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ON AUTOMORPHISM GROUPS AND ENDOMORPHISM RINGS OF ABELIAN p-GROUPS(1)

BY

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ABSTRACT. Let A be a noncyclic abelian p-group where $p \geqslant 5$, and let $p^{\infty}A$ be the maximal divisible subgroup of A. It is shown that $A/p^{\infty}A$ is bounded and nonzero if and only if the automorphism group of A contains a minimal noncentral normal subgroup. This leads to the following connection between the ideal structure of certain rings and the normal structure of their groups of units: if the noncommutative ring R is isomorphic to the full ring of endomorphisms of an abelian p-group, $p \geqslant 5$, then R contains minimal two-sided ideals if and only if the group of units of R contains minimal noncentral normal subgroups.

1. Throughout the following, A is a p-primary abelian group with endomorphism ring End A and automorphism group Aut A. The maximal divisible subgroup of A is denoted by $p^{\infty}A$. W. Liebert has proved the following result.

(1.1) THEOREM (LIEBERT [7, p. 94]). *If either $A/p^{\infty}A$ is unbounded or $A = p^{\infty}A$ then* End A *contains no minimal two-sided ideals. If $A/p^{\infty}A$ is bounded and nonzero then* End A *contains a unique minimal two-sided ideal.*

The purpose of this note is to show that, for $p \geqslant 5$, the same class of abelian p-groups has a similar characterization in terms of automorphism groups. The following theorem will be established. Note that Aut A is commutative if and only if A has rank at most one (for $p \neq 2$; [3, p. 264]).

(1.2) THEOREM. *Let A be a noncyclic abelian p-group where $p \geqslant 5$. Then $A/p^{\infty}A$ is bounded and nonzero if and only if* Aut A *contains a minimal noncentral normal subgroup.*

Whether or not this theorem holds true for all primes p is not known at

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the present time. However, in contrast to Liebert's result, minimal noncentral normal subgroups of Aut A need not be unique (see (3.2)).

The following corollary indicates the close connection between the ideal structure of certain endomorphism rings and the normal structure of their groups of units. It is an obvious consequence of (1.1) and (1.2) and the fact that End A is noncommutative if and only if A has rank at least two [3, p. 227, Exercise 6].

(1.3) COROLLARY. *Let R be a noncommutative ring such that $R \simeq$ End A for some abelian p-group A where $p \geqslant 5$. Then R contains minimal two-sided ideals if and only if the group of units of R contains minimal noncentral normal subgroups.*

2. From now on we assume that the prime p is at least 5. If B and C are fully invariant subgroups of A and $B \leqslant C$, the set of all $\epsilon \in$ End A such that $C\epsilon \leqslant B$ forms an ideal in End A which will be denoted by Ann C/B. Likewise, the set of all $\alpha \in$ Aut A which induce the identity mapping in C/B is a normal subgroup of Aut A, denoted by Fix C/B. Note that

$$\text{Fix } C/B = (1 + \text{Ann } C/B) \cap \text{Aut } A$$

where, for $S \subset$ End A a set, $1 + S$ denotes the set of all $\epsilon \in$ End A such that $\epsilon = 1 + \sigma$ for some $\sigma \in S$. The stabilizer of a subgroup B of A is defined as

$$\text{Stab } B = \text{Fix } B \cap \text{Fix } A/B.$$

It is well known that Stab $B \simeq$ Hom$(A/B, B)$ [3, p. 251]. If H is a subgroup of a group G, then $C(H)$ denotes the centralizer of H in G. The center of G is denoted by $Z(G)$. For other notation and terminology see [2] and [4].

(2.1) PROPOSITION. *If A is divisible then* Aut A *contains no minimal noncentral normal subgroup.*

PROOF. Assume the existence of a minimal noncentral normal subgroup Λ of Aut A. An automorphism α of an abelian p-group G, where $p \neq 2$, is central if and only if $x\alpha \in \langle x \rangle$ for all $x \in G$ [1, p. 110 f]. Hence $\Lambda \nleqslant Z(\text{Aut } A)$ implies the existence of an integer $N \geqslant 1$ such that

$$\Lambda_n = \Lambda | A[p^n] \nleqslant Z(\text{Aut } A[p^n]) \quad \text{for all } n \geqslant N. \tag{2.2}$$

Since Aut A is noncommutative, $A[p^{N+1}]$ has a decomposition

$$A[p^{N+1}] = \langle d_1 \rangle \oplus \langle d_2 \rangle \oplus C, \qquad o(d_i) = p^{N+1} \quad \text{for } i = 1, 2$$

[3, p. 254, Exercise 4]. Let $\delta \in$ Aut $A[p^{N+1}]$ be defined by

$$d_1(\delta - 1) = p^N d_2, \qquad (\langle d_2 \rangle \oplus C)(\delta - 1) = 0. \tag{2.3}$$

Using $p \neq 2, 3$ it follows that δ is contained in every noncentral normal subgroup of Aut $A[p^{N+1}]$ [6, p. 140], [8, p. 374], [5, Theorem 4.1]. Since every automorphism of $A[p^{N+1}]$ can be extended to an automorphism of A [3, p. 254, Exercise 1], (2.2) implies that Λ_{N+1} is a noncentral normal subgroup of Aut $A[p^{N+1}]$. Hence, there exists $\lambda \in \Lambda$ such that $\lambda | A[p^{N+1}] = \delta$. Because of (2.3), $\lambda | A[p^N] = \delta | A[p^N] = 1$ and $\lambda \notin Z(\text{Aut} A)$. Consequently, $\lambda \in (\Lambda \cap \text{Fix} A[p^N]) \setminus Z(\text{Aut} A)$ and the minimality of Λ implies $\Lambda \cap \text{Fix} A[p^N] = \Lambda$ and $\Lambda \leqslant \text{Fix} A[p^N]$. This contradiction to (2.2) completes the proof.

(2.4) THEOREM. *If* Aut A *contains a minimal noncentral normal subgroup then* $A/p^\infty A$ *is bounded and nonzero.*

PROOF. In view of (2.1) we may assume that $A/p^\infty A$ is unbounded. Then $Z(\text{Aut} A)$ is isomorphic to the group of p-adic units [3, p. 262] which contains no element of order p. Hence

$$\Pi \cap Z(\text{Aut} A) = 1, \tag{2.5}$$

where Π denotes the maximal normal p-subgroup of Aut A. Let Λ be a minimal noncentral normal subgroup of Aut A. From $\bigcap_n \text{Fix} A[p^n] = 1$ it follows that $\Lambda \nleqslant \textit{Fix}\, A[p^n]$ and

$$\Lambda \cap \text{Fix} A[p^n] \leqslant Z(\text{Aut} A) \quad \text{for large } n. \tag{2.6}$$

Since $\text{Fix} A[p^n] \geqslant \text{Stab} A[p^n]$ and $\text{Stab} A[p^n] \simeq \text{Hom}(A/A[p^n], A[p^n]) \leqslant \Pi$ [3, p. 251] is a p-group, (2.5) and (2.6) imply

$$\Lambda \cap \text{Stab} A[p^n] \leqslant \Pi \cap Z(\text{Aut} A) = 1$$

and

$$\Lambda \leqslant C(\text{Stab} A[p^n]) \quad \text{for large } n. \tag{2.7}$$

By Lemma 2.1 of [4],

$$C(\text{Stab} A[p^n]) \leqslant Z(\text{Aut} A) \cdot \text{Fix} A[p^n].$$

It follows from (2.7) that

$$\Lambda \leqslant Z(\text{Aut} A) \cdot \text{Fix} A[p^n] \quad \text{for large } n,$$

and hence

$$\Lambda \leqslant \bigcap_n (Z(\text{Aut} A) \cdot \text{Fix} A[p^n]) = Z(\text{Aut} A)$$

[1, p. 110f], contradicting the hypothesis.

The intersection, θ, of all noncentral normal subgroups of Aut A was determined in [5]. We shall need the following facts.

(2.8) *If A is a noncyclic bounded group containing only one independent element of maximal order, then* $\theta = \langle (1 + p^m) \cdot 1_A \rangle$ *where* p^{m+1} *is the exponent of* A [5, Theorem 2].

(2.9) *If A is a bounded group containing two independent elements of maximal order then θ is noncentral* [5, Theorem 3].

(2.10) PROPOSITION. *Let A be bounded of exponent p^{m+1} with only one independent element, h, of maximal order. Let $p^{k+1} \geqslant p$ be the exponent of $A/\langle h \rangle$. Then* $\Lambda = \operatorname{Fix} A[p^m] \cap \operatorname{Fix} A/p^kA$ *is a minimal noncentral normal subgroup of* Aut A.

PROOF. *Let* $J = \operatorname{Ann} A[p^m] \cap \operatorname{Ann} A/p^kA$. Then

$$\Lambda = 1 + J. \tag{2.11}$$

If $\lambda \in \Lambda$ has the property that $h(\lambda - 1) \in \langle h \rangle$ then $h\lambda = h + p^m qh$, for some integer q, and $\lambda = (1 + p^m \cdot q) \cdot 1_A$ is central. Hence, if $\delta \in \Lambda$ is noncentral then $h(\delta - 1) \notin \langle h \rangle$, and $\delta - 1 \in S$ where $S = \{\epsilon \in J | h\epsilon \notin \langle h \rangle\}$. Let $\Delta \leqslant \Lambda$ be a noncentral normal subgroup of Aut A. It follows that

$$\Delta \cap (1 + S) \neq \varnothing. \tag{2.12}$$

If $\epsilon \in S$ then $h\epsilon \notin \langle h \rangle$ and $h\epsilon \in p^kA[p]$. Since $\langle h \rangle$ is an absolute direct summand of A [2, p. 50, Exercise 8], A has a decomposition $A = \langle h \rangle \oplus \langle b \rangle \oplus C$ where $h\epsilon = p^k b$, $o(b) = p^{k+1}$. This implies that any two elements in the set $1 + S$ are conjugate [2, p. 89]. Hence, using (2.12),

$$1 + S \subset \Delta. \tag{2.13}$$

By (2.8), $1 + (J \backslash S) = \langle (1 + p^m) \cdot 1_A \rangle = \theta$ is contained in every noncentral normal subgroup of Aut A. This together with (2.13) and (2.11) completes the proof.

Following the notation of Liebert [7], E_0A denotes the set of all $\epsilon \in \operatorname{End} A$ such that $A\epsilon$ is finite.

(2.14) PROPOSITION. *Let $A/p^\infty A$ be bounded of exponent $p^{m+1} \geqslant p$. Then* $\Lambda = \operatorname{Fix}(A[p^m] + p^\infty A) \cap \operatorname{Fix} A/p^\infty A \cap (1 + E_0A)$ *is a minimal normal subgroup of* Aut A.

PROOF. Note that $\Lambda = 1 + J$ where

$$J = \operatorname{Ann}(A[p^m] + p^\infty A) \cap \operatorname{Ann} A/p^\infty A \cap E_0A \tag{2.15}$$

is the unique minimal two-sided ideal of End A [7, p. 94]. Let S be the set of all $\epsilon \in J$ such that $A\epsilon$ is cyclic. Then

$$J = \langle S \rangle \tag{2.16}$$

is generated by S as an additive group. Since $\epsilon\phi = 0$ for all $\epsilon, \phi \in S$, the map $\epsilon \longmapsto 1 + \epsilon$ is an isomorphism from J onto Λ. Hence

$$\Lambda = 1 + J = \langle 1 + S \rangle \tag{2.17}$$

is generated by the set $1 + S$ as a multiplicative group. Let $1 \neq \Delta \leqslant \Lambda$ be a normal subgroup of Aut A. Because of (2.17), it suffices to show that $1 + S \subset \Delta$. Let $0 \neq \epsilon \in S$. Then A has a decomposition $A = \langle h \rangle \oplus C$ where $o(h) = p^{m+1}$, $0 \neq h\epsilon \in p^\infty A[p]$, $C\epsilon = 0$. One easily verifies that any two elements $\neq 1$ in $1 + S$ are conjugate [2, pp. 89, 106, Exercise 10], [3, p. 250]. Therefore, the proof will be completed once we show that $\Delta \cap (1 + S) \neq \{1\}$. Let $1 \neq \delta \in \Delta$ and assume that $\delta \notin 1 + S$, i.e. $A(\delta - 1)$ has rank at least 2. Then A has a decomposition

$$A = \langle h_1 \rangle \oplus \langle h_2 \rangle \oplus B \tag{2.18}$$

where

$$o(h_i) = p^{m+1}, \quad 0 \neq h_i(\delta - 1) \in p^\infty A[p] \quad \text{for } i = 1, 2. \tag{2.19}$$

There exists $\tau \in$ End A such that

$$h_1\tau = h_2, \quad (\langle h_2 \rangle \oplus B)\tau = 0. \tag{2.20}$$

By construction, $\tau^2 = 0$ and $A(\delta - 1)\tau \leqslant p^\infty A\tau = 0$. It follows from Lemma 2.6 of [5] that

$$1 + \tau(\delta - 1) \in \Delta. \tag{2.21}$$

Because of (2.20), $A\tau(\delta - 1) \leqslant \langle h_2 \rangle\tau$ is cyclic and, using (2.19), $h_1\tau(\delta - 1) = h_2(\delta - 1) \neq 0$. This together with (2.21) implies $1 \neq 1 + \tau(\delta - 1) \leqslant \Delta \cap (1 + S)$ as desired.

3. The proof of (1.2) is now easily completed by combining (2.4) together with (2.9) and Propositions (2.10) and (2.14).

We collect the various characterizations of the class of p-groups under consideration. The equivalence of (i)–(iii) was proved by W. Liebert and is valid for all primes p and with or without the uniqueness in (ii) and (iii) [7, pp. 91, 94].

(3.1) THEOREM. *For an abelian p-group A, where $p \geqslant 5$, the following conditions are equivalent.*

(i) *$A/p^\infty A$ is bounded and nonzero.*

(ii) *A has a (unique) maximal fully invariant subgroup.*

(iii) End *A has a (unique) minimal two-sided ideal $Z \neq 0$.*

(iv) Aut *A has a minimal noncentral normal subgroup, Λ, or A is cyclic $\neq 0$.*

The following example shows that (3.1) cannot be improved to include the uniqueness of Λ in (iv).

(3.2) EXAMPLE. Let $A = \langle h \rangle \oplus \langle b \rangle$ where $o(h) = p^2$ and $o(b) = p$. Let $\Lambda_1 = \text{Stab}\, A[p]$ and $\Lambda_2 = \text{Stab}\, pA$. Then Λ_1 and Λ_2 both are noncentral normal subgroups of Aut A of order p^2. Since $\Lambda_1 \cap \Lambda_2 = \langle (1 + p) \cdot 1_A \rangle = \theta$ is the intersection of all noncentral normal subgroups of Aut A (see (2.8) and [5, Theorem 2]), Λ_1 and Λ_2 are two distinct minimal noncentral normal subgroups of Aut A.

(3.3) REMARK. If either A is not reduced or A contains two independent elements of maximal order, Λ in (3.1)(iv) can be shown to be unique.

(3.4) REMARK. If J is a two-sided ideal in End A then $\Lambda(J) = (1 + J) \cap$ Aut A is a normal subgroup in Aut A. For nonreduced A, the (unique) minimal noncentral normal subgroup, Λ, of Aut A is $\Lambda(Z)$ where Z denotes the unique minimal two-sided ideal in End A (see (2.14) and [7, p. 94]); in all other cases, however, $\Lambda(Z)$ is not a minimal noncentral normal subgroup of Aut A [7, p. 94], [5, Theorem 2].

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