phi}$ is a $C^2$ function compactly supported in $U$, and $\epsilon > 0$ is sufficiently small. Then $\phi - \epsilon\tilde{\phi}$ is strictly plurisubharmonic, and $\tilde{G} = \{z \in U | (\phi - \epsilon\tilde{\phi})(z) < 0\} \cup (G - U)$ is strongly pseudoconvex.

The expression

$$p_{z_0}(z) = \sum_{i=1}^{n} \frac{\partial \phi}{\partial z_i}(z_0)\,(z_i - (z_0)_i)$$

$$+ \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 \phi}{\partial z_i \partial z_j}(z_0)\,(z_i - (z_0)_i)\,(z_j - (z_0)_j)$$

is the Levi polynomial at $z_0$. Expanding $\phi$ about the boundary point $z_0$ we obtain

$$\phi(z) = 2\,\mathrm{Re}(p_{z_0}(z)) + L_{\phi,z_0}(z - z_0) + o(\|z - z_0\|^2).$$

Since $L_{\phi,z_0}$ is positive definite and $\phi(z) < 0$ in $G \cap U$ there is a neighborhood $V$ of $z_0$ for which $\mathrm{Re}\, p_{z_0}(z) < 0$ in $V \cap G$. This is of importance for the construction of peak functions.

§§ 3.2 and 4 depend on properties of certain solutions of the $\bar{\partial}$-problem for (0, 1) forms on strongly pseudoconvex domains. The interior estimates of the Kohn solution [19] suffice for the construction of holomorphic peak functions. The approximation theorem (§4) requires the $L^\infty$-estimates of one of the more recent solutions (Grauert-Lieb [9], Henkin [13]).

3.2. *Dependence of holomorphic peak functions on the boundary point.*

DEFINITION 5. Let $G$ be a strongly pseudoconvex domain in $\mathbf{C}^n$. Let $z_0 \in \partial G$. A peak function on $G$ at $z_0$ is a function $\psi$ such that

(i) $\psi$ is holomorphic on a neighborhood of $\bar{G}$;

(ii) $\psi(z_0) = 1$;

(iii) $|\psi(z)| < 1$ on $\bar{G} - \{z_0\}$.

The existence of peak functions is well known. It is their dependence on the boundary point $z_0$ which is of importance here.

PROPOSITION 4. *Let $G$ be a strongly pseudoconvex domain in $\mathbf{C}^n$ with $C^2$ boundary. There exist a neighborhood $\hat{G}$ of $G$, and a function $\Psi$: $\partial G \times \hat{G} \longrightarrow \mathbf{C}$, such that*

(i) *$\Psi(z_0, z)$ is jointly continuous in $z_0$ and $z$, and holomorphic in $z \in \hat{G}$ for fixed $z_0$;*

(ii) *for each* $z_0 \in \partial G$, $\Psi(z_0, \cdot)$ *is a peak function on* $G$ *at* $z_0$.

REMARKS. 1. The mapping $z_0 \longrightarrow \Psi(z_0, \cdot)$ is continuous in $L^\infty(G)$ by uniform continuity.

2. If $G$ has $C^{k+2}$ boundary then $\Psi \in C^k(\partial G \times \hat{G})$. By the Cauchy integral formula any derivative $D^\alpha_{z_0} D^\beta_z \Psi$ with $|\alpha| \leqslant k$, $\beta$ arbitrary, is continuous in $(z_0, z)$.

3. By replacing $\Psi$ by $\frac{1}{4}(3 + \Psi)$ one can assume $|\Psi| \geqslant \frac{1}{2}$ (in particular $\Psi \neq 0$) on $\partial G \times G$.

We recall the following facts from [19]: Let $\tilde{G}$ be a strongly pseudoconvex domain with $C^\infty$ boundary. Let $g$ be a $C^\infty$ differential form of type (0, 1) in $L^2(\tilde{G})$, $\bar{\partial} g = 0$. Then there is a unique $C^\infty$ function $u$ on $\tilde{G}$ such that

(i) $\bar{\partial} u = g$;

(ii) $u$ is orthogonal to the holomorphic functions on $\tilde{G}$.

In this section we shall write $u = Sg$ for this solution. The operator $S$ is linear and bounded in $L^2$:

$$\|u\|_{L^2(\tilde{G})} \leqslant C\|g\|_{L^2(\tilde{G})} \quad \text{where} \quad C = C(\tilde{G}).$$

We shall write $\|g\|_{L^\infty(\tilde{G})}$ for any of the equivalent $L^\infty$ norms on (0, 1) forms with bounded coefficients. Also $C$ will be used in this section to denote different constants.

LEMMA 1. *Let $U$ be an arbitrary open set in $\mathbf{C}^n$. Let $K$ be a compact subset of $U$. Any function $u \in C^\infty(U)$ satisfies an estimate*

$$\sup_K |u| \leqslant C(\|u\|_{L^2(U)} + \|\bar{\partial} u\|_{L^\infty(U)}) \tag{1}$$

*where $C = C(K, U)$.*

PROOF. In one variable this is obtained from the Cauchy integral formula. Since we can find $r > 0$ such that $r < d(K, \partial U)$, it suffices to show the following: if $u \in C^\infty(\overline{B_r})$ ($B_r$ is the ball of radius $r$ in $\mathbf{C}^n$ centered at 0), then

$$|u(0)| \leqslant C(\|u\|_{L^2(B_r)} + \|\bar{\partial} u\|_{L^\infty(B_r)}) \tag{2}$$

or

$$|u(0)|^2 \leqslant C(\|u\|^2_{L^2(B_r)} + \|\bar{\partial} u\|^2_{L^\infty(B_r)}). \tag{3}$$

We obtain (3) by integrating the corresponding result for the disk of radius $r$ over all directions.

LEMMA 2. *Let $\tilde{G} \subset\subset \mathbf{C}^n$ be strongly pseudoconvex with $C^\infty$ boundary. Let $K$ be a compact subset of $\tilde{G}$. Let $g$ be a $\bar{\partial}$-closed $C^\infty(0, 1)$-form on $\tilde{G}$, $g \in L^\infty(\tilde{G})$, and $u = Sg$. Then* $\sup_K |u| \leqslant C\|g\|_{L^\infty(\tilde{G})}$ *where $C = C(K, \tilde{G})$.*

PROOF. $u$ is a $C^\infty$ function. Lemma 1 and the $L^2$ boundedness of $S$ give

$$\begin{aligned}\sup_K |u| &\leqslant C(\|u\|_{L^2(\tilde{G})} + \|\bar{\partial}u\|_{L^\infty(\tilde{G})})\\ &\leqslant C(\|g\|_{L^2(\tilde{G})} + \|g\|_{L^\infty(\tilde{G})})\\ &\leqslant C\|g\|_{L^\infty(\tilde{G})}\end{aligned}$$

for (different) constants $C$.

LEMMA 3. *Let $M$ be a compact subset of $\mathbf{R}^m$. Let $\tilde{G} \subset\subset \mathbf{C}^n$ be strongly pseudoconvex with $C^\infty$ boundary. Let $\{g_x\}_{x\in M}$ be bounded $C^\infty\bar{\partial}$-closed* (0, 1) *forms on $\tilde{G}$ whose dependence on $x \in M$ is continuous in $L^\infty(G)$. Let $u_x = Sg_x$. Then $u(x, z) = u_x(z)$ is a continuous function on $M \times \tilde{G}$.*

PROOF. By Lemma 2 and the linearity of $S$, $u_x \longrightarrow u_{x_0}$ as $x \longrightarrow x_0$ uniformly on compact subsets of $\tilde{G}$. This implies the joint continuity of $u(x, z)$.

PROOF OF PROPOSITION 4. Let $z_0 \in \partial G$. There exists a neighborhood $V$ of $z_0$ of uniform size for which Re $p_{z_0}(z) < 0$ in $V \cap G$. (See §3.1. Uniform size means as usual that, for each $z_0 \in \partial G$, $V$ is the translate of a fixed neighborhood of the origin.)

Let $A(z_0) = \{z \in \mathbf{C}^n \mid p_{z_0}(z) = 0\}$. We claim there exist balls $B_1$, $B_2$ of uniform size centered at $z_0$ such that $B_2 \subset\subset B_1 \subset\subset V$, and a strongly pseudoconvex neighborhood $G'$ of $\bar{G}$, such that $(B_1 - B_2) \cap G' \cap A(z_0) = \varnothing$. (The subscripts 1, 2 do not refer to the radii of the balls.) For let $\epsilon > 0$ be smaller than the eigenvalues of $L_{\phi,z_0}$ for all $z_0 \in \partial G$. Choose $B_1$ of uniform size such that

$$A(z_0) \cap B_1 \cap \{z \in U \mid \phi(z) - \epsilon\|z - z_0\|^2 = 0\} = \{z_0\}.$$

($U$ is the domain of definition of $\phi$.) Now take $B_2$ to be any smaller ball, of radius $r$ say, and put

$$G' = \{z \in U \mid \phi(z) < \epsilon r^2\} \cup (G - U).$$

Finally let $\tilde{G}$ be a strongly pseudoconvex domain with $C^\infty$ boundary such that $G \subset\subset \tilde{G} \subset\subset G'$. It is on $\tilde{G}$ that we shall solve $\bar{\partial}$.

The balls $B_1$ and $B_2$ are obtained by translating fixed balls $B_1(0)$ and $B_2(0)$ centered at 0. Let $\chi$ be a $C^\infty$ function on $\mathbf{C}^n$ such that

(i) $\chi$ is supported in $B_1(0)$;

(ii) $\chi(z) = 1$ for $z \in B_2(0)$;

(iii) $0 \leqslant \chi(z) \leqslant 1$.

Define $\chi_{z_0}(z) = \chi(z - z_0)$.

Note that $\mathrm{Re}(\chi_{z_0}/p_{z_0}) \leqslant 0$ on $G$. Also the (0, 1) form $g_{z_0} = \bar{\partial}(\chi_{z_0}/p_{z_0})$ has bounded $C^\infty$ coefficients on $G'$ and is $\bar{\partial}$-closed. As $z_0$ varies over $\partial G$ the coefficients are jointly continuous in $z$ and $z_0$. Hence on $\partial G \times \tilde{G}$ they are uniformly continuous. The conditions of Lemma 3 are therefore satisfied with $M = \partial G$. The solutions $u_{z_0} = Sg_{z_0}$ give a function $u(z_0, z) = u_{z_0}(z)$ which is continuous on $\partial G \times \tilde{G}$. Replacing $\tilde{G}$ by a slightly smaller domain (with $G \subset\subset \tilde{G}$ still) we may assume $\sup_{\partial G \times \tilde{G}} |u| < k$ for some $k > 0$. Then the functions $\chi_{z_0}/p_{z_0} - u_{z_0} - k$ are meromorphic on $\tilde{G}$ and have negative real part on $\tilde{G}$.

The linear fractional transformation $h(w) = (w + 1)/(w \dot{-} 1)$ maps the left half plane onto the unit disk, taking $\infty$ to 1. Thus if we define

$$\Psi(z_0, z) = h(\chi_{z_0}/p_{z_0} - u_{z_0} - k),$$

properties (ii) and (iii) of peak functions are satisfied.

Each function $\psi_{z_0} = \Psi(z_0, \cdot)$ is holomorphic on $G \cup (\tilde{G} - B_2)$. On $B_2$,

$$\psi_{z_0} = \frac{1 - (u_{z_0} + k - 1)p_{z_0}}{1 - (u_{z_0} + k + 1)p_{z_0}}.$$

This is holomorphic when the denominator is not zero. Since $\|u_{z_0}\|_{L^\infty(\tilde{G})} < k$ and the Levi polynomials are equicontinuous on $\tilde{G}$, there is a neighborhood $\hat{G} \subset \tilde{G}$ of $\bar{G}$ on which the functions $\psi_{z_0}$ are all holomorphic. The continuity of $\Psi(z_0, z)$ on $\partial G \times \hat{G}$ follows from that of $u(z_0, z)$, $p_{z_0}(z)$, and $\chi_{z_0}(z)$.

3.3. *Completeness of the Carathéodory and Kobayashi distances.*

PROPOSITION 5. *Let $G \subset\subset \mathbf{C}^n$ be strongly pseudoconvex with $C^2$ boundary. For any $z \in G$ and $R \geqslant 0$, the subset $G_R(z) = \{w \in G \mid C(z, w) \leqslant R\}$ of $G$ is compact.*

PROOF. That $G_R(z)$ has compact closure contained in $G$ is shown in Satz 4 of [24] (a straightforward argument using peak functions).

$C(z, w)$ is a continuous function of $w$ by a normal families argument [21, pp. 17–18]. Hence $G_R(z)$ is closed in $G$.

REMARK. This implies the completeness of the Kobayashi distance on $G$, an unpublished result of Royden (verbal communication). For $C(z, w) \leqslant D(z, w)$, and on a bounded domain the Kobayashi distance defines the usual topology [18].

**4. Holomorphic approximation on the level sets of peak functions.** The idea for the following theorem is due to N. Kerzman.

THEOREM 2. *Let $G \subset\subset \mathbf{C}^n$ be strongly pseudoconvex with $C^2$ boundary. Let $z_0 \in \partial G$. Let $\psi$ be a peak function on $G$ at $z_0$. Let $0 < a < b < 1$, and*

*let* $A = \{z \in G \mid |\psi(z)| > a\}$, *and* $B = \{z \in G \mid |\psi(z)| > b\}$. *Choose an integer* $m \geqslant 0$ *and an arbitrary small positive* $\eta$. *Then there exists a constant* $L = L(G, a, b, m, \eta)$ *such that the following holds: given* $f \in H^\infty(A)$, *there exists* $\hat{f} \in H^\infty(G)$ *such that*

(i) $\|D^\alpha \hat{f} - D^\alpha f\|_{L^\infty(B)} \leqslant \eta \|f\|_{L^\infty(A)}$, $0 \leqslant |\alpha| \leqslant m$.

(ii) $\|\hat{f}\|_{L^\infty(G)} \leqslant L\|f\|_{L^\infty(A)}$.

*L can be chosen independently of* $z_0$ *if the peak function* $\psi$ *depends continuously on* $z_0$ *as in* §3.2.

REMARK. At a given point $w \in B$ we can require

(iii) $D^\alpha \hat{f}(w) = D^\alpha f(w)$, $0 \leqslant |\alpha| \leqslant m$,

by adding to $\hat{f}$ a polynomial of degree $m$ with small coefficients (and changing $L$).

We shall consider solutions of the equation $\bar{\partial} u = g$ on a strongly pseudoconvex domain $\hat{G} \supset G$ which satisfy $L^\infty$ estimates. ($g$ will be a (0, 1) form related to $f$.) Such solutions were first obtained by Grauert-Lieb [9] and Henkin ([12], [13]). By modifying Henkin's construction Ovrelid [20] reduced the differentiability assumption on $\partial G$ to $C^2$. We shall write $S$ for the Henkin-Ovrelid solution operator. Thus $\|Sg\|_{L^\infty(\hat{G})} \leqslant C\|g\|_{L^\infty(\hat{G})}$. A construction of Kerzman [17] which yields $L^\infty$ estimates for $\bar{\partial}$ also works for strongly pseudoconvex domains in Stein manifolds.

The key observation here is that $\psi^{-r}S(\psi^r g)$ also solves the given $\bar{\partial}$ problem for any positive integer $r$. (We can assume $|\psi| > \frac{1}{2}$ on $G$ by replacing $\psi$ by $\frac{1}{4}(3 + \psi)$.) $L^\infty$ estimates for $\psi^{-r}S(\psi^r g)$ *on* $B$ improve as $r$ increases. Estimates for the derivatives are obtained from the Cauchy estimates.

PROOF OF THEOREM 2. We may assume $b > \inf_{z \in G} |\psi(z)|$. Hence we may also assume $a > \inf_{z \in G} |\psi(z)|$. Choose $a'$ and $b'$ such that $a < a' < b' < b$. Set $A' = \{z \in G \mid |\psi(z)| > a\}$, $B' = \{z \in G \mid |\psi(z)| > b'\}$. Let $\hat{G}$ be a strongly pseudo convex perturbation of $G$ (§3.1) such that

(i) $B \subset\subset \hat{G}$,

(ii) $\hat{G} - B' = G - B'$,

(iii) $|\psi| \geqslant b'$ on $\hat{G} - G$.

*Note.* With $a, a', b', b$ fixed the same $\hat{G}$ satisfies these conditions for nearby boundary points of $G$.

Set $\hat{B} = (\hat{G} - G) \cup B'$. There is a function $\chi \in C^\infty(G)$ and positive constants $\delta$, $k$ such that $0 \leqslant \chi \leqslant 1$, $\chi = 1$ on $A'$, $\chi = 0$ on $G - A$, $|\nabla \chi| \leqslant k$ independently of $z_0 \in \partial G$, and $d(B, G - B') > \delta$ independently of $z_0 \in \partial G$. We may assume $d(B, \partial \hat{B}) > \delta$.

Given $f \in H^\infty(A)$, define a (0, 1) form $g$ on $\hat{G}$ by

$$g = \begin{cases} (\bar{\partial}\chi)f & \text{on } A, \\ 0 & \text{on } \hat{G} - A. \end{cases}$$

Then $g \in L^\infty(\hat{G})$ and $\bar{\partial}g = 0$. Note that $g = 0$ on $A'$, hence $|\psi| \leqslant a'$ on the support of $g$. Define $\hat{f} = \chi f - \psi^{-r}S(\psi^r g)$ where $r$ is an integer to be chosen. Clearly $\hat{f} \in H^\infty(\hat{G})$, and

$$\begin{aligned} \|\hat{f} - f\|_{L^\infty(\hat{B})} &= \|\psi^{-r}S(\psi^r g)\|_{L^\infty(\hat{B})} \\ &\leqslant C(b')^{-r}\|\psi^r g\|_{L^\infty(\hat{G})} \\ &\leqslant Ck(a'/b')^r\|f\|_{L^\infty(A)}. \end{aligned}$$

($C$ is the constant in the $L^\infty$ estimate for $S$. $k$ is related to the peak function $\psi$.) Since $\psi^{-r}S(\psi^r g)$ is holomorphic on $\hat{B}$ ($g = 0$ on $\hat{B}$) the Cauchy estimates give

$$\|D^\alpha \hat{f} - D^\alpha f\|_{L^\infty(B)} \leqslant \alpha!\, \delta^{-|\alpha|}Ck(a'/b')^r\|f\|_{L^\infty(A)}$$

By choosing $r$ such that $m!\, \delta^{-m}Ck(a'/b')^r < \eta$ we obtain the first statement in Theorem 2. Setting $t = \inf_{z\in G}|\psi(z)|$ we have

$$\begin{aligned} \|\hat{f}\|_{L^\infty(G)} &\leqslant \|f\|_{L^\infty(A)} + \|\psi^{-r}S(\psi^r g)\|_{L^\infty(G)} \\ &\leqslant [1 + t^{-r}Ck(a'/b')^r]\,\|f\|_{L^\infty(A)}. \end{aligned}$$

Thus $L = 1 + t^{-r}Ck(a'/b')^r$. As noted in §3.2 we may assume $t \geqslant ½$.

The constant $C$ depends on the domain $\hat{G}$. However since we can use a fixed $\hat{G}$ for all boundary points in a neighborhood of a fixed $z_0$, only finitely many domains $\hat{G}$ need be considered. Thus $L$ may be chosen independently of $z_0$ once the other parameters are fixed.

REMARK. The result holds for strongly pseudoconvex domains (with $C^4$ boundary) in Stein manifolds. One defines derivative norms with respect to a given covering of the compact set $\bar{G}$ by coordinate patches. For $S$ one uses the Kerzman solution operator [17] which works in the manifold case also.

## 5. Proof of Theorem 1.

THEOREM 1. *Let $G$ be a strongly pseudoconvex domain in $\mathbf{C}^n$ with $C^2$ boundary. Let $F(z, \xi)$ be either the Carathéodory or Kobayashi metric on $G$. Let $z_0 \in \partial G$. Let $\phi$ be a $C^2$ defining function for $\partial G$ such that $\|\nabla_z\phi(z_0)\| = 1$. Then*

$$\lim_{z\to z_0} F(z, \xi)d(z, \partial G) = ½\|\xi_N(z_0)\|.$$

*If $\xi_N(z_0) = 0$, i.e. $\xi$ is a holomorphic tangent vector to $\partial G$ at $z_0$, then*

$$\lim_{z\to z_0;\, z\in\Lambda} (F(z,\xi))^2 d(z,\partial G) = \tfrac{1}{2} L_{\phi,z_0}(\xi)$$
$$= \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2\phi}{\partial z_i \partial \bar{z}_j}(z_0)\xi_i\bar{\xi}_j .$$

*If the aperture of $\Lambda$ is fixed then both limits are approached uniformly in vectors $\xi$ of unit length and in the boundary point $z_0$.*

5.1. *Localization of the metrices on strongly pseudoconvex domains.* In the statement of Proposition 6, $F$ refers to either the Carathéodory or Kobayashi metric.

PROPOSITION 6. *Let $G \subset\subset \mathbf{C}^n$ be strongly pseudoconvex with $C^2$ boundary. Let $z_0 \in \partial G$. Let $P$ be any neighborhood of $z_0$. Then for all vectors $\xi \in \mathbf{C}^n$*

$$\lim_{z\to z_0} \frac{F_{G\cap P}(z,\xi)}{F_G(z,\xi)} = 1.$$

REMARK. The localization is uniform in the sense that:

(i) Suppose $G$ is fixed and $P$ is a neighborhhod of the (variable) boundary point $z_0$ of uniform size. Then given $\epsilon > 0$ there exists a neighborhood $Q$ of $z_0$ of uniform size, such that $F_{G\cap P}(z,\xi)/F_G(z,\xi) \leqslant 1+\epsilon$ for $z \in G\cap Q$ and all $\xi \in C^n$.

(ii) Suppose $P$ is a fixed neighborhood of 0 and $G$ varies over a family $\mathcal{E}(c, C)$ of analytic ellipsoids (§2.3). Then given $\epsilon > 0$ there is a fixed neighborhood $Q$ of 0 such that, for all $E \in \mathcal{E}(c, C)$ and all $\xi \in \mathbf{C}^n$, $F_{E\cap P}(z,\xi)/F_E(z,\xi) \leqslant 1+\epsilon$ when $z \in E\cap Q$.

PROOF. (a) *Carathéodory metric.* The proof combines peak functions with an approximation theorem (Theorem 2) in a manner similar to Lemma 12 of [4].

Let $\psi$ be a peak function for $G$ at $z_0$. We shall assume $\psi$ depends continuously on the boundary point $z_0$ as in §3.2. There exists $a \in (0, 1)$ such that the set $A = \{z \in G \mid |\psi(z)| > a\}$ is contained in $G \cap P$ independently of $z_0$. Choose $b \in (a, 1)$ and let $B = \{z \in G \mid |\psi(z)| > b\}$. Choose an arbitrary small $\eta > 0$.

For each $w \in B$ and $\xi \in \mathbf{C}^n$, there exists a function $f \in \Delta(G\cap P)$ such that $F_{G\cap P}(w,\xi) = |\Sigma_{i=1}^n \partial f(w)/\partial z_i \xi_i|$. Necessarily $f(w) = 0$. Apply Theorem 2 to $f|_A$ with $m = 1$ and $\eta$ as given: there exists $\hat{f} \in H^\infty(G)$ such that $\|\hat{f} - f\|_{L^\infty(B)} \leqslant \eta$, $\hat{f}(w) = f(w) = 0$, $\partial\hat{f}(w)/\partial z_i = \partial f(w)/\partial z_i$, $i = 1, \cdots, n$, and $\|\hat{f}\|_{L^\infty(G)} \leqslant L = L(\eta)$. ($L$ is independent of $z_0$. The other parameters on which

$L$ depends will be fixed.) There is a positive integer $\beta$ such that $b^\beta L < 1$. We take $Q$ to be a ball of uniform size centered at $z_0$ for which $|\psi^\beta(z)| \geqslant 1 - \eta$ on $G \cap Q$.

Assume $w \in G \cap Q$. Define a holomorphic function $\tilde{f}$ on $G$ by $\tilde{f} = \psi^\beta \hat{f}$. Since $\hat{f}(w) = f(w) = 0$, we have $\partial\tilde{f}(w)/\partial z_i = \psi^\beta(w)\,\partial f(w)/\partial z_i$, and hence

$$\left| \sum_{i=1}^{n} \frac{\partial \tilde{f}}{\partial z_i}(w)\xi_i \right| \geqslant (1-\eta)F_{G\cap P}(w, \xi).$$

Now $\tilde{f} \notin \Delta(G)$ but $\|\tilde{f}\|_{L^\infty(G-B)} \leqslant b^\beta L \leqslant 1$, and

$$\|\tilde{f}\|_{L^\infty(B)} \leqslant \|\hat{f}\|_{L^\infty(B)} \leqslant \|f\|_{L^\infty(B)} + \eta \leqslant 1 + \eta.$$

We conclude that $F_G(w, \xi) \geqslant (1-\eta)/(1+\eta)\, F_{G\cap P}(w, \xi)$ for $w \in G \cap Q$.

This proves the proposition and the first remark. The second remark follows from the fact that any two analytic ellipsoids are biholomorphic, and that bounds on the coefficients of the ellipsoids give bounds on the sequence complex affine transformations in 2.3.

(b) *Kobayashi metric.* The following estimate was drawn to my attention by H. L. Royden. It appears without proof as Lemma 2 in [22].

LEMMA 4. *Let $G$ and $P$ be subdomains of a hyperbolic manifold*(2) *$M$. For $z \in G \cap P$ define*

$$D^*(z) = D_G^*(z, G-P) = \inf_{w \in G-P} D_G^*(z, w).$$

*Then* $F_{G\cap P}(z, \xi) \leqslant \coth(D^*(z))F_G(z, \xi)$.

(The Kobayashi distance $D_G(z, w)$ was defined in §1. $D_G^*(z, w)$ is defined as follows:

$$D_G^*(z, w) = \inf\{\rho(a, b) \mid \exists f \in G(\Delta) \text{ such that } f(a) = z, f(b) = w\}.$$

Thus $D_G^*(z, w) \geqslant D_G(z, w)$.)

PROOF. Note that $\coth D_G^*(z, w) = \sup\{r^{-1} \mid \exists f \in G(\Delta)$ such that $f(0) = z, f(r) = w\}$. Since coth is a decreasing function we have

$$\coth D^*(z) = \sup_{w \in G-P} D_G^*(z, w)$$

$$= \sup\{r^{-1} \mid \exists f \in G(\Delta) \text{ such that } f(0) = z, f(r) \in G - P\}.$$

Let $\Delta_r$ denote the disk of radius $r$ centered at 0.

Suppose $f \in G(\Delta)$, $f(0) = z$, $f'(0) = \xi/\alpha$ where $\alpha > 0$. If $r^{-1} > \coth D^*(z)$

(2) I.e. a complex manifold on which the Kobayashi pseudodistance is a distance. The proposition applies to bounded subdomains of $\mathbf{C}^n$.

then $f(\Delta_r) \subset G \cap P$. We conclude that $F_{G\cap P}(z, \xi) \leqslant \alpha/r$, and hence $F_{G\cap P}(z, \xi) \leqslant \alpha \coth D^*(z)$ since $r$ was arbitrary. Taking the infimum over $\alpha$ now gives the lemma.

END OF PROOF OF PROPOSITION 7 FOR $F = F_K$. Let $D(z) = \inf_{w\in G-P} D(z, w)$. Since coth is decreasing we may replace $D^*(z)$ by $D(z)$ in the statement of Lemma 1. From the completeness of the Kobayashi distance on $G$ (§3.3) it follows that $\coth D(z) \longrightarrow 1$ as $z \longrightarrow z_0$.

However, remark (i) is more easily obtained using the Carathéodory distance and peak functions. With the numbers $a$, $b$ and sets $A$, $B$ defined as in part (a) of the proof we have

$$D_G(z, w) \geqslant C_G(z, w) \geqslant \frac{1}{2}\log\frac{1+b}{1-b} - \frac{1}{2}\log\frac{1+a}{1-a}$$

for $z \in B$, $w \in G - A$. The right-hand side may be made arbitrarily large by choosing $b$ close to 1, and a neighborhood $Q$ of $z_0$ of uniform size may be chosen so that $G \cap Q \subset B$.

Remark (ii) follows as before.

5.2. *Choice of coordinates at the boundary point $z_0$. The local domains of comparison.* We may assume $\phi$ is strictly plurisubharmonic. Let $U$ be the neighborhood of $\partial G$ in which it is defined.

By a translation followed by a unitary transformation we may take $z_0$ to 0 and the vector $\nabla_z\phi(z_0)$ to the negative real $z_1$ axis. The statements in Theorem 1 are invariant. Taking into account the normalization of $\phi$, we have

$$\begin{aligned}\phi(z) = -z_1 - \bar{z}_1 &+ \frac{1}{2}\sum_{i,j=1}^{n}(c_{ij}z_iz_j + \bar{c}_{ij}\bar{z}_i\bar{z}_j)\\ &+ \sum_{i,j=1}^{n} a_{ij}\, z_i\bar{z}_j + o(\|z\|^2)\end{aligned} \tag{1}$$

where $(a_{ij})_{i,j=1}^n$ is Hermitian positive definite. Also $\xi_N(z_0) = -\xi_1$, and

LEMMA 5.

$$\lim_{z\to 0;\, z\in\Lambda} \frac{\operatorname{Re} z_1}{d(z, \partial G)} = 1.$$

*Furthermore if the aperture of $\Lambda$ is fixed and $z_0$ (and hence the choice of coordinates) varies over $\partial G$, the limit is approached uniformly.*

PROOF. Follows from $d(z, \partial G) = \operatorname{Re} z_1 + O(\|z\|^2)$ and the definition of the cone $\Lambda$ (§2.4).

We again consider the expansion (1). The mapping $\Phi$: $\mathbf{C}^n \longrightarrow \mathbf{C}^n$ given by

$$w_1 = z_1 - \frac{1}{2}\sum_{i,j=1}^{n} c_{ij} z_i z_j, \quad w_k = z_k, \qquad k = 2, \cdots, n,$$

is biholomorphic in a neighborhood $P$ of 0 of fixed size (as $z_0$ runs over $\partial G$). In terms of the $w$ coordinates $\phi$ has the form

$$\phi(w) = -w_1 - \bar{w}_1 + H(w, w) + o(\|w\|^2)$$

where $H(w, w) = \Sigma_{i,j=1}^{n} a_{ij} w_i \bar{w}_j = L_{\phi,0}(w)$. (The Levi form at 0 is preserved while the other second order terms have been removed.) We shall also write $I(w, w) = \Sigma_{i=1}^{n} |w_i|^2$.

The analytic ellipsoids $E_{\pm\epsilon}$ with defining functions

$$\phi_{E_{\pm\epsilon}}(w) = -w_1 - \bar{w}_1 + H(w, w) \mp \epsilon I(w, w)$$

provide local domains of comparison for $\Phi(G \cap P)$. ($\epsilon$ will be chosen less than the eigenvalues of $H$.) We shall use their images under $\Phi^{-1}$ as local domains of comparison for $G$.

LEMMA 6. *Let $E = \{w \in \mathbf{C}^n \mid -w_1 - \bar{w}_1 + H(w, w) < 0\}$ be an analytic ellipsoid. Let $D' = E \cap \Phi(P)$, and $D = \Phi^{-1}(D')$. Let $F_D(z, \xi)$ be either the Carathéodory or Kobayashi metric on $D$. Then*

$$\lim_{z \to 0; z \in \Lambda} F_D(z, \xi) \operatorname{Re} z_1 = \tfrac{1}{2}|\xi_1| \tag{2}$$

*and if $\xi_1 = 0$,*

$$\lim_{z \to 0; z \in \Lambda} (F_D(z, \xi))^2 \operatorname{Re} z_1 = \tfrac{1}{2} H(\xi, \xi). \tag{3}$$

*Furthermore suppose $P$ and the aperture of $\Lambda$ are fixed, and $\Phi$ ranges over the transformations $\Phi_{z_0}$, $z_0 \in \partial G$, and $E$ ranges over a family $\mathcal{E}(c, C)$. Then the limits are approached uniformly in the domains $D$ and in unit vectors $\xi$.*

PROOF. (i) As in Satz 1 of [4] we first show that for fixed $\Lambda$ there exist a cone $\Lambda'$ in $w$ space and a neighborhood $Q$ of 0 in $z$ space such that $\Phi(\Lambda \cap Q) \subset \Lambda'$. $\Phi$ is allowed to range over all transformations $\Phi_{z_0}$. Suppose $\Lambda = \{z \in \mathbf{C}^n \mid \operatorname{Re} z_1 > k\|z\|\}$. We have $|\operatorname{Re} w_1(z)| \geqslant |\operatorname{Re} z_1| - C\|z\|^2$, and $\|w(z)\| \leqslant \|z\| + C\|z\|^2$, independently of $z_0$. Hence

$$\frac{\operatorname{Re} w_1(z)}{\|w(z)\|} \geqslant \frac{\operatorname{Re} z_1 - C\|z\|^2}{\|z\| + C\|z\|^2} \geqslant \frac{k - C\|z\|}{1 + C\|z\|}.$$

We may find a suitable $Q$ for any $\Lambda'$ with $k' < k$.

(ii) Similarly we note that

$$\lim_{z \to 0; z \in \Lambda} \frac{\operatorname{Re} w_1(z)}{\operatorname{Re} z_1} = 1$$

uniformly in the transformations $\Phi_{z_0}$ for fixed $\Lambda$.

(iii) Now $F_D(z, \xi) = F_{D'}(w(z), \Phi_*(\xi))$. Hence for (2) we must show

$$\lim_{z\to 0; z\in\Lambda} \operatorname{Re} w_1(z)\,[F_{D'}(w(z), \xi) - F_{D'}(w(z), \Phi_*\xi)] = 0$$

uniformly in $D'$ and in unit vectors $\xi$. For the Carathéodory metric which is subadditive in $\xi$ (§2.1) it suffices to show

$$\lim_{z\to 0; z\in\Lambda} \operatorname{Re} w_1(z) F_{D'}(w(z), \xi - \Phi_*\xi) = 0.$$

This is sufficient for the Kobayashi metric also, for by the Royden estimate $F_{D'}(w, \xi)/F_E(w, \xi) \longrightarrow 1$ as $w \longrightarrow 0$. Hence choosing $C > 1$, we will have

$$F_{D'}(w, \xi_1 + \xi_2) \leqslant CF_{E'}(w, \xi_1 + \xi_2) \leqslant C(F_E(w, \xi_1) + F_E(w, \xi_2))$$

$$\leqslant C(F_{D'}(w, \xi_1) + F_{D'}(w, \xi_2))$$

in the intersection of $D'$ with a neighborhood of 0 of uniform size.

Now the matrix of $\Phi_*$ can be written $I + O(\|z\|)$ where $O(\|z\|)$ has nonzero entries only in the first row. If $e$ denotes the column vector $(1\ 0 \cdots 0)^T$, we have $\xi - \Phi_*\xi = O(\|z\|)e$ where $\|O(\|z\|)e\| \leqslant C\|z\|$ for some $C > 0$ independently of $z_0$. Thus

$$F_{D'}(w(z), \xi - \Phi_*\xi) \leqslant C\|z\| F_{D'}(w(z), e). \tag{4}$$

Since $\lim_{w\to 0; w\in\Lambda} \operatorname{Re} w_1 F_{D'}(w, e)$ exists uniformly in $D'$ we are done.

(iv) For (3) we must show that if $\xi_1 = 0$,

$$\lim_{z\to 0; z\in\Lambda} \operatorname{Re} w_1(z)\,[(F_{D'}(w(z), \xi))^2 - F_{D'}(w(z), \Phi_*\xi)^2] = 0$$

uniformly in $D'$ and in unit vectors. Again it suffices to show

$$\lim_{z\to 0; z\in\Lambda} \operatorname{Re} w_1(z) F_{D'}(w(z), \xi - \Phi_*\xi)\,[F_{D'}(w, \xi) + F_{D'}(w, \Phi_*\xi)] = 0.$$

This is clear because of (4) and because

$$\lim_{z\to 0; z\in\Lambda} \operatorname{Re} w_1(z)\,(F_{D'}(w, \xi) + F_{D'}(w, \Phi_*\xi))$$

exists, uniformly in the usual quantities.

5.3. *Main part of proof.* We are reduced to showing, with the usual remarks about uniformness,

$$\lim_{z\to 0; z\in\Lambda} F(z, \xi) \operatorname{Re} z_1 = \tfrac{1}{2}|\xi_1| \tag{5}$$

and, if $\xi_1 = 0$,

$$\lim_{z\to 0; z\in\Lambda} (F(z, \xi))^2 \operatorname{Re} z_1 = \tfrac{1}{2} L_{\phi,0}(\xi) = \tfrac{1}{2}H(\xi, \xi). \tag{6}$$

For by Lemma 5 we can replace $\operatorname{Re} z_1$ by $d(z, \partial G)$. Then by uniformness in $z_0$ we can delete the restriction $z \in \Lambda$ in the first limit. We cannot in the second, for the complex tangent space varies with the boundary point $z_0$. (See the example in §5.4.)

Choose an arbitrary $\epsilon > 0$ which is smaller than the eigenvalues of $H(\xi, \xi)$ for all $z_0 \in \partial G$. Let $P$ be a neighborhood of $z_0 = 0$ such that the following two conditions are satisfied independently of the point $z_0 \in \partial G$:

(i) the transformation $\Phi$ (§5.2) is biholomorphic in $P$;

(ii) setting $G_\epsilon = \Phi^{-1}(E_\epsilon \cap \Phi(P))$, we have

$$G_{-\epsilon} \subset G \cap P \subset G_\epsilon. \tag{7}$$

The first inclusion together with Lemma 6 gives

$$\overline{\lim_{z \to 0; z \in \Lambda}} F_{G \cap P}(z, \xi) \operatorname{Re} z_1 \leqslant \tfrac{1}{2} |\xi_1| \tag{8}$$

and, if $\xi_1 = 0$,

$$\overline{\lim_{z \to 0; z \in \Lambda}} (F_{G \cap P}(z, \xi))^2 \operatorname{Re} z_1 \leqslant \tfrac{1}{2} [H(\xi, \xi) + \epsilon I(\xi, \xi)]. \tag{9}$$

The second inclusion in (7) and Lemma 6 give

$$\underline{\lim_{z \to 0; z \in \Lambda}} F_{G \cap P}(z, \xi) \operatorname{Re} z_1 \geqslant \tfrac{1}{2} |\xi_1| \tag{10}$$

and, if $\xi_1 = 0$,

$$\underline{\lim_{z \to 0; z \in \Lambda}} (F_{G \cap P}(z, \xi))^2 \operatorname{Re} z_1 \geqslant \tfrac{1}{2} [H(\xi, \xi) - \epsilon I(\xi, \xi)]. \tag{11}$$

By Proposition 6 we may replace $F_{G \cap P}$ by $F_G$ in (8)–(11). (5) follows immediately, and (6) follows on letting $\epsilon \longrightarrow 0$.

It remains only to note that (5) and (6) hold uniformly in the boundary point $z_0$ (taken to be 0) and in unit vectors $\xi$ if $\Lambda$ is fixed. This follows from the uniform statements in Proposition 6 and Lemma 6.

5.4. *A reformulation of Theorem* 1. The second statement in Theorem 1 does not hold unless the cone condition is imposed, as the following example illustrates. Let $G$ be the unit ball in $\mathbf{C}^2$, $z_0 = (1, 0)$, $\xi = (0, 1)$. Then

$$(F(z, \xi))^2 d(z, \partial G) = (1 - |z_1|^2)(1 - |z|^2)^{-2}(1 - |z|)$$

$$\geqslant \tfrac{1}{2}(1 - |z_1|^2)(1 - |z|^2)^{-1}.$$

The latter expression has the value $k$ when $|z_2|^2 = (1 - (2k)^{-1})(1 - |z_1|^2)$. Thus we may find a sequence of points converging to $z_0$ for which the left-hand side tends to infinity.

However by decomposing $\xi$ at points sufficiently near $\partial G$ as $\xi = \xi_N(z) + \xi_T(z)$, we can reformulate Theorem 1 so that unrestricted approach is permitted in both limits.

THEOREM 1′. *With hypotheses as in Theorem* 1,

$$\lim_{z \to z_0} F(z, \xi_N(z)) d(z, \partial G) = \tfrac{1}{2} \|\xi_N(z_0)\|,$$

$$\lim_{z \to z_0} F(z, \xi_T(z))^2 d(z, \partial G) = \tfrac{1}{2} L_{\phi, z_0}(\xi_T(z_0)).$$

*The limits are approached uniformly in the point* $z_0 \in \partial G$ *and in vectors* $\xi$ *of unit length.*

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