TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 210, 1975

# SOBOLEV-GALPERN EQUATIONS OF ORDER $n + 2$ IN $R^m \times R$, $m \geqslant 2$

BY

V. R. GOPALA RAO

ABSTRACT. Equations with mixed time and space derivatives play an important role in several branches of physics. Here we establish existence and uniqueness results for such equations. In addition, we also prove a regularity result which employs a regularity result for nonhomogeneous elliptic equations whose proof is also included.

1. **Introduction.** In this article we shall concern ourselves with linear partial differential equations of the form

$$\sum_{i=0}^{n} A_i \frac{\partial^{n-i}}{\partial t^{n-i}} u(x, t) = f(x, t), \tag{1.1}$$

where $A_0$ is a second order elliptic partial differential operator and $A_1, \ldots, A_n$ are linear partial differential operators of orders at most 2 for the unknown function $u(x, t)$, $(x, t) \in R^m \times R$, $m \geqslant 2$. The possibility that $A_1, \ldots, A_{n-1}$ are all zero operators is included. When $n = 1$, the equation takes the familiar form

$$A_0(\partial u/\partial t) + A_1 u = f \tag{1.2}$$

which is known as a pseudo-parabolic equation. It is known that solutions to initial-boundary value problems for (1.2) form approximations to similar problems associated with the parabolic equation $\partial u/\partial t + A_1 u = f$. Results of this type were proved by Showalter and Ting [17] and by Ting [20]. The Cauchy problem in the whole space and the mixed boundary problem in the bounded domain for the equation

$$(\partial^2/\partial t^2)\Delta u + (\partial^2/\partial x^2)u = 0$$

were studied by Sobolev [18]. Clearly this later equation is a special case of (1.1), when $n = 2$. Galpern [6] studied the Cauchy problem for a system of equations of the form

$$A\left(t, \frac{\partial}{\partial x_k}\right) \frac{\partial \bar{u}}{\partial t} + B\left(t, \frac{\partial}{\partial x_k}\right) \bar{u} = 0,$$

Presented to the Society, January 17, 1974; received by the editors January 17, 1974 and, in revised form, July 30, 1974.

*AMS (MOS) subject classifications* (1970). Primary 35Q99, 35J15.

*Key words and phrases.* Sobolev-Galpern equations, pseudo-parabolic partial differential equations, elliptic equations.

where $\bar{u}$ is a vector of functions and $A$ and $B$ are quadratic polynomial matrices depending on $t$. His method of proving the existence and regularity properties of solutions depended on the theory of Fourier transforms. Very recently the so-called pseudo-parabolic equations have been attacked systematically by several authors. For a partial listing of these see [16]. While Showalter [17] and subsequent works have more or less restricted his attention to problems in Hilbert spaces, Lagnese [8] and subsequent works have dealt with problems in $L^p$-spaces. Rao [14] under the direction of Professor T. W. Ting of the University of Illinois, and Rao and Ting [15], [16] have restricted their attention to proving existence, uniqueness and regularity theorems for pure initial value problems for pseudo-parabolic equations in $R^m \times R$, $m \geqslant 2$.

Equations of type (1.1) with $n = 1$ arose in the study of second order fluids by Ting [19] and in an article by Coleman and Noll [4]. For a brief history of other physical origins see [17]. We believe that in the theory put forward by Coleman and Noll [4], equations of type (1.1), $n > 1$, will arise if approximations to the memory functionals of orders greater than 2 are considered. Equations similar to (1.1) with $A_n$ equal to a constant, called weighted elliptic equations, were considered by Agmon and Nirenberg in [1].

Differential equations with operator coefficients of the type

$$Lu = A(t)\frac{d^2u}{dt^2} + B(t)\frac{du}{dt} + C(t)u = h(t) \tag{1.3}$$

were considered by M. I. Višik [21]. The operators $A(t)$, $B(t)$, $C(t)$ are operators defined on a Hilbert space $H$ and $u(t)$ and $h(t)$ are functions with values in $H$. The Cauchy problem solved by Višik for (1.3) is that of finding a solution $u(t)$ satisfying

$$u\Big|_{t=0} = u_0, \quad \frac{du}{dt}\Big|_{t=0} = u_1 \qquad (u_0, u_1 \in H). \tag{1.4}$$

In [9], H. A. Levine studied uniqueness and growth theorems for the solution to the Cauchy problem to (1.3) with $h = 0$ in the Hilbert space situation. The method employed there was based on a study of certain partial differential inequalities.

Physically, equations of type (1.1) with $n = 2$ arise in the theory of elasticity. In fact, in the study of external vibrations of thin rods (with lateral motion due to inertia taken into account) the equation for the displacement $w(s)$ parallel to the central line turns out to be

$$\rho\left(\frac{\partial^2 w}{\partial t^2} - \sigma^2 K^2 \frac{\partial^4 w}{\partial s^2 \partial t^2}\right) = E\frac{\partial^2 w}{\partial s^2},$$

where $\rho$ is the density of the material, $\sigma$ is Poisson's ratio, $E$ is Young's modulus,

$K$ is the radius of gyration of a cross section about the central line and $s$ is the distance measured from a chosen point of the line.

Similar equations also appeared in the study of internal gravity waves [2, pp. 65, 68]. It must be pointed out that in these last two examples the condition that guarantees the uniqueness of the solution to the Cauchy problem is not satisfied.

For other examples of physical situations see Levine [9].

2. **Notations, statement of the problem and some preliminary results.** We shall, from now onwards, write $x = (x_1, \ldots, x_m)$, $dx = dx_1 \cdots dx_m$, and for $\alpha = (\alpha_1, \ldots, \alpha_m)$, $|\alpha| = \Sigma\alpha_i$, $D_x^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_m)^{\alpha_m}$, where $\alpha_1, \ldots, \alpha_m$ are nonnegative integers. For $1 < p < \infty$, $W^{k,p}(R^m)$ is the Banach space of elements in $L^p(R^m)$ which together with their distribution derivatives up to order $k$ are also in $L^p(R^m)$. If $u \in W^{k,p}(R^m)$, then

$$\|u\|_{k,p} = \left[\sum_{0 \leqslant |\alpha| \leqslant k} \int |D_x^\alpha u|^p \, dx\right]^{1/p}.$$

We let $W^{n,k,p}$ denote the $n$-fold Cartesian product $W^{k,p}(R^m) \times \cdots \times W^{k,p}(R^m)$ which becomes a Banach space under the norm

$$\|U\|_{n,k,p} = \left[\sum \|u_i\|_{k,p}^2\right]^{1/2}$$

for all $U = (u_1, \ldots, u_n) \in W^{n,k,p}$. It is clear that $W^{n,k,p} = W^{k,p}(R^m)$ when $n = 1$ and $W^{0,p}(R^m) = L^p(R^m)$. For functions which depend on a space variable, say $x$, and a time variable $t$, all norms which we come across in this article refer only to the space variable and never to the time variable. Therefore, for such functions, say $\varphi(x, t)$, the norms may be written $\|\varphi(x, t)\|$, $\|\varphi(\cdot, t)\|$, $\|\varphi\|(t)$ or $\|\varphi\|$ depending on the situation, with suitable subscripts to indicate the space to which they belong. Such a function $\varphi(x, t)$ is sometimes simply denoted by $\varphi(t)$ or even $\varphi$.

For $0 \leqslant l \leqslant n$, let $A_l$ be $n + 1$ differential operators defined by

$$A_l u = \sum_{i,j=1}^m a_{l,ij}(x, t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^m a_{l,i}(x, t) \frac{\partial u}{\partial x_i} + a_{l,0}(x, t)u. \tag{2.1}$$

For the sake of convenience we write $a_{0,ij} = a_{ij}$, $a_{0,i} = a_i$ and $a_{0,0} = a_0$. The following is a list of assumptions on the coefficients of the operators $A_l$, $0 \leqslant l \leqslant n$, which will be referred to throughout this article.

($A_1$) The coefficients of the operator $A_0$ are uniformly continuous and bounded in $R^m \times R$. For some constant $\lambda$, $0 < \lambda < 1$, the coefficients of $A_0$ are Hölder continuous in $x$, the space variable, uniformly in $R^m \times R$ with exponent $\lambda$. For some constant $g > 0$, the coefficient $a_0 < -g^2$ for all $(x, t)$ in $R^m \times R$.

($A_2$) The operator $A_0$ is uniformly strongly elliptic, i.e., $\exists$ positive constants $\mu$ and $\eta$ such that for all real vectors $\xi = (\xi_1, \ldots, \xi_m)$, $\xi \neq 0$, and all $(x, t) \in R^m \times R$,

$$\mu|\xi|^2 \leqslant \sum a_{ij}(x, t)\xi_i\xi_j \leqslant \eta|\xi|^2, \tag{2.2}$$

where $|\xi|^2 = \Sigma \xi_i^2$.

($A_3$) The coefficients of the operators $A_l$, $1 \leqslant l \leqslant n$, are bounded measurable functions on $R^m \times R$ which are continuous in $t$ uniformly in $x \in R^m$ except possibly on a subset, of $R^m$, of measure zero.

Under the above three assumptions we wish to solve the following initial value problem in $R^m \times R$.

*Let $u_0 \in W^{2,p}(R^m)$ and $f: R \longrightarrow L^p(R^m)$ be two given functions with $1 < p < \infty$. Consider the problem of finding a function $u(x, t)$ which, as a function of $x$, belongs to $W^{2,p}(R^m)$, $1 < p < \infty$, for all $t \in R$ and which satisfies*

$$Lu = A_0 \frac{\partial^n u}{\partial t^n} + \cdots + A_{n-1} \frac{\partial u}{\partial t} + A_n u = f \tag{2.3}$$

*in $L^p(R^m)$ for each $t \in R$ together with the initial conditions*

$$\lim_{t \to 0} \frac{\partial^k}{\partial t^k} u(x, t) = u_{k+1}(x), \qquad 0 \leqslant k \leqslant n-1, \tag{2.4}$$

*in $W^{2,p}(R^m)$. Here $\partial u/\partial t$ is the limit as $t \longrightarrow 0$ of the difference quotient $h^{-1}[u(\cdot, t+h) - u(\cdot, t)]$ in $W^{2,p}(R^m)$, and similar meaning is attached to all other higher order derivatives with respect to $t$.*

The existence and uniqueness of solutions to this problem will be established in §3. In §4, a result on the smoothness of such solutions is proved with the aid of a result on the regularity of solutions of nonhomogeneous elliptic equations in $R^m$. A proof of this later result is included in §4. The method adopted for proving existence is constructive in nature and hence useful in applications. Our method relies heavily on the existence of the inverse of the operator $A_0$, which has been dealt with in our previous works quoted in the references. For the sake of reference and completeness we quote two results on the inverse of $A_0$.

THEOREM 2.1. *Let the operator $A_0$ satisfy* ($A_2$) *and* ($A_1$) *if $G(x, y; t)$ denotes the principal fundamental solution of the equation $A_0 u = 0$; then for $v \in L^p(R^m)$, $1 < p < \infty$, the function $u$, defined by*

$$u(x, t) = \int_{R^m} G(x, y; t) v(y)\, dy,$$

*as a function of $x$, belongs to $W^{2,p}(R^m)$ for all $t$ in $R$, and satisfies the equation $A_0 u = -v$ in $L^p(R^m)$ for all $t \in R$. Moreover, for $v \in L^p(R^m)$,*

$$\left\| \int_{R^m} G(\cdot, y; t) v(y)\, dy \right\|_{2,p} \leqslant \text{const} \|v\|_{0,p} \tag{2.5}$$

*where the constant can be chosen independent of t.*

The proof of this theorem for $t$ restricted to compact subsets of $R$ (see [16]) was based on the theory of singular integral operators and on the existence of unique principal fundamental solutions that decay exponentially at infinity together with their first derivatives. Existence of such solutions was demonstrated by Giraud [7]. This theorem may also be proved as in [16].

THEOREM 2.2. *Let the operator $A_0$ satisfy* ($A_1$) *and* ($A_2$). *If $f: R \longrightarrow L^p(R^m)$, $1 < p < \infty$, is continuous and bounded, then*

$$u(\cdot, t) = \int G(\cdot, y; t) f(y, t)\, dy \tag{2.6}$$

*is a continuous function from $R \longrightarrow W^{2,p}(R^m)$ satisfying $A_0 u = -f$. If $f: R \longrightarrow W^{2,p}(R^m)$, $1 < p < \infty$, is continuous and bounded, then*

$$\int G(\cdot, y; t)\, (A_0 f)\, (y, t)\, dy = f(\cdot, t). \tag{2.7}$$

In other words $A_0$ has a right inverse on $L^p(R^m)$ and a left inverse on $W^{2,p}(R^m)$ for all $t \in R$ and they coincide on $W^{2,p}(R^m)$. That the right inverse defines a continuous function from $R \longrightarrow W^{2,p}(R^m)$ has been proved [16]. It was pointed out to us that the existence of left inverse and the proof of (2.7) had to be made clear. This fact was used in the proof of the existence of right inverse [16]. The following is a

PROOF OF (2.7). By Miranda [11, Theorem 20, III, p. 71] (2.7) is true for all $t \in R$, for functions $f: R \longrightarrow C_0^\infty(R^m)$. Since for each $t \in R$ such functions are dense in $W^{2,p}(R^m)$, by use of (2.5) one can easily prove the validity of (2.7) by a standard argument for each $t \in R$.

Thus the right inverse exists on $L^p(R^m)$, while the inverse exists on the subspace $W^{2,p}(R^m)$, $\forall t \in R$ and they coincide on $W^{2,p}(R^m)$. Hence we shall write

$$A_0^{-1} f(x, t) = \int_{R^m} G(x, y; t) f(y, t)\, dy.$$

3. **Existence and uniqueness.** The problem posed in §2 by means of (2.3) and (2.4) will now be transformed into a first order equation in the variable $t$ with coefficients which are matrices of operators, and then this later equation will be solved via an equivalent integro-differential equation for functions in $W^{n,2,p}(R^m)$.

The transformation $u_1 = u$, $\partial u_1/\partial t = u_2, \ldots, \partial u_{n-1}/\partial t = u_n$ allows us to formulate the problem given by (2.3) and (2.4) as

$$AU_t + BU = F, \qquad \lim_{t \to 0} U(x, t) = U_0(x), \tag{3.1}$$

where

$$U(x, t) = (u_1(x, t), \dots, u_n(x, t))^t, \quad U_0(x, t) = (u_1(x), \dots, u_n(x))^t,$$

$$F(x, t) = (0, 0, \dots, f(x, t))^t,$$

$$A = \begin{bmatrix} 1 & 0 \cdots 0 & 0 \\ \cdot & \cdots\cdots & \cdot \\ \cdot & \cdots\cdots & \cdot \\ 0 & 0 \cdots 1 & 0 \\ 0 & 0 \cdots 0 & A_0 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0 & -1 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & -1 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & -1 \\ A_n & A_{n-1} & A_{n-2} & \cdot & \cdot & A_1 \end{bmatrix}$$

with $t$ as superscript indicating transpose and as a subscript indicating differentiation with respect to $t$. The problem given by (3.1) is clearly equivalent to the original problem. Both $A$ and $B$ are square matrices of order $n$ whose elements are operators. If $A_0^{-1}$ exists then $A$ has an obvious inverse, $A^{-1}$, which is essentially the same as $A$, but with the element $A_0$ replaced by $A_0^{-1}$ in the sense that $A^{-1}AU = U$ for any vector $U: R \longrightarrow W^{n,2,p}$ and that $AA^{-1}F = F$, $F: R \longrightarrow W^{n,0,p}$. Our objective is to establish

THEOREM 3.1. *Let the differential operator $A_0$ satisfy* ($A_1$) *and* ($A_2$) *and the differential operators $A_1$ through $A_n$ satisfy* ($A_3$). *If the function $f: R \longrightarrow L^p(R^m)$, $1 < p < \infty$, is continuous and bounded, then the problem described by* (3.1) *has a unique solution and so does the problem given by* (2.3) *and* (2.4).

Before we proceed to prove the theorem we first observe that part of the hypothesis on $A_0$ of Theorem 3.1 is the same as that of Theorems 2.1 and 2.2. Since $F = (0, 0, \dots, f)^t$ and $A^{-1}F = (0, 0, \dots, A_0^{-1}f)^t$, we now know that under the hypothesis of Theorem 3.1, $A^{-1}F: R \longrightarrow W^{n,2,p}$ is continuous. Note that $\|A^{-1}F\|_{n,2,p} = \|A_0^{-1}f\|_{2,p}$. The following proposition concerns an integro-differential equation equivalent to the problem given by (2.3). The equivalence will be proved shortly.

PROPOSITION 3.2. *Under the hypothesis of Theorem* 3.1, *the integral equation,*

$$U(\cdot, t) = U_0 - \int_0^t (A^{-1}BU)(\sigma)\, d\sigma + \int_0^t (A^{-1}F)(\sigma)\, d\sigma, \tag{3.2}$$

*for the function* $U : R \longrightarrow W^{n,2,p}(R^m)$ *has a solution given by*

$$U(t) = \sum_{l=0}^{\infty} (-1)^l (K_l V)(t) \tag{3.3}$$

*where*

$$V(t) = U_0 + \int_0^t (A^{-1}F)(\sigma)(\sigma)\, d\sigma = (K_0 V)(t), \tag{3.4}$$

$$(K_l V)(t) = \int_0^t (A^{-1}BK_{l-1}V)(\sigma)\, d\sigma, \qquad l \geq 1, \tag{3.5}$$

*and where the time integral is understood to be the limit in* $W^{n,2,p}(R^m)$ *of the Riemannian sums.*

The proof we adopt for this proposition is very much similar to that of Lemma 4.2 in [16]. While in [16] the spaces involved are Sobolev spaces, the spaces here are $n$-fold products of those Sobolev spaces. A look at the proof of Lemma 4.2 in [16] indicates that certain $W^{2,p}$-estimates used there should be extended to the $W^{n,2,p}$ spaces. Thus we shall show that in the present situation $K_1 V(t)$ is well defined and continuous and obtain an estimate on $\|K_1 V(t)\|_{n,2,p}$. The rest of the proposition can be proved exactly as in [16].

PARTIAL PROOF OF PROPOSITION 3.2. By the discussion immediately following the statement of Theorem 3.1 it is clear that $\|A^{-1}F\|_{n,2,p}$ is a continuous function of $t$. Consequently $V(t)$, defined by (3.4), is well defined and continuous. If $C[R, W^{n,k,p}(R^m)]$ denotes the space of continuous functions $U : R \longrightarrow W^{n,k,p}(R^m)$, then by Theorem 2.1 the right inverse

$$A^{-1} : C[R, W^{n,0,p}(R^m)] \longrightarrow C[R, W^{n,2,p}(R^m)]$$

is a bounded linear map. Hence if $C_1$ and $\beta$ are constants such that for all $t \in R$,

$$\|A^{-1}\| \leq C_1, \qquad \|f\|_{n,0,p} \leq \beta, \tag{3.6}$$

then we have the estimate

$$\begin{aligned}\|V(t)\|_{n,2,p} &\leq \|U_0\|_{n,2,p} + \int_0^{|t|} \|A^{-1}F\|_{n,2,p}(\sigma)\, d\sigma \\ &\leq \|U_0\|_{n,2,p} + C_1\beta|t| \quad \text{for all } t \in R.\end{aligned}$$

The hypothesis on the coefficients of the operators $A_1$ through $A_n$ implies that $BV : R \longrightarrow W^{n,0,p}(R^m)$ is continuous. This in turn implies that $A^{-1}BV : R \longrightarrow W^{n,2,p}(R^m)$ is continuous. Now

$$\begin{aligned}\|A^{-1}BV(t)\|^2_{n,2,p} = \sum_{i=2}^{n-1} \|u_i\|^2_{2,p} &+ \left\| u_n + \int_0^t (A_0^{-1}f)(\sigma)\, d\sigma \right\|^2_{2,p} \\ &+ \left\| A_0^{-1} \sum_{i=1}^{n} A_i u_{n-i+1} + A_1 \int_0^t (A_0^{-1}f)(\sigma)\, d\sigma \right\|^2_{2,p}.\end{aligned}$$

If $C_2$ denotes the maximum of all the norms of the operators $A_1, \ldots, A_n$ as operators from $C[R, W^{2,p}] \longrightarrow C[R, L^p]$, then the above inequality implies

$$\|A^{-1}BV(t)\|_{n,2,p}^2 \leqslant (C_1^2 C_2^2 + 1)\|V\|_{n,2,p}.$$

This means that $A^{-1}B$ as an operator from $C[R, W^{n,2,p}]$ is a bounded linear map. Hence, if $C = (C_1^2 C_2^2 + 1)^{1/2}$, then

$$\|A^{-1}B\|_{n,2,p} \leqslant C \quad \text{for all } t \in R. \tag{3.7}$$

From (3.5)–(3.7) it follows that

$$\|K_1 V(t)\|_{n,2,p} \leqslant C \int_0^{|t|} \|V(\sigma)\|\, d\sigma \leqslant C(\|U_0\|_{n,2,p}|t| + C_1\beta|t|^2/2!).$$

Proof of Theorem 3.1. First, we shall show that (3.1) is equivalent to the integral equation (3.2) for functions $U: R \longrightarrow W^{2,k,p}(R^m)$. Multiplying both sides of $AU_t + BU = F$ by $G(x, y; t)$ and integrating both sides of the equation with respect to $t$, we get

$$U = U_0 + \int_0^t (A^{-1}F)(\sigma)\, d\sigma - \int_0^t (A^{-1}BU)(\sigma)\, d\sigma, \tag{3.8}$$

where the initial condition of (3.1) was also used. Conversely by differentiating with respect to $t$ both sides of (3.2) in $W^{n,2,p}(R^m)$ and then applying the operator $A$ we get back the differential equation. Letting $t \longrightarrow 0$ in (3.2) we also get the initial condition.

The solution $U(t)$ obtained in Proposition 3.2 may be shown, as in [16], to satisfy

$$\|U(t) - U_0\|_{n,2,p} \leqslant (e^{C|t|} - 1)\|U_0\|_{n,2,p} + C_1\beta(e^{C|t|} - 1)/C$$

for all $t \in R$, where $C, C_1, \beta$ are as in the proof of Proposition 3.2. By letting $t \longrightarrow 0$ we observe that $U(t)$ satisfies the initial condition of the Cauchy problem.

To prove uniqueness we observe that as $U_1$ and $U_2$ are two solutions of (3.1), then $V = U_1 - U_2$ satisfies $V(t) = -\int_0^t (A^{-1}BV)(\sigma)\, d\sigma$. Therefore

$$\|V(t)\|_{n,2,p} \leqslant C \int_0^{|t|} \|V(\sigma)\|_{n,2,p}\, d\sigma,$$

where $C$ is as in Proposition 3.2. By Gronwall's inequality (see for example [17, Lemma 9.1]), this means $V(t) \equiv 0$.

Since $U = (u, \partial u/\partial t, \ldots, \partial^{n-1}u/\partial t^{n-1})^t$ and $U_t: R \longrightarrow W^{n,2,p}(R^m)$ are continuous, it follows that $u$ and its first $n$ derivatives exist and are continuous from $R \longrightarrow W^{2,p}(R^m)$. Therefore $U \in C^n(R, W^{2,p}(R^m))$ and is a unique solution to the problem given by (2.3) and (2.4) under $(A_1)$–$(A_3)$. This completes the proof of Theorem 3.1.

4. **Regularity of solutions.** We now wish to give a set of conditions on the coefficients of the operators $A_i$, $i = 0, 1, \ldots, n$, and the initial data $U_0$, so that

the solution $u$ lies in the Sobolev space $W^{k+2,p}(R^m)$ for each $t \in R$. The condition on $u_0$ is that it belongs to the same Sobolev space. The simple example $\Delta(\partial u/\partial t) - \Delta u = 0$, $u(x, 0) = u_0(x)$ shows that the condition is necessary. In the following the generic constant $\alpha$ (and $\beta$) need not necessarily be the same in any two different occurrences.

THEOREM 4.1. *Suppose that* ($A_1$)–($A_3$) *hold for the operators* $A_0, A_1, \ldots, A_n$ *and that the coefficients of these operators, together with their first k derivatives with respect to the coordinates of x are continuous and bounded in* $R^m \times R$. *Let* $f: R \longrightarrow W^{k,p}(R^m)$ *be continuous and bounded and* $u_0 \in W^{k+2,p}(R^m)$, *where* $k \geq 0$ *is an integer and* $1 < p < \infty$. *Then* $u(\cdot, t)$ *also belongs to* $W^{k+2,p}(R^m)$, $1 < p < \infty$, *for each* $t \in R$, *where* $u(x, t)$ *is the solution to the problem of* §2.

A proof of this theorem may be formulated along the lines of the proof given for a similar regularity result in [16]. The proof given here is based on a regularity result for solutions of nonhomogeneous elliptic equations in $R^m$.

LEMMA 4.2. *If the hypothesis of Theorem* 4.1 *pertaining to the operator* $A_0$ *and the function f holds good, then the solution to the equation* $A_0 u = -f$, *given by*

$$u(\cdot, t) = \int G(\cdot, y; t) f(y, t)\, dy \tag{4.1}$$

*belongs to* $W^{k+2,p}(R^m)$ *for all t, and*

$$\|u(\cdot, t)\|_{i+2,p} \leq \alpha \|u\|_{i+1,p} + \beta \|f\|_{i,p}, \qquad i = 1, \ldots, k, \tag{4.2}$$

*where* $\alpha$ *and* $\beta$ *are constants independent of u, f and t.*

PROOF. For any function $v(x, t)$ defined in $R^m \times R$ let $v^h(x, t)$ denote $h^{-1}[v(x + h, t) - v(x, t)]$, where $x + h = (x_1 + h, x_2, \ldots, x_m)$ and $A_0^h$ denote the operator $A_0$ with its coefficients replaced by the corresponding difference quotients. Then, since $A_0 u = f$, we have

$$A_0 u^h(x, t) = f^h(x, t) - A_0^h u(x + h, t).$$

The hypothesis on the coefficients of $A_0$ implies that

$$u^h(x, t) = \int G(x, y; t)\, [f^h(y, t) - A_0^h u(y + h, t)]\, dy.$$

Theorem 2.1 implies that for some constant $C_2$ independent of $t$,

$$\begin{aligned} \|u^h(\cdot, t)\|_{2,p} &\leq C_1 \|f^h(\cdot, t)\|_{0,p} + C_1 C_2 \|A^h u(\cdot, t)\|_{0,p} \\ &\leq \epsilon_1(h, t) + \epsilon_2(h, t)\|u(\cdot, t)\|_{2,p} + C_1 \|f(\cdot, t)\|_{1,p} + C_1 C_2 \|u(\cdot, t)\|_{2,p}, \end{aligned}$$

where $\epsilon_1$ and $\epsilon_2 \longrightarrow 0$ as $h \longrightarrow 0$ for all $t \in R$. Thus, for all $t \in R$, there exists a constant $h_0$ such that for $|h| \leqslant h_0$, the set of difference quotients $\{u^h, |h| \leqslant h_0\}$ is bounded in $W^{2,p}(R^m)$. The fact that $W^{2,p}(R^m)$ is reflexive (and hence weakly compact) implies that there exists a subsequence $\{h_\mu\}$, so that $\{u^{h_\mu}\}$ converges in $W^{2,p}(R^m)$. It is also clear that the limit function is the distribution of derivative of $u$ and hence a strong derivative of $u$, i.e. $\partial u/\partial x_1$. In a similar manner it can be shown that other first derivatives of $u$ exist and belong to $W^{2,p}(R^m)$. From this discussion it is clear that

$$\|\partial u/\partial x_i\|_{2,p} \leqslant \alpha\|u\|_{2,p} + \beta\|f\|_{1,p}, \qquad i = 1, \ldots, m,$$

which in turn implies

$$\|u(\cdot, t)\|_{3,p} \leqslant \alpha\|u(\cdot, t)\|_{2,p} + \beta\|f(\cdot, t)\|_{1,p}, \tag{4.3}$$

for some constants $\alpha$ and $\beta$ independent of $u$, $f$ and $t$. This completes the proof of (4.2) when $k = 1$. We prove the general case by induction on $k$. Thus, let us assume that (4.2) is true for $k - 1$. If we now differentiate $A_0 u = f$ $(k-1)$ times with respect to the coordinates of $x$ and rearrange the terms, we get

$$A_0 D_x^\gamma u = g(x, t), \qquad |\gamma| = k - 1,$$

where $g(x, t)$ is the sum of $D_x^\gamma f$ and products of derivatives of orders up to $k - 2$ of $u$ with derivatives of coefficients of $u$ up to order $k - 1$. By the hypothesis of this lemma $D_x^\gamma u$ and $g(x, t)$ are each differentiable one more time. It is also clear that $g(x, t) \in W^{1,p}(R^m)$. Thus, applying (4.3) to the solution $D_x^\gamma u$ of the elliptic equation above and the induction hypothesis we get the required inequality. The lemma is now proved.

PROOF OF THEOREM 4.1. Because of the special nature of the matrix $A$ of operators it is easy to extend (4.2) to the solution of $A\varphi = \psi$, in the form

$$\|\varphi\|_{n,i+2,p} \leqslant \alpha\|\varphi\|_{n,i+1,p} + \beta\|\psi\|_{n,i,p}, \qquad i = 1, \ldots, k, \tag{4.4}$$

where $\alpha$ and $\beta$ are constants independent of $\varphi$, $\psi$ and $t$. We now apply this result to the solution of (3.1) given by (3.3)–(3.5). More precisely, we will use (4.4) to estimate the $\|\cdot\|_{n,k+2,p}$-norms of the terms of the solution given by (3.3)–(3.5). First, we estimate $\|V\|_{n,k+2,p}$. Since $V(t) = U_0 + \int_0^t A^{-1}F(\sigma)\,d\sigma$ and $U_0$ and $A^{-1}F \in W^{n,k+2,p}$, it is clear that

$$\|V\|_{n,k+2,p} \leqslant \|U_0\|_{n,k+2,p} + |t|C_1 \sup_t \|f\|_{n,k,p}, \tag{4.5}$$

and

$$\|BV\|_{n,k,p} \leqslant C_2\|U_0\|_{n,k+2,p} + |t|C_1C_2 \sup_t \|f\|_{n,k,p}. \tag{4.6}$$

By definition of $K_1 V$, we have

$$\|K_1 V\|_{n,k+2,p} \leq \int_0^{|t|} \|A^{-1}BV\|_{n,k+2,p}.$$

Since $A^{-1}BV$ satisfies $A\varphi = BV$, we also have

$$\|K_1 V\|_{n,k+2,p} \leq \int_0^{|t|} [\alpha\|A^{-1}BV\|_{n,k+1,p} + \beta\|BV\|_{n,k,p}](\sigma)\, d\sigma.$$

By applying (4.4) recursively to the term $A^{-1}BV$ we get

$$\|K_1 V\|_{n,k+2,p} \leq \int_0^{|t|} [\alpha\|A^{-1}BV\|_{n,2,p} + \beta\|BV\|_{n,k,p}](\sigma)\, d\sigma$$

(4.7)
$$\leq \int_0^{|t|} [\alpha\|U_0\|_{n,k+2,p} + \beta\gamma\sigma]\, d\sigma$$

$$\leq C[\|U_0\|_{n,k+2,p}|t| + \gamma|t|^2/2],$$

where $\gamma = \sup\|f\|_{n,k,p}$ over $t \in R$, and $C = \max\{\alpha, \beta\}$. We now remark that it can be shown that

(4.8) $$\|K_l V\|_{n,k+2,p} \leq C^l[\|U_0\|_{n,k+2,p}|t|^l/l! + \gamma|t|^{l+1}/(l+1)!]$$

for integers $l > 2$ by induction. Since (4.7) and (4.5) are special cases of (4.8) for $l = 1$ and $l = 0$ respectively, it is clear that for a suitable constant $C$ independent of $U_0$ and $t$, we have

$$\|U\|_{n,k+2,p} = \left\|\sum_{l=0}^{\infty} K_l V\right\|_{n,k+2,p} \leq \|U_0\|_{n,k+2,p} e^{C|t|} + (l/C^2)(e^{C|t|} - 1).$$

This proves that $U \in W^{n,k+2,p}$ for all $t \in R$ and gives an estimate on its norm. Since $u$ is the first component of the vector $U$, our theorem is completely proved.

ACKNOWLEDGEMENT. The author would like to thank the referee for his valuable suggestions.

REFERENCES

1. S. A. Agmon and L. Nirenberg, *Properties of solutions of ordinary differential equations in Banach space*, Comm. Pure Appl. Math. 16 (1963), 121–239. MR 27 #5142.

2. F. P. Bretherton, *Linearised theory of wave propagation*, Lectures in Appl. Math., vol. 13, Amer. Math. Soc., Providence, R.I., 1971, pp. 61–102.

3. R. Carroll, *Abstract methods in partial differential equations*, Harper and Row, New York, 1969.

4. B. D. Coleman and W. Noll, *An approximation theorem for functionals, with applications in continuum mechanics*, Arch. Rational. Mech. Anal. 6 (1960), 355–370. MR 22 #10359.

5. Avner Friedman, *Partial differential equations*, Holt, Reinhart and Winston, New York, 1969.

6. C. A. Galpern, *Cauchy problem for general systems of linear partial differential equations*, Dokl. Akad. Nauk SSSR 119 (1958), 640–643. (Russian) MR 21 #218.

7. G. Giraud, *Généralisation des problèmes sur les opérations du type elliptic*, Bull. Sci. Math. 56 (1932), 248–272, 281–312, 316–352.

8. J. E. Lagnese, *General boundary value problems for differential equations of Sobolev-Galpern type*, SIAM J. Math. Anal. 3 (1972), 105–119.

9. H. A. Levine, *Some uniqueness and growth theorems in the Cauchy problem for* $Pu_{tt} + Mu_t + Nu = 0$ *in Hilbert space,* Math. Z. 126 (1972), 345–360. MR 47 #3865.

10. N. G. Meyers and J. Serrin, *The exterior Dirichlet problem for second order elliptic partial differential equations,* J. Math. Mech. 9 (1960), 513–518. MR 22 #8200.

11. C. Miranda, *Partial differential equations of elliptic type,* Springer-Verlag, New York, 1966.

12. C. B. Morrey, Jr., *Multiple integrals in the calculus of variations,* Die Grundlehren der math. Wissenschaften, Band 130, Springer-Verlag, New York, 1966. MR 34 #2380.

13. L. Nirenberg, *Remarks on strongly elliptic partial differential equations,* Comm. Pure Appl. Math. 8 (1955), 649–675. MR 17, 742.

14. V. R. Gopala Rao, *A Cauchy problem for pseudo-parabolic partial differential equations in whole space,* Thesis, University of Illinois at Urbana-Champaign, 1972.

15. V. R. Gopala Rao and T. W. Ting, *Solutions of pseudo-heat equations in whole space,* Arch. Rational. Mech. Anal. 49 (1972), 57–78.

16. ———, *Initial value problems for pseudo-parabolic partial differential equations,* Indiana Univ. Math. J. 23 (1973), 131–153.

17. R. E. Showalter and T. W. Ting, *Pseudo-parabolic partial differential equations,* SIAM J. Math. Anal. 1 (1970), 1–26.

18. S. L. Sobolev, *On a new problem of mathematical physics,* Izv. Akad. Nauk SSSR Ser. Mat. 18 (1954), 3–50. (Russian) MR 16, 1029.

19. T. W. Ting, *Certain non-steady flows of second-order fluids,* Arch. Rational Mech. Anal. 14 (1963), 1–26. MR 27 #3224.

20. ———, *Parabolic and pseudo-parabolic partial differential equations,* J. Math. Soc. Japan 21 (1969), 440–453. MR 41 #8827.

21. M. I. Višik, *The Cauchy problem for equations with operator coefficients; mixed boundary value problem for systems of differential equations and approximation methods for their solution,* Mat. Sb. 39 (81) (1956), 51–148; English transl., Amer. Math. Soc. Transl. (2) 24 (1963), 173–278. MR 18, 215.

DEPARTMENT OF MATHEMATICS, LEHIGH UNIVERSITY, BETHLEHEM, PENNSYLVANIA 18015