ON SOME CLASSES OF MULTIVALENT STARLIKE FUNCTIONS

BY

RONALD J. LEACH

ABSTRACT. Classes of multivalent functions analogous to certain classes of univalent starlike functions are defined and studied. Estimates on coefficients and distortion are made, using a variety of techniques.

1. Let \( S_t \) denote the class of all functions \( f(z) = z + \ldots \) analytic, univalent and starlike in the unit disc \( U \). Such functions satisfy the condition
\[
\text{Re}(zf'(z)/f(z)) > 0, \quad z \in U.
\]

The problem of defining a corresponding class of multivalent starlike functions has been studied by several authors. Hummel [5] distinguishes six commonly used definitions, a typical one being \( f(z) \) belongs to the class \( S(p) \) if \( f \) has at most \( p \) zeros in \( U \) and
\[
\lim_{r \to 1^+} \min_{|z| = r} \frac{zf'(z)}{f(z)} > 0.
\]

In this note we will study three classes of multivalent starlike functions which are analogues of certain subclasses of \( S_t \).

2. Let \( S_x(p, a) \), \( p \) a positive integer, \( 0 < a < 2p \), denote the class of all functions \( f(z) = a_0 + a_1z + \ldots \) analytic in \( U \) with precisely \( p \) zeros there such that
\[
\lim_{r \to 1^+} \min_{|z| = r} \frac{zf'(z)}{f(z)} > a.
\]

\( S_x(p, a) \) is the generalization of the class \( S(a) \) of starlike functions of order \( a \) introduced by Robertson [11].

Theorem 2.1. Let \( f(z) \) belong to the class \( S_x(p, a) \) and suppose \( f \) has zeros at \( z_1, z_2, \ldots, z_p \). Then \( f(z) \) is \( p \)-valent in \( U \) and there is a function \( g \) in the class \( S(\alpha/p) \) and a constant \( A \) such that
\[
f(z) = A \prod_{j=1}^{p} \Psi(z, z_j) \, [g(z)]^p,
\]

Received by the editors May 6, 1974.

AMS (MOS) subject classifications (1970). Primary 30A32, 30A34.

Key words and phrases. Multivalent starlike functions, coefficients, distortion theorems.

Copyright © 1975, American Mathematical Society

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

267
where 
\[ \Psi(z, z_j) = (z - z_j) (1 - z_j^p z)/z. \]

(The asterisk denotes conjugation.)

**Proof.** Since inequality (2.1) implies inequality (1.1), it follows from [5] that \( f(z) \) is \( p \)-valent and that there is a function \( g \in St \) such that (2.2) holds. We compute

\[ \frac{zf'(z)}{f(z)} = \sum_{j=1}^{p} \frac{z\Psi'(z, z_j)}{\Psi(z, z_j)} + p \frac{zg'(z)}{g(z)}. \]

Since \( \Re(z\Psi'/\Psi) = 0 \) on \(|z| = 1\), it follows from (2.3) and the definition of \( S_1(p, \alpha) \) that \( \lim \sup \min \Re(zg'(z)/g(z)) \geq \alpha/p \) and the result follows from the maximum principle.

For any \( f \in S(p) \), let \( z_1, \ldots, z_p \) be the zeros of \( f(z) \), let \( r_i = |z_i|, R_M = \max \{r_i\}, R_m = \min \{r_i | r_i \neq 0\} \), and let \( r = |z| \). We also assume that the constant \( A \) of Theorem 2.1 equals 1.

**Theorem 2.2.** Let \( S_1(p, \alpha; z_1, \ldots, z_p) \) denote the subclass of \( S_1(p, \alpha) \) of functions with zeros at \( z_1, \ldots, z_p \). Then the extreme points of the closed convex hull of \( S_1(p, \alpha; z_1, \ldots, z_p) \) are precisely the functions of the form

\[ f(z) = \prod_{j=1}^{p} \Psi(z_1 z_j) z^p (1 - z x)^{\alpha - 2p}, \quad |x| = 1. \]

**Proof.** It follows from (2.2) and the compactness of \( S(\alpha/p) \) that \( S_1(p, \alpha; z_1, \ldots, z_p) \) is compact. For \( z_1, \ldots, z_p \) fixed the mapping

\[ T: [g(z)]^p \to \prod_{j=1}^{p} \Psi(z_1 z_j)[g(z)]^p \]

is a linear homomorphism; thus it suffices to find the extreme points of \( \{ [g(z)]^p | g \in S(\alpha/p) \} \). Since \( p > 0 \), the argument in [3] applies and we are done.

**Corollary 2.3.** Let \( f(z) = a_0 + \ldots \in S_1(p, \alpha; z_1, \ldots, z_p) \). Then

(i) \( |f(z)| \leq \prod_{j=1}^{p} (r + r_j) (1 + r_j r) (1 - r)^{\alpha - 2p}, \quad |z| < 1, \)

(ii) \( |f(z)| \geq \prod_{j=1}^{p} (r - r_j) (1 - r_j r) (1 + r)^{\alpha - 2p}, \quad |z| > R_M, \)

(iii) \( |f(z)| \geq \prod_{j=1}^{p} (r_j - r) (1 - r_j r) (1 + r)^{\alpha - 2p}, \quad |z| < R_m, \)

(iv) \( |a_n| \leq A_n \), where \( A_n \) is the coefficient of \( z^n \) in

\[ F(z) = \prod_{j=1}^{p} (z + r_j)(1 + r_j z)(1 - z)^{\alpha - 2p}, \]

(v) \( |f^{(k)}(z)| \leq F^{(k)}(r), \quad k = 1, 2, \ldots \).
We note that it is possible to obtain sharp upper and lower bounds for
\( \text{Re}(zf'(z)/f(z)) \) using the estimates in [6], but we do not state them here.

The problem of determining \( \max|a_n| \) when \( |a_1|, \ldots, |a_p| \) are fixed was
first studied for the class \( S(p) \) by Goodman [4]. We are only able to obtain a
partial result.

**Lemma 2.4.** Let \( f(z) = (z - z_0) (1 - \bar{z}_0 z) \cdot z^{-1} (z + \sum_{n=2}^{\infty} b_n z^n)^p \). Then
if \( f(z) = \sum_{n=p-1}^{\infty} a_n z^n \),

\[
(2.4) \quad a_{p+1} = \left[ pb_3 - \frac{p(p+1)}{2} b_2^2 + \frac{z_0}{z_0} \right] a_{p-1} + pb_2 a_p.
\]

**Proof.** Let \( \sum_{n=1}^{\infty} c_n z^n = (\sum_{n=1}^{\infty} b_n z^n)^p, \) \( b_1 = 1 \). Then

\[ f(z) = -z_0 c_p z^{p-1} + \left[ -z_0 c_{p+1} + (1 + |z_0|^2) c_p \right] z^p \]
\[ + \left[ -z_0 c_{p+2} + (1 + |z_0|^2) c_{p+1} - \bar{z}_0 c_p \right] z^{p+1} + \cdots. \]

Comparing coefficients, we have

\[ a_{p+1} = -z_0 c_p, \quad a_p = -z_0 c_{p+1} + (1 + |z_0|^2) c_p, \]
\[ a_{p+1} = -z_0 c_{p+2} + (1 + |z_0|^2) c_{p+1} - \bar{z}_0 c_p. \]

An easy calculation yields

\[
(2.5) \quad a_{p+1} = \frac{c_{p+1}}{c_p} a_p + \left[ \frac{c_{p+2}}{c_p} - \left( \frac{c_{p+1}}{c_p} \right)^2 + \frac{z_0}{z_0} \right] a_{p-1}. \]

Since \( c_p = 1, c_{p+1} = p b_2, \) and \( c_{p+2} = p(p-1) b_2^2/2 + pb_3, \) substitution
of (2.5) yields (2.4).

**Theorem 2.5.** Let \( f(z) = a_{p-1} z^{p-1} + a_p z^p + \ldots \in S_1(p, \alpha) \) with \( p \geq 2, (2p^2 - 3p - (5p^2 - 4p)^{1/2})/2(p-1) \geq \alpha \geq 0, a_p \) real. Then

\[ |a_{p+1}| \leq [2(p-\alpha)^2 - (p-\alpha) - 1] |a_{p-1}| + (p-\alpha) |a_p|. \]

**Proof.** It follows from Theorem 2.1 that \( f(z) \) has a single real zero \( z_0 \)
not at the origin and that \( g(z) = z + \sum_{n=2}^{\infty} b_n z^n \) has real coefficients. From
(2.4) we have

\[ |a_{p+1}| \leq \left( 1 + p \left| b_3 - \frac{(p+1)}{2} b_2^2 \right| \right) |a_{p-1}| + p|b_2| |a_p|. \]

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
For $\alpha/p \leq 1 - 1/p$, it follows from a result of Keogh and Merkes [7] that $|b_3 - (p + 1)b_2^2/2| \leq (1 - \alpha/p)(2p - 2\alpha - 1)$ and thus

$$1 + p\left(b_3 - \frac{p + 1}{2}b_2^2\right) \geq 1 - (p - \alpha)(2p - 2\alpha - 1).$$

(2.6)

It remains to find an upper bound for the left-hand side of (2.6). It suffices to prove

$$(p - \alpha)(2p - 2\alpha - 1) - 1 + p\left(\frac{p + 1}{2}b_2^2\right) \geq 1 + pb_3,$$

which is certainly true if

$$1 + p(b_3 - \frac{\alpha}{p}) > 1 - (p - \alpha)(2p - 2\alpha - 1).$$

(2.7)

Since $|b_3| \leq (1 - \alpha/p)(3 - 2\alpha/p)$ [3], [11], (2.7) is certainly true if

$$(p - \alpha)(2p - 2\alpha - 1) \geq 2 + (p - \alpha)(3 - 2\alpha/p),$$

or

$$\alpha \leq (2p^2 - 3p - (5p^2 - 4p)^{1/2})/(2p - 1).$$

This proves the theorem, since $|b_2| \leq (1 - \alpha/p)$ [3], [11].

3. Let $S_2(p, \alpha)$ denote the subclass of $S(p)$ consisting of all functions $f(z)$ for which

$$\limsup \min \Re \left[ \frac{1 + zf''(z)}{f'(z)} + (1 - \alpha) \frac{zf'(z)}{f(z)} \right] > 0.$$

The class $S_2(p, \alpha)$ is the analog of the class $C(\alpha)$ of $\alpha$-convex functions defined by Mocanu [10]. A short calculation gives

**Theorem 3.1.** Let $f(z) \in S_2(p, \alpha)$. Then $f(z) = \prod_{j=1}^{\infty} \Psi(z, z_j) [g(z)]^p$, where $g(z) \in C(\alpha/p)$.

We cannot state an analogue of Theorem 2.2 since as yet the extreme points of $C(\beta)$ are not known for all $\beta$. However, we can obtain the results of Corollary 2.3.

**Theorem 3.2.** Let $f(z) = a_0 + \ldots \in S_2(p, \alpha), \alpha > 0$. Then:

(i) $|f(z)| \leq \prod_{j=1}^{\infty} (r + r_j)(1 + r_j^p) r^{-p} \left[ \frac{p}{\alpha} \int_0^r (1 - t)^{-p/(\alpha + p)} \alpha^{-1} dt \right]^\alpha$, $|z| < 1,$

(ii) $|f(z)| \geq \prod_{j=1}^{\infty} (r - r_j)(1 - r_j^p) r^{-p} \left[ \frac{p}{\alpha} \int_0^r (1 + t)^{-p/(\alpha + p)} \alpha^{-1} dt \right]^\alpha$, $|z| > R_M,$

(iii) $|f(z)| \geq \prod_{j=1}^{\infty} (r - r_j)(1 - r_j^p) r^{-p} \left[ \frac{p}{\alpha} \int_0^r (1 + t)^{-p/(\alpha + p)} \alpha^{-1} dt \right]^\alpha$, $|z| < R_m,$
(iv) \(|a_n| \leq A_n\); where
\[
F(z) = \prod_{j=1}^{p} \Psi(z, -r_j) \left[ \frac{p}{\alpha} \int_{0}^{z} (1 - t) \frac{1}{t^{\alpha + 1}} dt \right]^{\alpha} = \sum_{n=0}^{\infty} A_n z^n,
\]

(v) \(|f^{(k)}(z)| \leq F^{(k)}(r), k = 1, 2, \ldots\)

Proof. Inequalities (i), (ii) and (iii) follow from Theorem 3.1 and the distortion theorem for \(\alpha\)-convex functions (see Miller [9]). Statements (iv) and (v) follow from the recent result of P. Kulshrestha [8].

We mention without proof that a result similar to Theorem 2.5 holds for \(S_2(p, \alpha)\) using the technique of Theorem 2.5 and a result of J. Syzmal [12].

4. Let \(S_3(p, \alpha), 0 < \alpha \leq 1\), denote the subclass of \(S(p)\) consisting of all functions \(f\) for which

\[
\lim \sup \max |\arg(z f'(z)/f(z))| < \alpha \pi/2.
\]

This extends the class \(S^*(\alpha)\) of strongly starlike functions of order \(\alpha\) defined by D. Brannan and W. Kirwan [2]. Note that a single valued branch of \(\arg(z f'(z)/f(z))\) can be defined in some annulus \(\rho < |z| < 1\).

Theorem 4.1. Let \(f(z) \in S_3(p, \alpha)\). Then there is a function \(g(z) \in S^*(\alpha)\) such that \(f(z) = \prod_{j=1}^{p} \Psi(z, z_j) g(z)^p\).

Proof. This follows from the equation (2.3) since \(\text{Re}(z \Psi'(z, z_j)/\Psi(z, z_j)) = 0\) on \(|z| = 1\).

Corollary 4.2. If \(f \in S_3(p, \alpha)\), then \(f\) is bounded in \(U\).

Proof. This follows from [2, Theorem 2.1] and the previous theorem.

Lemma 4.3. Let \(g(z) = z + b_2 z^2 + \ldots \in S^*(\alpha)\). Then if either \(\lambda \geq 3/4\) or \(3/4 - 1/4\alpha \geq \lambda, |b_3 - \lambda^2| \leq |3\alpha^2 - 4\lambda\alpha^2|, and this result is sharp.

Proof. Using the notation of [1, Theorem 2.1], we have

\[
b_3 - \lambda b_2^2 = \frac{\alpha}{2} \left[ p_2 + \frac{3\alpha - 1 - 4\lambda\alpha}{2} p_1^2 \right],
\]

where \(P(z) = 1 + p_1 z + p_2 z^2 + \ldots\) has \(\text{Re} P(z) > 0\) in \(U\). We have

\[
b_3 - \lambda b_2^2 = \frac{\alpha}{2} \left( \int_{0}^{\pi} e^{-2i\theta} d\mu(\theta) + \frac{\alpha}{2} \left( \frac{3\lambda - 1 - 4\lambda\alpha}{2} \right) \left( \int_{0}^{\pi} e^{-i\theta} d\mu(\theta) \right)^2 \right),
\]

where \(\mu(\theta)\) is an increasing function on \([0, 2\pi]\) with \(\mu(2\pi) - \mu(0) = 1\). Hence
\[ \frac{2}{\alpha} \text{Re}(b_3 - \lambda b_2^2) = 2 \int_0^{2\pi} \cos \theta \, d\mu(\theta) \]

\[ + (6\alpha - 2 - 8\lambda\alpha) \left[ \left( \int_0^{2\pi} \cos \theta \, d\mu(\theta) \right)^2 - \left( \int_0^{2\pi} \sin \theta \, d\mu(\theta) \right)^2 \right]. \]

Suppose first that \( 6\alpha - 2 - 8\lambda\alpha \geq 0 \). Then

\[ \frac{2}{\alpha} \text{Re}(b_3 - \lambda b_2^2) \leq 2 \int_0^{2\pi} \cos 2\theta \, d\mu(\theta) + (6\alpha - 2 - 8\lambda\alpha) \left( \int_0^{2\pi} \cos \theta \, d\mu(\theta) \right)^2 \]

\[ \leq 4 \int_0^{2\pi} \cos^2 \theta \, d\mu(\theta) - 2 + (6\alpha - 2 - 8\lambda\alpha) \int_0^{2\pi} \cos^2 \theta \, d\mu(\theta) \]

\[ \leq 6\alpha - 8\lambda\alpha, \]

where we have used Jensen’s inequality and the estimate \( \int_0^{2\pi} \cos^2 \theta \, d\mu(\theta) \leq 1 \).

The case \( 6\alpha - 2 - 8\lambda\alpha < 0 \) is treated in a similar manner.

To show that the inequality is sharp, we consider the function \( g(z) \) defined by

\[ (4.1) \quad zg'(z)/g(z) = ((1 + z)/(1 - z))^\alpha \]

for which \( a_2 = 2\alpha, b_3 = 3\alpha^2 \).

**Theorem 4.4.** Let \( f(z) = a_{p-1}z^{p-1} + a_p z^p + \text{in} \in S_3(p, \alpha) \) and suppose each \( a_n \) is real. If \( p \geq 3 \) and \( \alpha \geq \min(1/3, (p^2 - 2p)^{-1/2}) \),

\[ |a_{p+1}| \leq (2p^2\alpha^2 - p\alpha^2 - 1) |a_{p-1}| + 2\alpha p |a_p|, \]

and this result is sharp.

**Proof.** By Lemma 2.4, it is sufficient to show that

\[ \left| 1 + pb_3 - p \left( \frac{p+1}{2} \right) b_2^2 \right| \leq 2p^2\alpha^2 - p\alpha^2 - 1, \]

since \( |b_2| \leq 2\alpha \) [1]. By Lemma 2.3, with \( \lambda = (p + 1)/2 \),

\[ 1 + pb_3 - p((p + 1)/2) b_2^2 \geq 1 + p\alpha^2 - 2p^2\alpha^2 \]

and hence it suffices to show

\[ (4.2) \quad 1 + pb_3 \leq 2p^2\alpha^2 - p\alpha^2 - 1. \]

Since \( |b_3| \leq 3\alpha^2 \) if \( \alpha \geq 1/3 \), (4.2) follows if \( 1 + 3p\alpha^2 \leq 2p^2\alpha^2 - p\alpha^2 - 1 \), which is certainly true if \( p \geq 3 \).

The sharpness of the result follows from the fact that the function \( g(z) \) defined by (4.1) simultaneously maximizes \( b_2^2 \) and \( b_3 - (p + 1)b_2^2/2 \).
Notes. 1. If \( p \geq 4 \), the result holds for all \( \alpha \geq 1/3 \).

2. For \( 0 < \alpha \leq 1/3 \), a similar (but not sharp) result holds for \( p > \left[ \alpha^2 + \alpha + (17\alpha^4 + 2\alpha^3 + \alpha^2) \right] (4\alpha^2)^{-1} \).

REFERENCES


