

A CLASSIFICATION THEOREM FOR ABELIAN p -GROUPS

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ABSTRACT. A new class of Abelian p -groups, called S -groups, is studied, and the groups in this class are classified in terms of cardinal invariants. The class of S -groups includes Nunke's totally projective p -groups. The invariants consist of the Ulm invariants (which Hill has shown can be used to classify the totally projective groups) together with a new sequence of invariants indexed by limit ordinals which are not cofinal with ω . The paper includes a fairly complete discussion of dense isotype subgroups of totally projective p -groups, including necessary and sufficient conditions for two of them to be congruent under the action of an automorphism of the group. It also includes an extension of Ulm's theorem to a class of mixed modules over a discrete valuation ring.

The work reported here was partly announced in [21]. To study the S -groups, it will be necessary to prove a classification theorem for certain modules over the ring Z_p of integers localized at p . We will therefore regard p -groups as torsion modules over this ring, and all modules in this paper will be modules over the ring Z_p . The ring of p -adic integers could have been used just as well. The results are valid for modules over any discrete valuation ring. If A is any Z_p -module, we define $p^\alpha A$ for any ordinal α by $p^{\alpha+1}A = p(p^\alpha A)$, and, if α is a limit ordinal, $p^\alpha A = \bigcap_{\beta < \alpha} p^\beta A$. A module is *reduced* if, for some ordinal α , $p^\alpha A = 0$. The smallest ordinal α such that $p^\alpha A = 0$ is called the *length* of A . $A[p]$ is the submodule of A generated by the elements of order p . We define

$$U_\alpha(A) = (p^\alpha A)[p]/(p^{\alpha+1}A)[p] \quad (\alpha \geq 0)$$

and let $f(\alpha, A) = \dim(U_\alpha(A))$, where the dimension is taken over the field Z/pZ . The cardinals $f(\alpha, A)$ are the *Ulm invariants* of A .

If A is a Z_p -module, a submodule H is *isotype* in A if, for all ordinals α , $p^\alpha H = H \cap p^\alpha A$. If λ is an ordinal, H is λ -dense in A if, for all $\alpha < \lambda$, $A = H + p^\alpha A$. If λ is a limit ordinal, λ is cofinal with ω if there is a sequence λ_i ,

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$0 < i < \omega$, of smaller ordinals, such that λ is the supremum of the λ_i . We can now state a theorem which summarizes our results on λ -dense, isotype subgroups of totally projective groups.

THEOREM A. *Let G be a totally projective p -group, λ a limit ordinal such that $p^\lambda G = 0$, and H and K λ -dense, isotype subgroups of G .*

(i) *If $\dim((G/H)[p]) = \dim((G/K)[p])$ then there is an automorphism of G which takes H isomorphically onto K .*

(ii) *If λ is cofinal with ω , H and K are isomorphic to G .*

(iii) *If λ is not cofinal with ω , then H and K are isomorphic if and only if $\dim((G/H)[p]) = \dim((G/K)[p])$.*

The proof of Theorem A is contained in 1.6 and 3.1 below. It is a consequence of (iii) above that the groups arising in this way are not necessarily totally projective. They are all contained in a larger class, which we call S -groups. To study the S -groups, we first must study a class of mixed Z_p -modules. We give a rather unsatisfactory treatment of these modules, which has the virtue of being brief and of containing all that is needed for the application to p -groups. For a more intrinsic treatment, we refer to [22] and a subsequent paper.

DEFINITION [22]. If λ is a limit ordinal, we say a module M is a λ -elementary KT -module if $p^\lambda M \cong Z_p$ and $M/p^\lambda M$ is torsion and totally projective. A module is a KT -module if it is a direct sum of a totally projective p -group and λ -elementary KT -modules for various λ .

To state the form of Ulm's theorem that holds for KT -modules, we need a new invariant, introduced in [22]. If M is a module, we let $h(0, M) = \dim(M/(pM + T))$ where T is the torsion submodule of M . Note that if M is a direct sum of cyclic modules, then $h(0, M)$ is the number of infinite cyclic summands in a decomposition of M . More generally, $h(0, M)$ is the number of infinite cyclic summands in a basic submodule of M [11, Lemma 21]. For any limit ordinal λ , we define $h(\lambda, M) = h(0, p^\lambda M)$. If M is a KT -module, it is clear that $h(\lambda, M)$ is the number of λ -elementary summands in a decomposition of M .

THEOREM K. *The class K of KT -modules has the following properties:*

(i) *A module isomorphic to a module in K is in K .*

(ii) *A direct sum of modules in K is in K .*

(iii) *If M is a module (over the ring Z_p) and α an ordinal, then M is in K if M is in K for all $\beta < \alpha$, and $h(\alpha, M) = h(\alpha, M)$.*

(iv) *If C is any class of modules containing Z_p and having the closure properties (i), (ii), and (iii), then C contains K .*

(v) *If A and B are in K and $f(\alpha, A) = f(\alpha, B)$ for all ordinals α , and*

$h(\lambda, A) = h(\lambda, B)$ for all limit ordinals λ , then $A \cong B$.

This theorem is proved in 1.3, 4.6, and 5.1 below. The KT -modules possess a number of other nice properties [22], which will be proved in a subsequent paper.

DEFINITION. A p -group is an S -group if it is isomorphic to the torsion submodule of a KT -module. It is a λ -elementary S -group if it is the torsion submodule of a λ -elementary KT -module.

In §2 we will define a family of functors K_λ , where λ is any ordinal. K_λ will associate with any module a Z/pZ -vector space. If λ is a limit ordinal which is not cofinal with ω , we let $k(\lambda, M) = \dim(K_\lambda(M))$. If G is an S -group, we will show that G is a direct sum of a totally projective group and γ -elementary S -groups for various limit ordinals γ which are not cofinal with ω , and that $k(\lambda, G)$ is the number of λ -elementary summands in such a representation.

The invariants $k(\lambda, G)$ and the Ulm invariants will classify the S -groups. To obtain a result for S -groups analogous to Theorem K, we will need a stronger closure property to get an analogue of condition (iv) of that theorem. We say a submodule H of a Z_p -module M is λ -high if $H \cap p^\lambda M = 0$, and H is maximal with respect to this property.

THEOREM S. *The class S of S -groups has the following properties:*

- (i) *A group isomorphic to a group in S is in S .*
- (ii) *A direct sum of groups in S is in S .*
- (iii) *If G is a p -group and α an ordinal, then G is in S if and only if $p^\alpha G$ and $G/p^\alpha G$ are in S .*
- (iv) *If G is in S and λ is a limit ordinal, and H is a λ -high subgroup of G , then H is in S .*
- (v) *If C is any class of p -groups containing Z/pZ and satisfying properties (i), (ii), (iii), and (iv) above, then C contains S .*
- (vi) *If A and B are in S and $f(\alpha, A) = f(\alpha, B)$ for every ordinal α and $k(\lambda, A) = k(\lambda, B)$ for every limit ordinal λ which is not cofinal with ω , then $A \cong B$.*
- (vii) *If G is in S then G is totally projective if and only if for every limit ordinal λ which is not cofinal with ω , $k(\lambda, G) = 0$.*

This theorem is an immediate consequence of the definition, and Theorems 3.2, 5.2, and 5.3 below.

A historical remark is in order concerning the form of Theorems K and S. Nunke introduced totally projective p -groups in [16] in order to give a homological description of direct sums of countable p -groups. It was E. Walker who sug-

gested that Ulm's theorem might generalize to this class of groups. Walker showed that if this could be done, then the class of totally projective p -groups would be the smallest class containing Z/pZ and satisfying conditions (i), (ii) and (iii) of Theorem S. Part (iv) of Theorem K and part (v) of Theorem S were motivated by this example. In [19], Parker and Walker proved Ulm's theorem for totally projective p -groups of length less than $\Omega\omega$, thus giving the first major extension of the theory of p -groups to groups of uncountable length. The complete proof of Ulm's theorem for totally projective p -groups required a different technique and an entirely new characterization of these groups, and this was proved by P. Hill (unpublished). A short proof (using Hill's basic ideas) has recently been given by Walker [21].

The theory presented in this paper is still very incomplete. In fact, the results open a veritable Pandora's box of unsolved and, apparently, difficult problems. §6 of this paper contains a list of ten of these.

§1 below uses Hill's methods to classify KT -modules and to construct certain automorphisms for totally projective p -groups. The second section reviews the theory of cotorsion envelopes and defines the functors K_λ . In §3 we classify S -groups and dense isotype subgroups of totally projective groups. In §4, we determine what values the invariants of a KT -module or S -group can have. This includes a new proof of the corresponding result for totally projective p -groups, and a proof that any subgroup of a totally projective p -group has the Ulm invariants of some totally projective p -group. §5 completes the proof of Theorems K and S.

1. Extending isomorphisms and constructing automorphisms. If G is an Abelian group, H a subgroup and $\phi: G \rightarrow G/H$ the natural map, we say H is a p -nice subgroup if, for all ordinals α , $p^\alpha(G/H) = \phi(p^\alpha G)$.

We remark that if G is a group, $p^\alpha G$ is a p -nice subgroup; and if, for all $i \in I$, H_i is a p -nice subgroup of G_i , then $\bigoplus_{i \in I} H_i$ is p -nice in $\bigoplus_{i \in I} G_i$. These are the only cases we will need to prove our theorems.

We follow E. A. Walker [21] in using Hill's analysis of totally projective p -groups as the definition of these groups.

DEFINITION. If G is a reduced p -group, G is totally projective if it has a family \mathcal{C} of subgroups such that

- (i) If $H_i \in \mathcal{C}$, $i \in I$, then $\sum_{i \in I} H_i \in \mathcal{C}$,
- (ii) If $H \in \mathcal{C}$, and X is a countable subset of G , there is a subgroup $H' \in \mathcal{C}$ such that $H \subseteq H'$, $X \subseteq H'$ and H'/H is countable.
- (iii) $\{0\} \in \mathcal{C}$.
- (iv) The elements of \mathcal{C} are p -nice subgroups of G .

We now summarize the known results in several theorems.

1.1. THEOREM ([7], [19], [21]). *Let \mathcal{T} be the class of totally projective p -groups.*

- (i) *A group isomorphic to a group in \mathcal{T} is in \mathcal{T} .*
- (ii) *A direct sum of groups in \mathcal{T} is in \mathcal{T} .*
- (iii) *If G is a p -group, and α an ordinal, then $G \in \mathcal{T}$ if and only if $p^\alpha G \in \mathcal{T}$ and $G/p^\alpha G \in \mathcal{T}$.*
- (iv) *If \mathcal{C} is any class of p -groups containing Z/pZ and having properties (i), (ii) and (iii), then $\mathcal{C} \supseteq \mathcal{T}$.*
- (v) *If A and B are in \mathcal{T} , and $f(\alpha, A) = f(\alpha, B)$ for all ordinals α , then $A \cong B$.*

Conditions (i) and (ii) are clear. For a proof of (v) we refer to [7] or [21]. The fact that if G is totally projective then $p^\alpha G$ and $G/p^\alpha G$ are also is clear. For the other half of (iii) we remark that the basic extension theorem (1.2 below, proved in [7] and [21]) shows that if $p^\alpha G$ and $G/p^\alpha G$ are totally projective and if the Ulm invariants of G are those of a totally projective group, then G is totally projective. This is easy to see if α is a finite ordinal, and the rest of the proof can be given as in that of 5.1 below. An alternative proof is in [3, 81.9 and 81(A)]. (iv) is an easy consequence of (v) and a suitable existence theory, and a proof of it is contained in our proof of 4.6 below. There are a number of other easily established properties of totally projective groups which we do not list because their generalizations for KT -modules and S -groups are either difficult or unknown. Condition (v) has a generalization which we will need, also essentially due to Hill, and stated in this form in [21].

DEFINITION. If G is a group whose torsion subgroup is a p -group, A is a subgroup, and $A_\alpha = A \cap p^\alpha G$, we define

$$I_\alpha(A) = (A_\alpha + p^{\alpha+1}G)[p]/(p^{\alpha+1}G)[p].$$

We regard $I_\alpha(A)$ as a subspace of the Ulm invariant $U_\alpha(G)$.

1.2. THEOREM ([7], [21]). *If G and H are groups, A and B p -nice subgroups of G and H respectively such that G/A and H/B are totally projective p -groups, $\phi: A \rightarrow B$ an isomorphism such that, for all α , $\phi(A_\alpha) = B_\alpha$, and if, for all α , $U_\alpha(G)/I_\alpha(A) \cong U_\alpha(H)/I_\alpha(B)$, then ϕ extends to an isomorphism of G onto H .*

1.3. COROLLARY. *If M and N are KT -modules, $f(\alpha, M) = f(\alpha, N)$ for all ordinals α , and $h(\lambda, M) = h(\lambda, N)$ for all limit ordinals λ , then $M \cong N$.*

PROOF. We write $M = T \oplus (\bigoplus_{i \in I} M_i)$, where T is a totally projective p -

group and M_i is a $\lambda(i)$ -elementary KT -module. For each $i \in I$, choose x_i to be a generator of $p^{\lambda(i)}M_i$ and let $X = \{x_i: i \in I\}$. We similarly choose a subset $Y = \{y_j: j \in J\}$ of N . For a given ordinal λ , the number of elements of I with $\lambda(i) = \lambda$ is exactly $h(\lambda, M)$. There is, therefore, a bijective map $\phi: I \rightarrow J$ such that $\lambda(i) = \lambda(\phi(i))$ for all $i \in I$.

From the comments following the definition of p -nice subgroups, it is clear that the submodules $[X]$ and $[Y]$, generated by X and Y , are p -nice submodules of M and N , and it is clear by the construction that $M/[X]$ and $N/[Y]$ are totally projective p -groups. It is easily computed that for all α , $I_\alpha([X]) = 0$ and $I_\alpha([Y]) = 0$. Since X and Y are free generators of the free modules $[X]$ and $[Y]$, ϕ defines an isomorphism $\phi': [X] \rightarrow [Y]$. It is now clear that all the hypotheses of 1.2 are satisfied, from which we conclude that $M \cong N$.

1.4. LEMMA. *If $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is an exact sequence of modules, then multiplication by p in B induces a homomorphism $\sigma: C[p] \rightarrow A/pA$. If we regard A as a submodule of B and $\nu: B \rightarrow C$ is the homomorphism of the sequence, then $\ker(\sigma) = \nu(B[p])$. For any ordinal $\alpha > 0$, if $A \subseteq p^\alpha B$ then σ is surjective and $C[p] = \ker(\sigma) + (p^\beta C)[p]$ for all $\beta < \alpha$. If H is a submodule of B maximal with respect to the property that $H \cap A = 0$, then $\nu(H)[p] = \ker(\sigma)$.*

PROOF. We define σ as follows: if $x \in C[p]$ choose a $y \in B$ such that $\nu(y) = x$ and let $\sigma(x) = py + pA$. This is clearly well defined, and it is clear that $\ker(\sigma) = \nu(B[p])$. If $\alpha > 0$, then $A \subseteq p^\alpha B$ if and only if for $\beta < \alpha$ and for any $x \in A$, there is a $y \in p^\beta B$ such that $py = x$. This implies that $x + pA = \sigma(\nu(y))$ and $\nu(y) \in \nu(p^\beta B)[p] \subseteq (p^\beta C)[p]$, which proves all of the result but the last statement, which is trivial.

If M is a Z_p -module and N a submodule, we follow Nunke in saying that N is p^λ -pure in M if the corresponding extension $0 \rightarrow N \rightarrow M \rightarrow C \rightarrow 0$ is in $p^\lambda \text{Ext}(C, N)$.

1.5. LEMMA. *Let G be a p -group and H a subgroup, λ a limit ordinal, $\lambda > 0$, and suppose that $p^\lambda G = 0$. Then the following conditions are equivalent.*

- (i) H is λ -dense and isotype.
- (ii) H is p^λ -pure and G/H is divisible.
- (iii) $H \cap pG = pH$ and, for all ordinals $\alpha < \lambda$, $G[p] = H[p] + (p^\alpha G)[p]$.
- (iv) There is a Z_p -module M and an imbedding $\phi: H \rightarrow M$ and a surjective homomorphism $\psi: M \rightarrow G$ such that $\ker(\psi) = p^\lambda M$, $\phi(H)$ is λ -high in M , and $\psi\phi$ is the natural imbedding of H into G .
- (v) There is a Z_p -module M and an imbedding $\phi: H \rightarrow M$ and a surjective homomorphism $\psi: M \rightarrow G$ such that $\ker(\psi) = p^\lambda M$, $p^\lambda M$ is free, $\phi(H)$ is the

torsion submodule of M , and $\psi\phi$ is the natural imbedding of H into G .

PROOF. The equivalence of (i) and (iii) seems to be due to R. S. Pierce [20]. We show that (i) implies (iii). The first condition of (iii) is trivial. For the second, by λ -density, if $x \in G[p]$, there is a $y \in H$ and a $z \in p^\alpha G$ such that $x = y - z$. It follows that $py = pz \in p^{\alpha+1}G$, so, since H is an isotype subgroup, $py \in p^{\alpha+1}H$, so there is a $y' \in p^\alpha H$ such that $py = py'$. Hence, $x = (y + y') - (z + y')$, where $y + y' \in H[p]$ and $z + y' \in (p^\alpha G)[p]$.

The equivalence of (ii) and (iii) is due to Nunke. (ii) implies (iii) by [16, 2.9]. Conversely, if (iii) holds, then since $pH = H \cap pG$, any element in $(G/H)[p]$ is in the image of $G[p]$, and hence in the image of $(p^\alpha G)[p]$ for any $\alpha < \lambda$. It follows that G/H is divisible, from which (ii) follows, again by [16, 2.9].

That (iii) implies (v) seems to be due (in an equivalent form) to Kulikov [12]. For a proof see also [8, 1.7]. (The basic idea is to look at the sequence $0 \rightarrow H \rightarrow G \rightarrow D \rightarrow 0$ and an epimorphism $f: L \rightarrow D$, where L is divisible and torsion-free and $\text{Ker}(f)$ is free. The induced extension $0 \rightarrow H \rightarrow M \rightarrow L \rightarrow 0$ works.) (v) implies (iv) trivially.

To complete the proof, we will show that (iv) implies (i). By [10, Theorem 2], $\phi(H)$ is an isotype submodule of M . We recall that if G is a group and N a subgroup, H is N -high [9] in G if $H \cap N = 0$ and H is maximal with respect to this property. It is easy to verify that this implies that the image of N is essential in G/H , and if $N \subseteq p^\omega G$, it follows from this that G/N is p -divisible. It is clear that if H is λ -high in M , then for any $\alpha < \lambda$, $H \cap p^\alpha M$ is $(p^\lambda M)$ -high in $p^\alpha M$. Since λ is a limit ordinal, the elements of $p^\lambda M$ still have infinite height in $p^\alpha M$ for any $\alpha < \lambda$, which implies that $p^\alpha M / (\phi(H) \cap p^\alpha M)$ is divisible. Therefore, the image of $p^\alpha M$ in $M/\phi(H)$ is all of $M/\phi(H)$ (since it is a divisible submodule of $M/\phi(H)$ containing the image of $p^\lambda M$, an essential submodule). This implies that $M = \phi(H) + p^\alpha M$, which implies that $G = H + p^\alpha G$, since the map ψ is surjective.

1.6. THEOREM. *Let G be a totally projective p -group of length λ , λ a limit ordinal. Let H and K be two λ -dense, isotype subgroups of G . Then there is an automorphism σ of G such that $\sigma(H) = K$, if and only if $H[p]$ and $K[p]$ have the same codimension in $G[p]$.*

PROOF. The condition on H and K is clearly necessary.

By 1.5, there are modules M and N and imbeddings $\phi: H \rightarrow M$, $\phi': K \rightarrow N$, such that $p^\lambda M$ and $p^\lambda N$ are free, $\phi(H)$ and $\phi'(K)$ are the torsion submodules of M and N , and there are homomorphisms $\psi: M \rightarrow G$ and $\psi': N \rightarrow G$, satisfy-

ing the conditions of 1.5 (v). If $F = p^\lambda M$, then multiplication by p induces an isomorphism $G[p]/H[p] \rightarrow F/pF$, by 1.4, so that if $H[p]$ and $K[p]$ have the same codimension, then $p^\lambda M \cong p^\lambda N$, since they are free and of the same rank. The conditions of 1.2 are satisfied, so we get an isomorphism of M onto N , which clearly carries $\phi(H)$ onto $\phi'(K)$. The isomorphism between M and N induces an isomorphism $M/p^\lambda M \rightarrow N/p^\lambda N$, and since ψ and ψ' can be used to identify both of these modules with G , we have an induced automorphism of G with the desired properties.

Theorem 1.6 admits of a generalization, using the λ -Zippin property.

DEFINITION. If M is a module, λ a limit ordinal, and $p^\lambda M = 0$, then M has the λ -Zippin property if for every triple (G, H, f) , where G and H are modules, $G/p^\lambda G \cong H/p^\lambda H \cong M$ and $f: p^\lambda G \rightarrow p^\lambda H$ is an isomorphism; there is an extension of f to an isomorphism $f': G \rightarrow H$.

1.7. LEMMA. *If G is an S -group or a KT -module, and $p^\lambda G = 0$, then G has the λ -Zippin property.*

This is proved by Nunke in [18]. The proof of 1.6 immediately implies the following.

1.8. THEOREM. *If G is a p -group, λ a limit ordinal, $p^\lambda G = 0$, G has the λ -Zippin property, H and K are λ -dense, isotype subgroups of G , and $\dim((G/H)[p]) = \dim((G/K)[p])$, then there is an automorphism of G taking H onto K .*

2. Cotorsion envelopes and the functors K_λ . In this section we will define a family of functors K_λ , where λ is any ordinal. We will need the theory of cotorsion modules, which were introduced independently by Fuchs [2], Nunke [14] and Harrison [4]. We follow Fuchs in saying that a module C is cotorsion if and only if $\text{Ext}(F, C) = 0$ for all torsion-free modules F . (It is sometimes required that a cotorsion module be reduced.) If we identify a module M with $\text{Hom}(Z_p, M)$, then we get a natural map $M \rightarrow \text{Ext}(Q/Z_p, M)$, which is injective if M is reduced. If M is reduced, we let $c(M) = \text{Ext}(Q/Z_p, M)$, and regard M as contained in $c(M)$. We refer to Nunke [14] or Harrison [4] for the proofs that $c(M)$ is cotorsion and reduced and that $c(M)/M$ is torsion-free and divisible. This last implies easily that M is isotype in $c(M)$, and that the torsion submodule of $c(M)$ is contained in M . We recall that a submodule N of M is *pure* in M if $p^n N = N \cap p^n M$ for all n , $0 \leq n < \omega$.

2.1. LEMMA. *If M is a reduced module and N a submodule such that M/N is reduced, then the sequence*

$$0 \rightarrow c(N) \rightarrow c(M) \rightarrow c(M/N) \rightarrow 0$$

is exact. If M is cotorsion, then so is N . In particular, if M is a reduced cotorsion module then so is $p^\lambda M$, for any ordinal λ .

PROOF. The first follows from the exact sequence for Ext and the fact that $\text{Hom}(Q/Z_p, M/N) = 0$. For the second statement, we use the fact that a module C is cotorsion if and only if $\text{Ext}(Q, C) = 0$ (since this implies that $\text{Ext}(D, C) = 0$ for any torsion-free divisible module D , and any torsion-free module can be imbedded in such a D). If $\text{Ext}(Q, M) = 0$, then since $\text{Hom}(Q, M/N) = 0$, $\text{Ext}(Q, N) = 0$.

2.2. LEMMA. Let M be a reduced module and N a submodule such that M/N is torsion-free and divisible. Then (i) if C is a reduced cotorsion module and $f: N \rightarrow C$ a homomorphism, there is a unique extension of f to a homomorphism $g: M \rightarrow C$, (ii) if f is injective and $C/f(N)$ is torsion-free, then g will also be injective, and (iii) if M is cotorsion then there is a natural isomorphism $M \cong c(N)$.

PROOF. We have an exact sequence

$$\text{Hom}(M/N, C) \rightarrow \text{Hom}(M, C) \rightarrow \text{Hom}(N, C) \rightarrow \text{Ext}(M/N, C).$$

The hypotheses imply that the extremal terms of this sequence vanish, which implies (i). For (ii), we remark that the kernel K of g will be isomorphic to a subgroup of M/N , and if $C/f(N)$ is torsion-free, then $M/N + K$ will also be torsion-free, so K must be isomorphic to a pure subgroup of M/N . A pure subgroup of M/N is divisible, and since M has no divisible subgroups except 0, $K = 0$. For (iii) we apply (i) in both directions to the imbeddings of N into M and $c(N)$.

A torsion-free module is cotorsion if and only if it is algebraically compact [11], which means that it is a direct sum of a divisible module and a module which is complete and Hausdorff in its p -adic topology. Part (iii) of 2.2 implies that if M is a reduced torsion-free module, then $c(M)$ can be identified with the p -adic completion of M . We recall that if A is a torsion-free, reduced module which is complete in its p -adic topology, and $\nu: A \rightarrow A/pA$ is the natural map, and if X is a subset of A which is taken by ν bijectively onto a Z/pZ -basis of A/pA , then the submodule $[X]$ generated by X is free, with X as a set of free generators, and A is the p -adic completion of $[X]$. In particular, if M is a module such that $p^\lambda M = 0$, then $p^\lambda c(M)$ is complete, torsion-free module, since it is cotorsion by 2.1 and torsion-free (since $M \cap p^\lambda c(M) = p^\lambda M = 0$). The above analysis then motivates the following definition.

DEFINITION. If M is a Z_p -module, and λ is an ordinal, we define

$$K_\lambda(M) = p^\lambda c(M/p^\lambda M)/p^{\lambda+1} c(M/p^\lambda M).$$

If we let $k(\lambda, M)$ be the Z/pZ dimension of $K_\lambda(M)$, we obtain a new family of invariants indexed by ordinals. We remark that in [5] Harrison defined a new sequence of invariants which, in our terminology, were the invariants $h(\lambda, c(M))$. In general it is easy to verify that $h(\lambda, c(M)) = h(\lambda, M) + k(\lambda, M)$, so our present invariants are more delicate than Harrison's for modules in general, but equivalent for p -groups.

Though we define K_λ for any ordinal λ , it is clear that if $0 < n < \omega$, then K_λ and $K_{\lambda+n}$ are equivalent functors, so that for most purposes we can restrict ourselves to limit ordinals.

2.3. LEMMA. *Let M be a reduced Z_p -module and $\lambda > 0$ a limit ordinal which is not cofinal with ω . Suppose that $M/p^\lambda M$ is a totally projective p -group and $p^\lambda M$ is torsion-free. Then there is a natural isomorphism $p^\lambda M/p^{\lambda+1} M \rightarrow K_\lambda(H)$ where H is the torsion submodule of M .*

PROOF. M/H is divisible and torsion-free, so by 2.2, there is an imbedding $\phi: M \rightarrow c(H)$. Since M/H is torsion-free and divisible, $c(H)/M$ is torsion-free and divisible, so $p^\lambda M$ is an isotype submodule of $p^\lambda c(H)$. Since $p^\lambda c(H)$ is reduced and torsion-free, it is complete in its p -adic topology, and if L is the p -adic completion of the pure submodule $p^\lambda M$, then L is a summand of $p^\lambda c(H)$ by [11, Theorem 23]. The quotient $c(H)/L$ is a reduced cotorsion module with $M/p^\lambda M$ as its torsion submodule. If $C = c(H)/L$ and G is the image in C of $M/p^\lambda M$, then C/G is torsion-free and divisible, so we can identify C with $c(G)$ by 2.2. Since G is totally projective, it is a direct sum of groups of smaller length, so by [16, 3.10], $p^\lambda c(G) = 0$ (using the fact that λ is not cofinal with ω). Hence $L = p^\lambda c(H)$, which gives the desired result.

It is clear that the functors K_λ are additive in the sense that there is a natural isomorphism $K_\lambda(M \oplus N) = K_\lambda(M) \oplus K_\lambda(N)$. This additivity does not extend to infinite direct sums, as trivial examples show. However, if λ is a limit ordinal which is not cofinal with ω , Nunke has shown that K_λ is strongly additive in this sense. We will not need this general result, since the special case which occurs when all of the modules involved are S -groups is quite easy.

3. Classification theorems. We classify dense isotype subgroups of totally projective p -groups up to isomorphism, and use this to classify S -groups. The groups classified in 3.1 actually are S -groups, but to prove this, one needs some considerations concerning admissible invariants which we postpone till §4.

3.1. THEOREM. *Let G be a totally projective p -group of length λ , λ a limit*

ordinal. If λ is cofinal with ω then all λ -dense, isotype subgroups of G are isomorphic to G . If λ is not cofinal with ω , then two λ -dense, isotype subgroups H and K are isomorphic if and only if $H[p]$ and $K[p]$ have the same codimension in $G[p]$.

PROOF. If λ is cofinal with ω , then by Nunke's theorem [16, 4.4], $p^\lambda \text{Ext}$ is hereditary, which implies that any p^λ -pure subgroup of G is p^λ -projective. If H is a λ -dense, isotype subgroup of G , then H is p^λ -projective by 1.5, which certainly implies that H is p^α -projective for any $\alpha > \lambda$. If $\alpha < \lambda$, then $H/p^\alpha H \cong G/p^\alpha G$, so $H/p^\alpha H$ is p^α -projective. This shows that H is totally projective, and since its Ulm invariants are the same as those of G , $H \cong G$.

To show the second half of the theorem, we must show that if H is a λ -dense, isotype subgroup of G and is not cofinal with ω , then the codimension of $H[p]$ in $G[p]$ is an isomorphism invariant of H , and the rest will follow by Theorem 1.6. Using part (v) of 1.5, we construct a Z_p -module M and a mapping $\psi: H \rightarrow M$ taking H isomorphically onto the torsion submodule of M . We also have a surjective mapping $\psi: M \rightarrow G$ with kernel $p^\lambda M$ such that $\psi\phi$ is the natural imbedding of H into G . It follows, from 1.4, that there is a natural isomorphism $G[p]/H[p] \rightarrow p^\lambda M/p^{\lambda+1}M$, given by multiplication by p . Since λ is not cofinal with ω , 2.3 implies that $p^\lambda M/p^{\lambda+1}M \cong K_\lambda(H)$, which shows that the codimension of $H[p]$ in $G[p]$ is indeed an isomorphism invariant of H .

DEFINITION. If M is a module and λ is a limit ordinal which is not cofinal with ω , we define $k(\lambda, M) = \dim(K_\lambda(M))$, where $K_\lambda(M)$ is regarded as a Z/pZ -vector space.

3.2. THEOREM. *If G and H are S -groups, then $G \cong H$ if and only if for every ordinal α and every limit ordinal λ which is not cofinal with ω , $f(\alpha, G) = f(\alpha, H)$ and $k(\lambda, G) = k(\lambda, H)$. Any S -group G is a direct sum of a totally projective p -group and λ -elementary S -groups where the ordinals λ are not cofinal with ω . The number of λ -elementary summands in such a representation is exactly $k(\lambda, G)$. In particular, G is totally projective if and only if $k(\lambda, G) = 0$ for all limit ordinals λ which are not cofinal with ω .*

PROOF. We first remark that if G is totally projective then $k(\lambda, G) = 0$ for all limit ordinals λ which are not cofinal with ω , by [16, 3.10], since $G/p^\lambda G$ is a direct sum of groups of smaller length. We remark that this makes the last statement of the theorem a trivial consequence of the rest.

We next complete the proof of the second half of the theorem. Any S -group is a direct sum of a totally projective group and λ -elementary S -groups for various ordinals λ , by the definition. By 3.1, if λ is a limit ordinal which is co-

final with ω , then a λ -elementary S -group is totally projective. Therefore, any S -group is the direct sum of a totally projective group and λ -elementary S -groups, where the λ 's may all be assumed to be not cofinal with ω . If Λ is the set of limit ordinals (not dofinal with ω) for which summands appear, then we can write $G = T \oplus (\bigoplus_{\lambda \in \Lambda} G_\lambda)$ where each G_λ is a direct sum of λ -elementary S -groups. Fixing λ , we write $G = T \oplus A \oplus G_\lambda \oplus B$, where A is the direct sum of the G_γ for $\gamma > \lambda$ and B is the direct sum of the G_γ for $\gamma < \lambda$. $k(\lambda, T) = 0$ by the first remark of the proof. $k(\lambda, B) = 0$, since $p^\lambda c(B) = 0$. $k(\lambda, A) = 0$ since if E is a γ -elementary S -group and $\lambda < \gamma$, then $E/p^\lambda E$ is totally projective. Hence $k(\lambda, G) = k(\lambda, G_\lambda)$. G_λ is the torsion submodule of a KT -module M_λ such that $p^\lambda M_\lambda$ is a free module whose rank is the number of λ -elementary summands of G_λ . By 2.3, $k(\lambda, G)$ is equal to the rank of $p^\lambda M_\lambda$, which establishes the computation of $k(\lambda, G)$ stated in the second part of the theorem.

We turn now to the first part of the theorem. We have shown that an S -group G can be regarded as the maximal torsion subgroup of a KT -module M , where M is the direct sum of a totally projective p -group and λ -elementary KT -module, $\lambda > 0$, where the ordinals λ are not cofinal with ω , and if this is done, then the number of λ -elementary summands of M is just $k(\lambda, G)$. It is also $h(\lambda, M)$. We note also that the Ulm invariants of M and G agree, and that if λ is a limit ordinal which is cofinal with ω , then $h(\lambda, M) = 0$. Hence we can imbed G and H as the torsion submodules of KT -modules M and N with the same invariants. By 1.3, $M \cong N$, and since G and H are the maximal torsion submodules, $G \cong H$.

4. Admissible sets of invariants.

DEFINITION. Let f and h be functions associating to each ordinal number a cardinal number, and such that for some ordinal γ , $f(\alpha) = h(\alpha) = 0$ for all α , $\alpha > \gamma$. Let

$$\lambda(f, h) = \sup\{\alpha + 1, f(\alpha) \neq 0; \alpha + \omega, h(\alpha) \neq 0\}.$$

We say (f, H) is an *admissible pair* if $h(\alpha) = 0$ unless α is a limit ordinal, and if α is a limit ordinal with $\alpha + \omega < \lambda(f, h)$, and $0 \leq n < \omega$, then

$$(*) \quad \sum_{\alpha+n \leq \beta < \alpha+\omega} f(\beta) \geq \sum_{\alpha+\omega \leq \beta < \lambda(f,h)} (f(\beta) + h(\beta)).$$

4.1. THEOREM. *If f and h are functions associating to each ordinal a cardinal number, then there is a KT -module M such that $f(\alpha, M) = f(\alpha)$ and $h(\beta, M) = h(\beta)$ for all ordinals α and all limit ordinals β if and only if (f, h) is an admissible pair.*

Before proving this theorem, we investigate a large class of modules whose Ulm invariants satisfy the above inequalities. The next lemma is motivated by a method used by Nunke in [15, pp. 167–168]. For any module A , we let $\#(A)$ be the smallest cardinal number which is the cardinality of a set of generators of A . (For the countable ring Z_p , this number is just the cardinality of A , except when A is finite, but the version presented here is valid over any discrete valuation ring.)

DEFINITION. A module M is said to be in the class Θ if and only if for every submodule A of M , $\#(A) = \#(A/pA)$.

4.2. LEMMA. *If $M_i, i \in I$, are in the class Θ then $\bigoplus_{i \in I} M_i$ is also in Θ . If M is a module and C a submodule which is finitely generated, and if M/C is in Θ , then M is in Θ .*

PROOF. In the first case, let A be a submodule of $\bigoplus_{i \in I} M_i$. Let X be a subset of A which is taken bijectively onto a basis of A/pA under the natural map, and let $B = [X]$. Clearly, $\#(B) = \#(A/pA)$. Let π_i be the projection of M onto the summand M_i . Since $A = B + pA$, $\pi_i(A) = \pi_i(B) + p\pi_i(A)$, and since $M_i \in \Theta$, $\#(\pi_i(A)) = \#(\pi_i(B))$. If B is not finitely generated, then $\#(B) = \sum_{i \in I} \#(\pi_i(B))$, and A is also infinitely generated, so $\#(A) = \sum_{i \in I} \#(\pi_i(A))$ and the result follows. If B is finitely generated, then since $\pi_i(A)/\pi_i(B)$ is divisible, and since it is clear that an element of Θ cannot have a divisible submodule, $\pi_i(A) = 0$ for all but finitely many i , since the same holds for B . Since each $\pi_i(A)$ is finitely generated, it follows that A must be finitely generated, in which case a standard computation shows that $\#(A) = \#(A/pA)$.

For the second statement, we may restrict ourselves to infinitely generated submodules of M entirely. If $\phi: M \rightarrow M/C$ is the natural map, and A is a submodule of M which is not finitely generated, then $\#(A) = \#(\phi(A)) = \#(\phi(A)/p\phi(A)) \leq \#(A/pA)$, from which the result follows.

4.3. LEMMA. *If $M \in \Theta$ and for any ordinal α and limit ordinal β we define $f(\alpha) = f(\alpha, M)$ and $h(\beta) = h(\beta, M)$, and set $h(\alpha) = 0$ for any ordinal α which is not a limit ordinal, then (f, h) is an admissible pair.*

PROOF. All that we need to prove is the inequality (*) of the definition. If α is a limit ordinal, we can write $p^\alpha M = B \oplus A$ where $p^n B = 0$ and A has no summands which are cyclic of order less than p^{n+1} . Let $v: A \rightarrow A/p^{\alpha+\omega} M$ be the natural map and let N be the submodule consisting of all elements of A which are mapped into the torsion submodule of $A/p^{\alpha+\omega} M$. We note that $p^{\alpha+\omega} M = p^\omega A = p^\omega N$, and if $k < \omega$, we compute the Ulm invariants $f(k, N) = 0, k < n$, and $f(k, N) = f(\alpha + k), k \geq n$. From this, it is clear that

$$\#(N/pN) = \sum_{\alpha+n \leq \beta < \alpha+\omega} f(\beta).$$

Since $N \in \Theta$ and $p^{\alpha+\omega}M \subseteq N$, we can prove the inequality by finding a submodule C of $p^{\alpha+\omega}M$ such that

$$\#(C) = \sum_{\alpha+\omega \leq \beta < \lambda} (f(\beta) + h(\beta))$$

(since $\#(C) \leq \#(N) = \#(N/pN)$). (Here, λ is the length of M .)

If γ is a limit ordinal and B_γ is a basic submodule of $p^\gamma M$ then

$$\#(B_\gamma) = \sum_{\gamma \leq \beta < \gamma+\omega} (f(\beta) + h(\beta)).$$

Since $B_\gamma \cap p^{\gamma+\omega}M = 0$, the submodules B_γ of M are independent, and if we let $C = \bigoplus B_\gamma$, where the direct sum ranges over all limit ordinals γ , $\alpha + \omega \leq \gamma < \lambda$, then C has the desired properties.

4.4. LEMMA. *If G is a p -group of length α , there is a Z_p -module M such that $p^\alpha M \cong Z_p$ and $M/p^\alpha M \cong G$.*

This is a special case of a well-known existence theorem, first proved by Pierce (unpublished); see [20], [18, Theorem 1.6].

PROOF OF THEOREM 4.1. The existence proof is by induction on $\lambda = \lambda(f, h)$. We distinguish two cases: (i) when there is a limit ordinal γ such that $\lambda = \gamma + \omega$ or $\lambda = \gamma + n$ for some positive integer n , and (ii) when no such γ exists. Case (ii) is easy. By a standard infinite combinatorial argument, we can find functions f_i and h_i ($i \in I$) such that $f = \sum_{i \in I} f_i$, $h = \sum_{i \in I} h_i$, and for each $i \in I$, (f_i, h_i) is an admissible pair with $\lambda(f_i, h_i) < \gamma$. By induction, there are KT -modules M_i ($i \in I$) with invariants given by the functions (f_i, h_i) , and $M = \bigoplus_{i \in I} M_i$ corresponds to the pair (f, h) .

In case (i), we use a similar combinatorial argument to write f and h as sums of functions f_i and h_i ($i \in I$), such that each pair (f_i, h_i) is admissible and we impose the additional restriction that for each $i \in I$,

$$\sum_{\gamma \leq \alpha < \gamma+\omega} (f_i(\alpha) + h_i(\alpha)) = 1.$$

By induction, there are KT -modules H_i , $i \in I$, such that $p^\gamma H_i = 0$ and such that for $\alpha < \gamma$, $f(\alpha, H_i) = f_i(\alpha)$; and for limit ordinals β , $\beta < \gamma$, $h(\beta, H_i) = h_i(\beta)$. Let C_i be a cyclic module, $C_i \cong Z_p$ if $h_i(\gamma) \neq 0$ and $C_i \cong Z/p^{n+1}Z$ if $f_i(\gamma + n) \neq 0$. By 4.4, there are KT -modules M_i such that $p^\gamma M_i \cong C_i$ and $M_i/p^\gamma M_i \cong H_i$. The module $M = \bigoplus_{i \in I} M_i$ now satisfies our conditions.

All of the modules produced in the existence part of the proof of 4.1 are in

Θ , by 4.2. If M is any KT -module, we can get from this existence proof a KT -module N in Θ such that the invariants of N and $N \oplus M$ agree, from which it follows (from 1.3) that $N \cong N \oplus M$. Since a summand of an element of Θ is in Θ , this shows that any KT -module is in Θ , thus (with another reference to 4.3) completing the proof of 4.1.

4.5. COROLLARY. *Let C be a class of modules containing the KT -modules, and such that the direct sum of two modules in C is in C . Suppose that for all pairs of elements M and N of C , $M \cong N$ if for all ordinals α and all limit ordinals λ , $f(\alpha, M) = f(\alpha, N)$ and $h(\lambda, M) = h(\lambda, N)$. Then C coincides with the class of KT -modules.*

PROOF. If M is in C there is a KT -module N such that the invariants of N and $N \oplus M$ are the same. It follows that $N \oplus M \cong N$, so that M is a summand of a KT -module, and hence in Θ . M therefore has the same invariants as some KT -module K , from which it follows by hypothesis that $M \cong K$, thus proving the result.

4.6. COROLLARY. *Let C be a class of Z_p -modules such that*

- (i) *any module isomorphic to an element of C is in C ;*
- (ii) *a direct sum of modules in C is in C ;*
- (iii) *if for some ordinal α , $p^\alpha G$ and $G/p^\alpha G$ are in C , then $G \in C$.*

Then (a) if Z/pZ is in C , C contains all totally projective p -groups, and (b) if Z_p is in C , C contains all KT -modules.

This follows from the proof of 4.1. Notice that in (iii), if we had restricted $p^\alpha G$ to be cyclic of order p , we still would get all totally projective p -groups. Part (a) was proved by Parker and Walker in [19].

4.7. THEOREM. *If G is a p -group which can be imbedded in a totally projective group, then there is a totally projective p -group H such that for all α , $f(\alpha, G) = f(\alpha, H)$.*

PROOF. This is an immediate consequence of 4.1, 4.3, and the fact that all totally projective p -groups are in Θ (which follows from the existence proof in 4.1).

4.8. THEOREM. *Let f and k be two functions associating to each ordinal a cardinal and such that there is some ordinal τ such that $f(\alpha)$ and $k(\alpha)$ vanish for $\alpha > \tau$. Suppose also that for any ordinal α , $k(\alpha) = 0$ unless α is a limit ordinal which is not cofinal with ω , $\alpha > 0$. Then there is an S -group G such that $f(\alpha, G) = f(\alpha)$ and $k(\lambda, G) = k(\lambda)$ for all ordinals α and all limit ordinals λ which*

are not cofinal with ω , if and only if for any limit ordinal and integer n , $0 \leq n < \omega$,

$$\sum_{\gamma+n \leq \beta < \gamma+\omega} f(\beta) \geq \sum_{\gamma+\omega \leq \beta < \tau} (f(\beta) + k(\beta)).$$

PROOF. By the argument used in the proof of 3.2, any S -group is the torsion submodule of a KT -module M such that $f(\alpha, M) = f(\alpha, G)$ for all ordinals α , $h(\lambda, M) = 0$ for any limit ordinal λ which is cofinal with ω , and $h(\lambda, M) = k(\lambda, G)$ for any limit ordinal which is not cofinal with ω . 4.1 therefore implies that the above inequality must hold. Conversely, if the above inequality holds, then there is a KT -module M with these invariants, and its torsion submodule is the desired G .

5. Closure properties of the classes of S -groups and KT -modules.

5.1. THEOREM. *If M is a Z_p -module and δ an ordinal, then M is a KT -module if and only if $p^\delta M$ and $M/p^\delta M$ are both KT -modules.*

PROOF. If M is a KT -module, it is clear from the definition and the corresponding property for totally projective groups that $p^\delta M$ and $M/p^\delta M$ are KT -modules.

For the converse, we first remark that we may assume that δ is a limit ordinal. This is because we already know the corresponding result for totally projective p -groups, which makes it clear from the definition that if M is a module and $p^n M$ is a KT -module, then so is M .

We next show that the invariants of M are those of some KT -module. Let $\sigma: (M/p^\delta M)[p] \rightarrow p^\delta M/p^{\delta+1} M$ be the surjective homomorphism defined in Lemma 1.4. Let L be a subgroup of $(M/p^\delta M)[p]$ which is mapped isomorphically onto $p^\delta M/p^{\delta+1} M$ by σ . Clearly, since $p^\delta M \in \Theta$,

$$\#(L) = \sum_{\delta \leq \beta < \lambda} (f(\beta, M) + h(\beta, M))$$

where λ is the length of M . We wish to check the inequality (*) for a limit ordinal α , $\alpha < \delta$ (since for other limit ordinals, the result is clear). For every limit ordinal γ such that $\alpha < \gamma < \delta$, let B_γ be a basic submodule of $p^\gamma M$ and B'_γ the image of B_γ in $M/p^\delta M$. As in the proof of 4.3, if C is the direct sum of the submodules B'_γ , then

$$\#(C) = \sum_{\alpha+\omega \leq \beta < \delta} (f(\beta, M) + h(\beta, M)).$$

Since $C \cap L = 0$ by construction (this is why we did not choose the modules B'_γ as arbitrary basic submodules of $p^\gamma(M/p^\delta M)$), as in the proof of 4.3,

$$\#(C) + \#(L) \leq \sum_{\alpha+n \leq \beta < \alpha+\omega} f(\beta, M).$$

Putting these inequalities together, we obtain condition (*).

From this we know that there is a KT -module N with the same invariants as M . Since $p^\delta N$ and $p^\delta M$ are KT -modules with the same invariants, there is an isomorphism $f: p^\delta M \rightarrow p^\delta N$. Since KT -modules of length δ have the δ -Zippin property (1.7), f extends to an isomorphism of M onto N , proving the result.

5.2. THEOREM. *If α is an ordinal and G a group, then G is an S -group if and only if $p^\alpha G$ and $G/p^\alpha G$ are S -groups.*

PROOF. If G is an S -group then G is the torsion subgroup of a KT -module M . $p^\alpha G$ is the torsion subgroup of the KT -module $p^\alpha M$, and therefore is an S -group. If G is a λ -elementary S -group, then if $\alpha \geq \lambda$, $G/p^\alpha G = G$, and is therefore an S -group, while if $\alpha < \lambda$, $G/p^\alpha G$ is totally projective. Since any S -group, G , is a direct sum of λ -elementary S -groups and a totally projective group, it follows that $G/p^\alpha G$ is an S -group.

We suppose now that G is a p -group, and that $p^\alpha G$ and $G/p^\alpha G$ are S -groups. Write $\alpha = \delta + n$, where δ is a limit ordinal. If $\phi: p^\alpha G \rightarrow M$ is an imbedding of $p^\alpha G$ as the torsion subgroup of a KT -module M (in which it may be assumed that $M/p^\alpha G$ is divisible), then there is a module M' and an imbedding $\phi': p^\delta G \rightarrow M'$ such that $p^n M' = M$. By 5.1, M' is a KT -module, and by construction, $p^\delta G$ is its torsion submodule. We therefore need only show that G is an S -group given that $p^\delta G$ and $G/p^\delta G$ are S -groups, where δ is a limit ordinal.

By the same argument as used in the proof of 5.1, the invariants of G are those of an S -group H . $p^\delta G$ and $p^\delta H$ are therefore isomorphic S -groups and there is an isomorphism $\phi: p^\delta G \rightarrow p^\delta H$. By 1.7, this extends to an isomorphism of G onto H , proving the result.

5.3. THEOREM. *If G is an S -group and λ a limit ordinal, a λ -high subgroup of G is also an S -group. If G is an S -group, λ a limit ordinal and $p^\lambda G = 0$, then a λ -dense, isotype subgroup of G is an S -group.*

PROOF. By 1.5, the first stated result is a consequence of the second. We assume, then, that $p^\lambda G = 0$ and that H is a λ -dense, isotype subgroup. By 1.5, there is a Z_p -module M and maps $\phi: H \rightarrow M$, $\psi: M \rightarrow G$ such that ϕ imbeds H as the torsion submodule of M , $\ker(\psi) = p^\lambda M$, $p^\lambda M$ is free, ψ is surjective, and $\psi\phi$ is the natural imbedding of H into G .

We can imbed G into a KT -module M' as its torsion submodule, by a map $f: G \rightarrow M'$. Since the map $\text{Ext}(M', p^\lambda M) \rightarrow \text{Ext}(G, p^\lambda M)$ is surjective, M is a submodule of a module M'' such that the map $\psi: M \rightarrow G$ extends to a surjective

map $\psi': M'' \rightarrow M'$ with kernel $p^\lambda M$. Clearly, $p^\lambda M''$ is free and $M''/p^\lambda M'' \cong M'/p^\lambda M'$, so, by 5.1, M'' is a KT -module. By construction, it is clear that ϕ imbeds H as the torsion submodule of M'' , so (by definition) H is an S -group.

6. Unsolved problems.

Problem 1. Is a summand of an S -group necessarily an S -group?

Problem 2. If not, can summands of S -groups be classified by the same invariants?

Problem 3. Is there a class C of reduced p -groups such that the direct sum of two groups in C is in C , such that C properly contains the class of S -groups, and such that the groups in C can be classified by the same invariants?

Before Ulm's theorem was proved for totally projective groups, Walker showed [19] that if this could be done, Ulm's invariants could not be used to classify the elements of any larger class of reduced p -groups closed with respect to finite direct sums. His argument shows that any group in such a class C (as described in Problem 3) would be a summand of an S -group. An affirmative answer to Problem 1 would therefore imply a negative answer to Problem 3, but possibly Problem 3 could be answered independently (compare 4.5 above). In [23], Wick gives a "projective" characterization of summands of S -groups.

Problem 4. Is there a reasonable structure theory for a larger class of groups which are isotype subgroups of totally projective groups?

Hill showed [6] that an isotype subgroup of a totally projective group of countable length is again totally projective. However, C. Megibben has pointed out to the author that an isotype subgroup of a direct sum of countable p -groups need not be an S -group. (This is easily obtained from an example in the last paragraph of [13, p. 109], using the fact that if G is an S -group, then $L_\Omega G/G$ is divisible.)

Problem 5. Is there a reasonable structure theory for the groups which arise as $\text{Tor}(A, B)$, when A and B are totally projective?

Nunke showed [14] that if A and B are totally projective of countable length, then $\text{Tor}(A, B)$ is totally projective, but Wick [23] shows that there are totally projective groups A and B such that $\text{Tor}(A, B)$ is not an S -group.

Problem 6. If G is totally projective and H a subgroup such that $(G/H)[p]$ is finite, is H necessarily an S -group?

Edington [1] gives necessary and sufficient conditions, in this situation, for H to be totally projective.

If λ is a limit ordinal and G is a p -group with $p^\lambda G = 0$, we say G has the λ -Zippin property if for all pairs H and K of p -groups such that $H/p^\lambda H \cong K/p^\lambda K \cong G$ and all isomorphisms $f: p^\lambda H \rightarrow p^\lambda K$, there is an isomorphism $\phi:$

$H \rightarrow K$ whose restriction to $p^\lambda H$ is f . Nunke shows [18] that if G is an S -group and $p^\lambda G = 0$, then G has the λ -Zippin property. The converse fails in a nontrivial way.

Problem 7. If G is a p -group and for all limit ordinals λ , $G/p^\lambda G$ has the λ -Zippin property, is G necessarily an S -group?

If u is a function associating to each nonnegative integer an ordinal or ∞ , and G is a p -group, we define $uG = \{x \in G: \text{for all } n \geq 0, p^n x \in p^{u(n)}G\}$. Kaplansky showed in [11] that if G is a reduced countable p -group, then the fully invariant subgroups of G are exactly the subgroups uG obtained in this way. The methods now available make it trivial to extend this result to totally projective p -groups. L. Fuchs and E. A. Walker (unpublished) have shown that if G is totally projective and u is such a function, then uG and G/uG are again totally projective.

Problem 8. If G is an S -group, are all fully invariant subgroups of the form uG ?

Problem 9. If G is an S -group, and u is one of the above functions, are uG and G/uG necessarily S -groups?

Problem 10. Let G and H be reduced p -groups and $\phi: G \rightarrow H$ a homomorphism such that the induced maps $U_\alpha(G) \rightarrow U_\alpha(H)$ and $K_\lambda(G) \rightarrow K_\lambda(H)$ are isomorphisms for all ordinals α and all limit ordinals λ . Is ϕ necessarily an isomorphism?

The answer to Problem 10 is yes for groups with no elements of infinite height.

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